Bell's inequalities
and non-locality

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Introduction

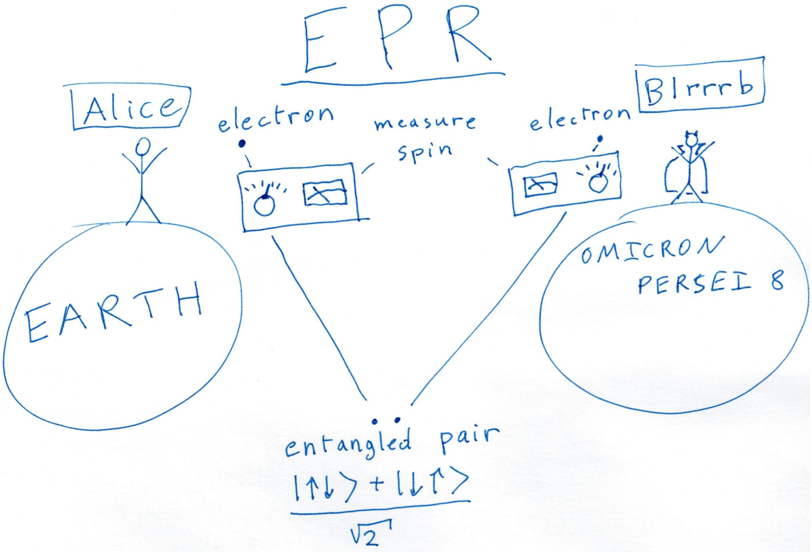
- Bell's inequalities: [Bell, 1964], [Clauser, Horne, Shimony, Holt, 1969], [Bell, 1975]...
- Prove that quantum mechanics is a non-local theory.
- Actually, they are extremely general, and only assume some carefully chosen assumptions about the probabilistic nature of quantum theory.
- Often characterized as forbidding "local hidden variable theories", as we shall see, this is not exactly untrue, but is misleading.
- Nobel prize 2022: Aspect, Clauser, Zeilinger: experimental verification of the predictions of quantum mehcanics.
- Reference: *Scholarpedia* article by Goldstein, Norsen, Tausk, Zanghi.

Part I

Bell, 1964

Bell's theorem

- Step 1: the EPR argument: [Einstein, Podolsky, Rosen, 1935].


EPR
Alice
electron
measure spin
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entangled pair
$\frac{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle}{\sqrt{2}}$

EPR

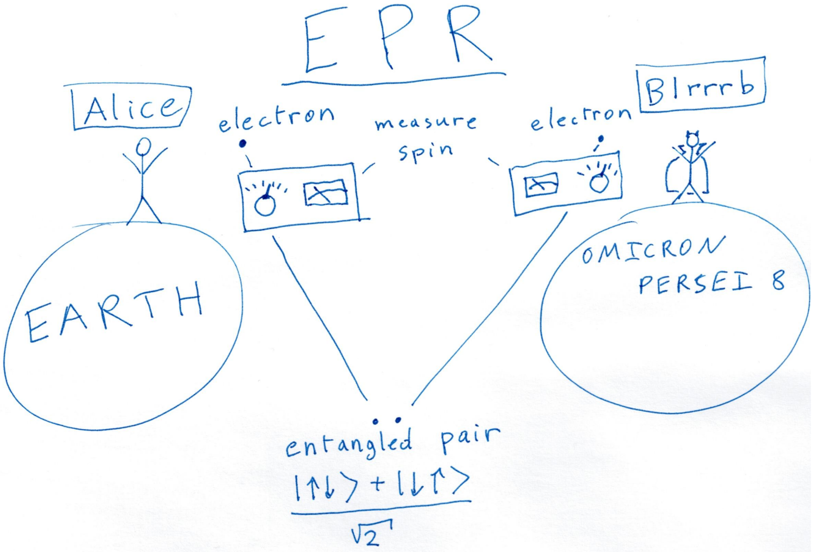
- Two observers at distance Δx measure the spin of electrons.
- Electron spin: can be measured in any direction in $\mathcal{S}^2$, returns $+1$ or -1.
- Before measurement, the electrons are in an entangled state: they are anti-correlated (if one returns $+1$, the other returns -1).
- In the usual approach to quantum mechanics, observables do not have values before they are measured.
- The observers perform their measurement at the same time. If the world were local, then, since the observers are spatially separated, one measurement cannot affect the other.
- But the outcomes are perfectly anticorrelated. Therefore, the outcomes had to be determined before the measurement was done ("hidden variables").

