# An effective equation to study Bose gasses at all densities

**Ian Jauslin**

joint with **Eric A. Carlen**, **Markus Holzmann**, **Elliott H. Lieb**

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http://ian.jauslin.org

# Bose-Einstein condensation

- System of many Bosons, e.g. Helium atoms, Rubidium atoms, etc...
- Bose-Einstein condensate: most particles are in the same quantum state.

- Related to the phenomena of superfluidity (flow with zero viscocity) and superconductivity (currents with zero resistance).
- Predicted theoretically in 1924-1925, experimentally observed in 1995.
- Mathematical understanding: still no proof of the existence of a condensate (at finite density, in the presence of interactions and in the continuum).

# Repulsive Bose gas

- Potential: $v(r) \geqslant 0$ and $v \in L_1(\mathbb{R}^3)$, Hamiltonian:

$$H_N := -\frac{1}{2}\sum_{i=1}^{N}\Delta_i + \sum_{1\leqslant i<j\leqslant N} v(|x_i - x_j|)$$

- Ground state: $\psi_0$, energy $E_0$.
- Observables in the thermodynamic limit: ground state energy per particle and condensate fraction: $P_i$: projection onto condensate state

$$e_0 := \lim_{\substack{V,N\to\infty \\ \frac{N}{V}=\rho}} \frac{E_0}{N}, \quad \eta_0 := \lim_{\substack{V,N\to\infty \\ \frac{N}{V}=\rho}} \frac{1}{N}\sum_{i=1}^{N} \langle\psi_0|\, P_i \,|\psi_0\rangle\,.$$

# Low density

- Bogolubov theory: approximation scheme that reduces the problem to an effective 1-particle problem.
- Predictions [Lee, Huang, Yang, 1957]:
  - Energy:
$$e_0 = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)$$
  - Condensate fraction:
$$1 - \eta_0 \sim \frac{8\sqrt{\rho a^3}}{3\sqrt{\pi}}$$

# Low density

- Energy asymptotics: proved: [Lieb, Yngvason, 1998], [Yau, Yin, 2009], [Fournais, Solovej, 2020].
- Condensate fraction: still open in the theormodynamic limit, but there are proofs of condensation in the Gross-Pitaevskii regime (ultra-dilute): [Lieb, Seiringer, 2002], [Boccato, Brennecke, Cenatiempo, Schlein, 2018].
- In Gross-Pitaevskii limit, can also compute excitation spectrum: [Boccato, Brennecke, Cenatiempo, Schlein, 2019].

# High density

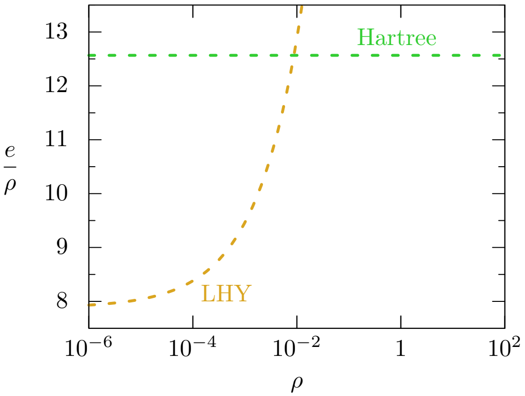
- [Bogolubov, 1947]: if $\hat{v} \geqslant 0$.

$$e_0 \sim \frac{\rho}{2} \int v$$

- Proved in [Lieb, 1963].
- The ultra-concentrated limit is the Hartree regime, in which the excitation spectrum can also be computed: [Seiringer, 2011].

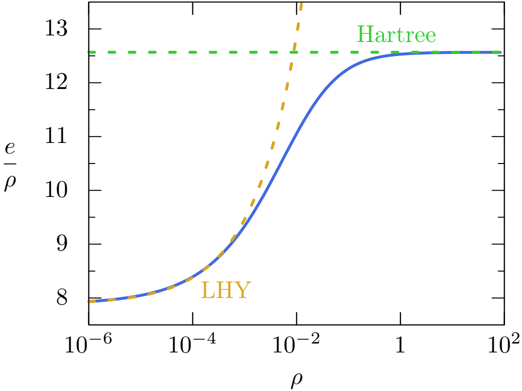
# Energy as a function of density

For $v(x) = e^{-|x|}$:



# Energy as a function of density

For $v(x) = e^{-|x|}$:



# Derivation of the equation

- [Lieb, 1963].
- Integrate $H_N\psi_0 = E_0\psi_0$:

$$\int dx_1 \cdots dx_N \, \left( -\frac{1}{2}\sum_{i=1}^{N} \Delta_i \psi_0 + \sum_{1\leqslant i<j\leqslant N} v(x_i - x_j)\psi_0 \right) = E_0 \int dx_1 \cdots dx_N \, \psi_0$$

- Therefore,

$$\frac{N(N-1)}{2} \int dx_1 dx_2 \; v(x_1 - x_2) \frac{\int dx_3 \cdots dx_N \;\, \psi_0}{\int dx_1 \cdots dx_N \;\, \psi_0} = E_0$$

