# Many interacting quantum particles:

## open problems, and a new point of view on an old problem

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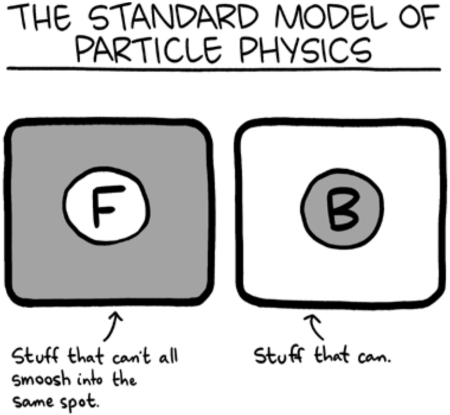
# Fermions/Bosons

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# Bose-Einstein condensation

- System of Bosons, e.g. Helium atoms, Rubidium atoms, etc...
- At low temperatures, superfluidity (flow with zero viscocity) and superconductivity (currents with zero resistance).
- Bose-Einstein condensate: most particles are in the same quantum state.

- Predicted theoretically in 1924-1925, experimentally observed in 1995.
- Mathematical understanding: still no proof of the existence of a condensate (at finite density, in the presence of interactions and in the continuum).

# Repulsive Bose gas

- $N$-particle quantum state in a volume $V$:

$$\psi(x_1, \cdots, x_N) \in L^2_{\text{symmetric}}((V\mathbb{T}^3)^N)$$

- $|\psi|^2$: probability distribution on the positions of the $N$ particles.
- Hamiltonian operator acting on $\psi$:

$$H_N\psi := -\frac{1}{2}\sum_{i=1}^{N}\Delta_i\psi + \sum_{1\leqslant i<j\leqslant N} v(|x_i - x_j|)\psi$$

$v(r) \geqslant 0$.

# Energy and condensate fraction

- Zero-temperature: Ground state: $\psi_0$, energy $E_0 = \inf \mathrm{spec} H_N$:

$$H_N \psi_0 = E_0 \psi_0$$

- In the thermodynamic limit:

$$e_0 := \lim_{\substack{V,N\to\infty \\ \frac{N}{V}=\rho}} \frac{E_0}{N}.$$

- Condensate fraction: proportion of particles in the Bose-Einstein condensate: in the thermodynamic limit: ($P_i$: projection onto condensate wavefunction)

$$\eta_0 := \lim_{\substack{V,N\to\infty \\ \frac{N}{V}=\rho}} \frac{1}{N} \langle \psi_0 | \sum_{i=1}^{N} P_i | \psi_0 \rangle .$$

# Ground state energy

- At low density: Bogolyubov theory: [Bogolyubov, 1947], [Lee, Huang, Yang, 1957]:

$$e_0 = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)$$

- Proved: [Lieb, Yngvason, 1998], [Yau, Yin, 2009], ([Basti, Cenatiempo, Schlein, 2021]), [Fournais, Solovej, 2020].
- At high density: Hartree theory: (Proved in [Lieb, 1963]).

$$e_0 \sim \frac{\rho}{2}\int v.$$

# Condensate fraction

- At low density: Bogolyubov theory: [Bogolyubov, 1947], [Lee, Huang, Yang, 1957]:

$$1 - \eta_0 \sim \frac{8\sqrt{\rho a^3}}{3\sqrt{\pi}}$$

- Still open in the thermodynamic limit. (No proof of Bose-Einstein condensation, in the continuum, at finite density.)
- At high density: Hartree theory: (open)

$$\eta_0 \to 1$$

# Effective equations

- Boltzmann equation: $N$ classical hard particles with an infinitely small radius (dilute limit) [Lanford, 1976].
- Thomas-Fermi theory: $Z$ electrons orbiting a nucleus in the $Z \to \infty$ limit [Lieb, Simon, 1973].
- Hartree-Fock equation: dynamics of many Fermions in the weakly-interacting limit [Benedikter, Porta, Schlein, 2015].
- Hartree-Fock-Bogolyubov equation: dynamics of many Bosons in the weakly-interacting limit [Bach, Breteaux, Chen, Fröhlich, Sigal, 2016].