Bell's theorem

- Step 1: the EPR argument: [Einstein, Podolsky, Rosen, 1935]:

$$\boxed{\text{QM is local} \quad \Longrightarrow \quad \text{observables have predetermined values}}$$

- Step 2: Bell's inequality.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Bell's inequality (pigeonhole)

- In the EPR setting, the observers each make three measurements (e.g. choosing three different directions of spin). The outcomes of the measurements are random.
- We assume that the outcomes of the measurements exist independently of the measurement (predetermined values). In other words, we can represent the outcome of measurements as random variables that exist simultaneously:

$$Z_1^{(A)}, Z_2^{(A)}, Z_3^{(A)} \in \{-1, +1\}, \quad Z_1^{(B)}, Z_2^{(B)}, Z_3^{(B)} \in \{-1, +1\}$$

  that are distributed according to a probability distribution $\mathbb{P}$.
- We assume perfect anticorrelation:

$$\mathbb{P}(Z_i^{(A)} \neq Z_i^{(B)}) = 1.$$

Bell's inequality (pigeonhole)

- By the pigeonhole principle, at least two of the measurements must give the same answer:

$$\mathbb{P}(Z_1^{(A)} = Z_2^{(A)}) + \mathbb{P}(Z_1^{(A)} = Z_3^{(A)}) + \mathbb{P}(Z_2^{(A)} = Z_3^{(A)}) \geqslant 1$$

- Since A and B are anticorrelated: $\mathbb{P}(Z_i^{(A)} = Z_j^{(A)}) = \mathbb{P}(Z_i^{(A)} \neq Z_j^{(B)})$, so

$$\boxed{\mathbb{P}(Z_1^{(A)} \neq Z_2^{(B)}) + \mathbb{P}(Z_1^{(A)} \neq Z_3^{(B)}) + \mathbb{P}(Z_2^{(A)} \neq Z_3^{(B)}) \geqslant 1}$$

Bell's theorem

- Step 1: the EPR argument: [Einstein, Podolsky, Rosen, 1935]:

$$\boxed{\text{QM is local} \quad \Longrightarrow \quad \text{spins have predetermined values}}$$

- Step 2: Bell's inequality:

$$\boxed{\begin{array}{l}\text{spins have predetermined values} \Longrightarrow \\ \qquad \Longrightarrow \mathbb{P}(Z_1^{(A)} \neq Z_2^{(B)}) + \mathbb{P}(Z_1^{(A)} \neq Z_3^{(B)}) + \mathbb{P}(Z_2^{(A)} \neq Z_3^{(B)}) \geqslant 1\end{array}}$$

- Step 3: According to the laws of quantum mechanics,

Quantum mechanical prediction

- Suppose the three measurements are done at angles $\frac{2\pi}{3}$ from each other, then, for $i \neq j$,

$$\mathbb{P}(Z_i^{(A)} \neq Z_j^{(B)}) = \frac{1 + \cos\frac{2\pi}{3}}{2} = \frac{1}{4}.$$

- Therefore,

$$\mathbb{P}(Z_1^{(A)} \neq Z_2^{(B)}) + \mathbb{P}(Z_1^{(A)} \neq Z_3^{(B)}) + \mathbb{P}(Z_2^{(A)} \neq Z_3^{(B)}) = \frac{3}{4} < 1.$$

Bell's theorem

- Step 1: the EPR argument: [Einstein, Podolsky, Rosen, 1935]:

$$\boxed{\text{QM is local} \quad \Longrightarrow \quad \text{spins have predetermined values}}$$

- Step 2: Bell's inequality:

$$\boxed{\begin{array}{l}\text{spins have predetermined values} \Longrightarrow \\ \qquad \Longrightarrow \mathbb{P}(Z_1^{(A)} \neq Z_2^{(B)}) + \mathbb{P}(Z_1^{(A)} \neq Z_3^{(B)}) + \mathbb{P}(Z_2^{(A)} \neq Z_3^{(B)}) \geqslant 1\end{array}}$$

- Step 3: According to the laws of quantum mechanics,

$$\boxed{\mathbb{P}(Z_1^{(A)} \neq Z_2^{(B)}) + \mathbb{P}(Z_1^{(A)} \neq Z_3^{(B)}) + \mathbb{P}(Z_2^{(A)} \neq Z_3^{(B)}) \ngeqslant 1}$$

Bell's theorem

- Step 1: the EPR argument: [Einstein, Podolsky, Rosen, 1935]:

