Many interacting quantum particles:
open problems, and a new point of view on an old problem

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Fermions/Bosons

THE STANDARD MODEL OF
PARTICLE PHYSICS

 Stuff that can’t all
smoosh into the
same spot.

 Stuff that can.
Bose-Einstein condensation

- System of Bosons, e.g. Helium atoms, Rubidium atoms, etc...
- At low temperatures, superfluidity (flow with zero viscosity) and superconductivity (currents with zero resistance).
- Bose-Einstein condensate: most particles are in the same quantum state.
- Predicted theoretically in 1924-1925, experimentally observed in 1995.
- Mathematical understanding: still no proof of the existence of a condensate (at finite density, in the presence of interactions and in the continuum).
Repulsive Bose gas

- $N$-particle quantum state in a volume $V$:
  \[ \psi(x_1, \cdots, x_N) \in L^2_{\text{symmetric}}((V\mathbb{T}^3)^N) \]

- $|\psi|^2$: probability distribution on the positions of the $N$ particles.

- Hamiltonian operator acting on $\psi$:
  \[ H_N \psi := -\frac{1}{2} \sum_{i=1}^{N} \Delta_i \psi + \sum_{1 \leq i < j \leq N} v(|x_i - x_j|) \psi \]
  \[ v(r) \geq 0. \]
Energy and condensate fraction

- Zero-temperature: Ground state: $\psi_0$, energy $E_0 = \inf \text{spec} H_N$:
  
  $$H_N \psi_0 = E_0 \psi_0$$

- In the thermodynamic limit:
  
  $$e_0 := \lim_{V,N \to \infty} \frac{E_0}{N}. \quad \frac{N}{V} = \rho$$

- Condensate fraction: proportion of particles in the Bose-Einstein condensate: in the thermodynamic limit: ($P_i$: projection onto condensate wavefunction)
  
  $$\eta_0 := \lim_{V,N \to \infty} \frac{1}{N} \langle \psi_0 \mid \sum_{i=1}^{N} P_i \mid \psi_0 \rangle.$$
Ground state energy

• At low density: **Bogolyubov theory**: [Bogolyubov, 1947], [Lee, Huang, Yang, 1957]:

\[
e_0 = 2\pi \rho a \left( 1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3} + o(\sqrt{\rho}) \right)
\]

• **Proved**: [Lieb, Yngvason, 1998], [Yau, Yin, 2009], ([Basti, Cenatiempo, Schlein, 2021]), [Fournais, Solovej, 2020].

• At high density: **Hartree theory**: (Proved in [Lieb, 1963]).

\[
e_0 \sim \frac{\rho}{2} \int v.
\]
Condensate fraction

- At low density: **Bogolyubov theory**: [Bogolyubov, 1947], [Lee, Huang, Yang, 1957]:
  \[
  1 - \eta_0 \sim \frac{8\sqrt{\rho a^3}}{3\sqrt{\pi}}
  \]

- Still open in the thermodynamic limit. (No proof of Bose-Einstein condensation, in the continuum, at finite density.)

- At high density: **Hartree theory**: (open)
  \[
  \eta_0 \rightarrow 1
  \]
Effective equations

- **Boltzmann equation:** $N$ classical hard particles with an infinitely small radius (dilute limit) [Lanford, 1976].

- **Thomas-Fermi theory:** $Z$ electrons orbiting a nucleus in the $Z \to \infty$ limit [Lieb, Simon, 1973].

- **Hartree-Fock equation:** dynamics of many Fermions in the weakly-interacting limit [Benedikter, Porta, Schlein, 2015].

- **Hartree-Fock-Bogolyubov equation:** dynamics of many Bosons in the weakly-interacting limit [Bach, Breteaux, Chen, Fröhlich, Sigal, 2016].
Simple equation

- Simple equation

\[-\Delta u(x) = (1 - u(x))v(x) - 4eu(x) + 2e\rho \, u \ast u(x)\]

\[e = \frac{\rho}{2} \int dx \, (1 - u(x))v(x)\]

- \( \rho > 0 \), \( v(x) \geq 0 \), \( v \in L_1(\mathbb{R}^3) \).

- Non-linear and non-local partial differential equation.

- Effective equation for the ground state of a Bose gas.

- Main idea: think of \( \psi \) as a probability distribution instead of \( |\psi|^2 \).
Energy as a function of density for the Simple equation

For \( v(x) = e^{-|x|} \):

\[
\begin{array}{c|c|c|c|c|c}
\text{Energy} & 10^{-6} & 10^{-4} & 10^{-2} & 1 & 10^2 \\
\hline
\text{LHY} & 8 & 9/13 & & & \\
\text{Hartree} & & & & & \\
\end{array}
\]
Energy as a function of density for the **Simple equation**

For $v(x) = e^{-|x|}$:
$v(x) = e^{-|x|}$, Blue: simple equation; purple: big equation; red: Monte Carlo
\[
\bullet \; x \in \mathbb{R}^3, \quad \nabla^2 u(x) = (1 - u(x)) \left( v(x) - 2\rho K(x) + \rho^2 L(x) \right)
\]

\[
K := u * S, \quad S(y) := (1 - u(y))v(y)
\]

\[
L := u * u * S - 2u * (u(u * S)).
\]
Condensate fraction

\[ v(x) = e^{-|x|} \], Blue: simple equation; purple: big equation; red: Monte Carlo
Conclusions

• Two effective equations: the big equation and the simple equation, which are non-linear 1-particle equations.

• Reproduce the known results for both small and large densities.

• Their derivation is different from Bogolyubov theory, so they may give new insights onto studying the Bose gas in these asymptotic regimes.

• The big equation is quantitatively accurate at intermediate densities.

• This opens up the possibility of studying the physics of the Bose gas at intermediate densities.