# Simple equation

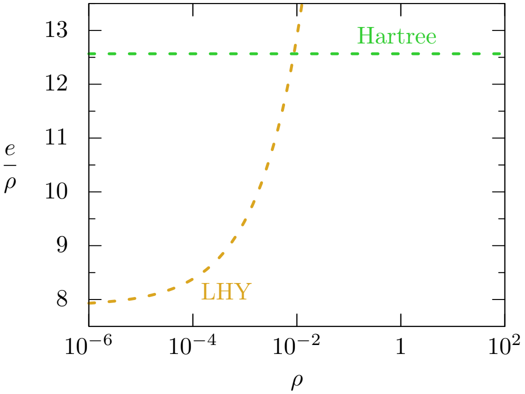
- Simple equation

$$-\Delta u(x) = (1-u(x))v(x) - 4eu(x) + 2e\rho\ u*u(x)$$
$$e = \frac{\rho}{2}\int dx\ (1-u(x))v(x)$$

- $\rho > 0$, $v(x) \geqslant 0$, $v \in L_1(\mathbb{R}^3)$.
- Non-linear and non-local partial differential equation.
- Effective equation for the ground state of a Bose gas.
- Main idea: think of $\psi$ as a probability distribution instead of $|\psi|^2$.

# Energy as a function of density for the Simple equation

For $v(x) = e^{-|x|}$:



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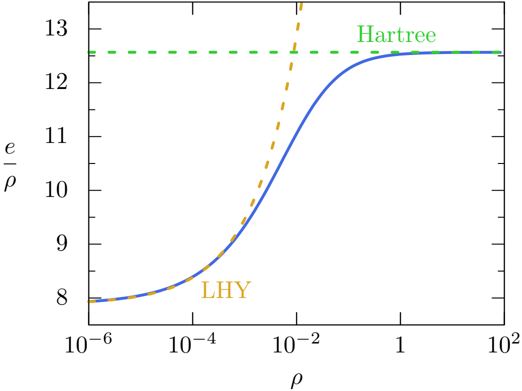
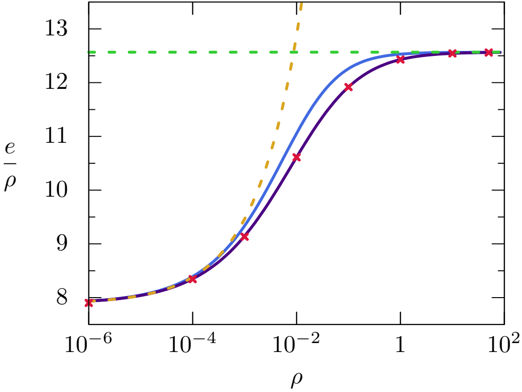


# Energy

$v(x) = e^{-|x|}$, Blue: simple equation; purple: big equation; red: Monte Carlo



# Big equation

- $x \in \mathbb{R}^3$,

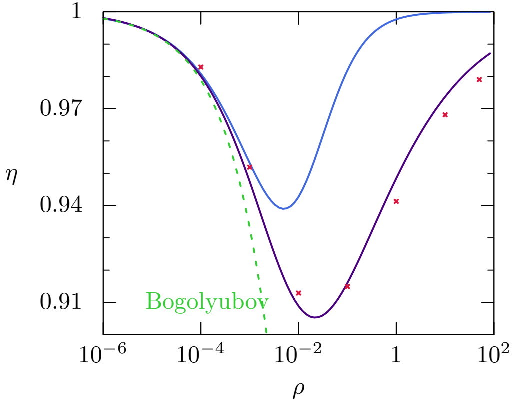
$$-\Delta u(x) = (1 - u(x))\left(v(x) - 2\rho K(x) + \rho^2 L(x)\right)$$

$$K := u * S, \quad S(y) := (1 - u(y))v(y)$$

$$L := u * u * S - 2u * (u(u * S)).$$

# Condensate fraction

$v(x) = e^{-|x|}$, Blue: simple equation; purple: big equation; red: Monte Carlo



# Conclusions

- Two effective equations: the big equation and the simple equation, which are non-linear 1-particle equations.
- Reproduce the known results for both small and large densities.
- Their derivation is different from Bogolyubov theory, so they may give new insights onto studying the Bose gas in these asymptotic regimes.
- The big equation is quantitatively accurate at intermediate densities.
- This opens up the possibility of studying the physics of the Bose gas at intermediate densities.