  QM is not local $\Longleftarrow$ spins do not have predetermined values

- Step 2: Bell's inequality:

  spins do not have predetermined values $\Longleftarrow$

  $$\Longleftarrow \mathbb{P}(Z_1^{(A)} \neq Z_2^{(B)}) + \mathbb{P}(Z_1^{(A)} \neq Z_3^{(B)}) + \mathbb{P}(Z_2^{(A)} \neq Z_3^{(B)}) \ngeq 1$$

- Step 3: According to the laws of quantum mechanics,

  $$\mathbb{P}(Z_1^{(A)} \neq Z_2^{(B)}) + \mathbb{P}(Z_1^{(A)} \neq Z_3^{(B)}) + \mathbb{P}(Z_2^{(A)} \neq Z_3^{(B)}) \ngeq 1$$

Reactions to Bell's theorem

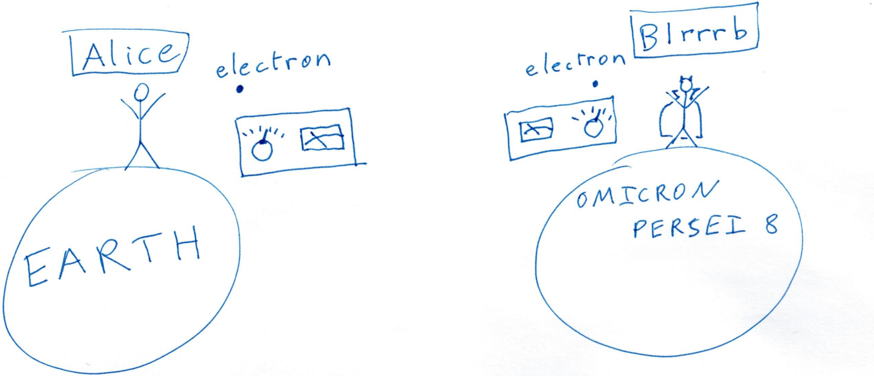
- The violation of Bell's inequality shows that the values of the spin in different directions do not simultaneously exist. More generally, the values of non-commuting operators do not simultaneously exist.
- This has lead many to state that Bell's theorem shows there are no hidden variables in quantum mechanics. This is not true: it is only true that there non-commuting observables do not simultaneously have values ("no non-contextual hidden variables").
- There is a theory of quantum mechanics with "hidden variables": Bohmian.
- Others have said that Bell's theorem shows there are no local hidden variables in quantum mechanics. While this is technically true, it misses the point: there is no locality in quantum mechanics.

More general statement

- The theorem discussed earlier requires perfect anticorrelation between spins in an entangled singlet. What if quantum mechanics is only very slightly wrong, and the anticorrelation is not exactly perfect?
- The proof of Bell's theorem goes through showing there are no pre-existing values, which has caused some confusion. Can we prove the theorem without any reference to pre-existing values?

Part II

Bell, revisited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General setup

- Observers A and B
- Each observer can set a tunable parameter on their measurement device: α, β (for instance, the direction of spin).
- Outcomes of a measurement: random variables $X_A, X_B \in [-1, 1]$, distributed according to a joint probability distribution $\mathbb{P}_{\alpha,\beta}$ that depends on α, β (the outcomes of measurements with different parameters α, β are not required to simultaneously exist).

Locality

- Naive definition:

$$\mathbb{P}_{\alpha,\beta}(X_A, X_B) = \mathbb{P}_{\alpha}^{(A)}(X_A)\mathbb{P}_{\beta}^{(B)}(X_B)$$

- This does not allow for any correlation between A and B (in particular, it excludes the anticorrelation considered earlier).

General setup

- Observers A and B
- Parameters α, β.
- Outcomes of a measurement: random variables $X_A, X_B \in [-1, 1]$, $\sim \mathbb{P}_{\alpha,\beta}$.
- Allow for an extra parameter λ, whose value is shared by both A and B and may affect the outcome of the measurements. For instance, λ could be a shared state of the electrons (e.g. an anticorrelated entangled state). λ is a random variable distributed according to $\mathbb{Q}$ (independent of α, β).
- Locality:

$$\mathbb{P}_{\alpha,\beta}(X_A, X_B|\lambda) = \mathbb{P}_{\alpha}^{(A)}(X_A|\lambda)\mathbb{P}_{\beta}^{(B)}(X_B|\lambda)$$

Bell's inequality

Theorem: Under these assumptions, $\forall \alpha, \alpha', \beta, \beta'$,

$$|\mathbb{E}_{\alpha,\beta}(X_A X_B) - \mathbb{E}_{\alpha,\beta'}(X_A X_B)| + |\mathbb{E}_{\alpha',\beta}(X_A X_B) + \mathbb{E}_{\alpha',\beta'}(X_A X_B)| \leqslant 2$$

where $\mathbb{E}_{\alpha,\beta}$ is the expectation value in the probability distribution $\mathbb{P}_{\alpha,\beta}$.

Proof of Bell's inequality

$$\mathbb{E}_{\alpha,\beta}(X_A X_B) = \int d\mathbb{Q}(\lambda)\ \mathbb{E}_{\alpha,\beta}(X_A X_B|\lambda)$$

by the locality condition,

$$\mathbb{E}_{\alpha,\beta}(X_A X_B|\lambda) = \mathbb{E}_{\alpha}^{(A)}(X_A|\lambda)\mathbb{E}_{\beta}^{(B)}(X_B|\lambda)$$

so

$$|\mathbb{E}_{\alpha,\beta}(X_A X_B) \pm \mathbb{E}_{\alpha,\beta'}(X_A X_B)| \leqslant \int d\mathbb{Q}(\lambda)\ \left|\mathbb{E}_{\alpha}^{(A)}(X_A|\lambda)\right| \left|\mathbb{E}_{\beta}^{(B)}(X_B|\lambda) \pm \mathbb{E}_{\beta'}^{(B)}(X_B|\lambda)\right|.$$

Since $|X_A| \leqslant 1$, $|\mathbb{E}_{\alpha}^{(A)}(X_A|\lambda)| \leqslant 1$.

Proof of Bell's inequality

If $x := \mathbb{E}_{\beta}^{(B)}(X_B|\lambda)$ and $y := \mathbb{E}_{\beta'}^{(B)}(X_B|\lambda)$, then

$$|\mathbb{E}_{\alpha,\beta}(X_A X_B) - \mathbb{E}_{\alpha,\beta'}(X_A X_B)| + |\mathbb{E}_{\alpha',\beta}(X_A X_B) + \mathbb{E}_{\alpha',\beta'}(X_A X_B)| \leqslant \\ \leqslant \int d\mathbb{Q}(\lambda)\ |x-y| + |x+y|$$

and since $|x| \leqslant 1$ and $|y| \leqslant 1$,

$$|x-y| + |x+y| \leqslant 2.$$

$\square$

Quantum mechanical prediction

- $\alpha, \beta \in \mathcal{S}^2$: direction of spin:

$$\mathbb{E}_{\alpha,\beta}(X_A X_B) = -\alpha \cdot \beta.$$

-

$$|\mathbb{E}_{\alpha,\beta}(X_A X_B) - \mathbb{E}_{\alpha,\beta'}(X_A X_B)| + |\mathbb{E}_{\alpha',\beta}(X_A X_B) + \mathbb{E}_{\alpha',\beta'}(X_A X_B)| =$$

$$= |\alpha \cdot (\beta - \beta')| + |\alpha' \cdot (\beta + \beta')|$$

- Choose $\beta' \cdot \beta = 0$, $\alpha = (\beta - \beta')/\sqrt{2}$, $\alpha' = (\beta + \beta')/\sqrt{2}$:

$$|\mathbb{E}_{\alpha,\beta}(X_A X_B) - \mathbb{E}_{\alpha,\beta'}(X_A X_B)| + |\mathbb{E}_{\alpha',\beta}(X_A X_B) + \mathbb{E}_{\alpha',\beta'}(X_A X_B)| = 2\sqrt{2} > 2.$$

- Quantum mechanics violates Bell's inequality.

General setup

- Observers A and B
- Parameters α, β.
- Outcomes of a measurement: random variables $X_A, X_B \in [-1, 1]$, $\sim \mathbb{P}_{\alpha,\beta}$.
- Allow for an extra parameter λ, distributed according to $\mathbb{Q}$ (independent of α, β).
- ~~Locality~~.

EPR

Theorem: If $X_A, X_B \in \{-1, 1\}$, if

$$\mathbb{P}_{\alpha,\alpha}(X_A \neq X_B) = 1$$

and $\mathbb{P}$ is local, then $\forall \lambda, \alpha$, there exists $Z_\alpha(\lambda) \in \{-1, 1\}$ such that

$$\mathbb{P}_{\alpha,\beta}(X_A = Z_\alpha(\lambda)|\lambda) = \mathbb{P}_{\alpha,\beta}(X_B = -Z_\alpha(\lambda)|\lambda) = 1$$

and

$$\mathbb{P}_\alpha^{(A)}(X_A = Z_\alpha(\lambda)|\lambda) = \mathbb{P}_\beta^{(B)}(X_B = -Z_\alpha(\lambda)|\lambda) = 1.$$