## Derivation of the equation

- Thus,

$$\frac{E_0}{N} = \frac{N-1}{2V} \int dx \ v(x) g_2(0,x)$$

- $\psi_0 \geqslant 0$, so it can be thought of as a probability distribution.
- $g_n$: correlation functions of $V^{-N}\psi_0$

$$g_n(x_1, \cdots, x_n) := \frac{V^n \int dx_{n+1} \cdots dx_N \ \psi_0(x_1, \cdots, x_N)}{\int dx_1 \cdots dx_N \ \psi_0(x_1, \cdots, x_N)}$$

# Hierarchy

- Equation for $g_2$: integrate $H_N\psi_0 = E_0\psi_0$ with respect to $x_3, \cdots, x_N$:

$$-\frac{1}{2}(\Delta_x + \Delta_y)g_2(x,y) + \frac{N-2}{V}\int dz\ (v(x-z) + v(y-z))g_3(x,y,z)$$

$$+v(x-y)g_2(x,y) + \frac{(N-2)(N-3)}{2V^2}\int dzdt\ v(z-t)g_4(x,y,z,t) = E_0 g_2(x,y)$$

- Factorization assumption:

$$g_3(x_1,x_2,x_3) = g_2(x_1,x_2)g_2(x_1,x_3)g_2(x_2,x_3)$$

$$g_4(x_1,x_2,x_3,x_4) = \prod_{i<j}(g_2(x_i,x_j) + O(V^{-1}))$$

# Big equation

- In the thermodynamic limit, if $u(x) := 1 - g_2(0,x)$,

$$-\Delta u(x) = (1-u(x))\left(v(x) - 2\rho K(x) + \rho^2 L(x)\right)$$

$$K := u * S, \quad S(y) := (1-u(y))v(y)$$

$$L := u*u*S - 2u*(u(u*S)) + \frac{1}{2}\int dydz\ u(y)u(z-x)u(z)u(y-x)S(z-y).$$

- "Big" equation:

$$L \approx u*u*S.$$

# Simple equation

- Further approximate $S(x) \approx \frac{2e}{\rho}\delta(x)$ and $u \ll 1$.
- Simple equation

$$-\Delta u(x) = (1 - u(x))v(x) - 4eu(x) + 2e\rho\ u * u(x)$$

$$e = \frac{\rho}{2}\int dx\ (1 - u(x))v(x)$$

- **Theorem 1**: If $v(x) \geqslant 0$ and $v \in L_1 \cap L_2(\mathbb{R}^3)$, then the simple equation has an integrable solution (proved constructively), with $0 \leqslant u \leqslant 1$.

## Energy for the simple equation

- **Theorem 2**:

$$\frac{e}{\rho} \underset{\rho\to\infty}{\longrightarrow} \frac{1}{2}\int dx\ v(x)$$

(note that there is no condition that $\hat{v} \geqslant 0$). This coincides with the Hartree energy.

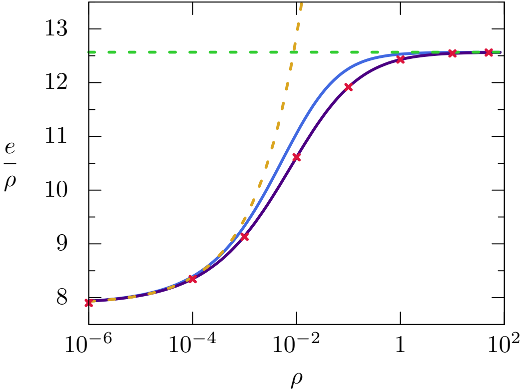
- **Theorem 3**:

$$e = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)$$

This coincides with the Lee-Huang-Yang prediction.

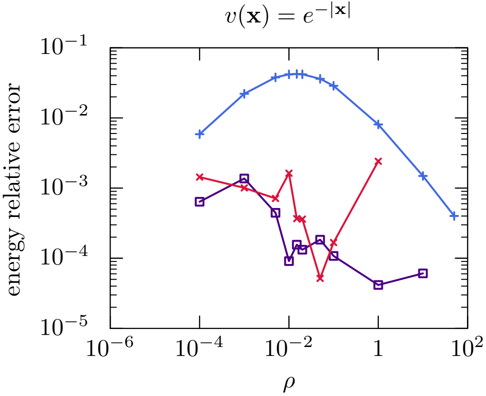
# Energy

$v(x) = e^{-|x|}$, Blue: simple equation; purple: big equation; red: Monte Carlo



# Energy

$v(x) = e^{-|x|}$, Blue: simple equation; red: Jastrow; purple: big equation



# Condensate fraction

- Add a parameter $\mu$ to the Hamiltonian:

$$H_N(\mu) := -\frac{1}{2}\sum_{i=1}^{N}\Delta_i + \sum_{1\leqslant i<j\leqslant N} v(x_i - x_j) - \mu\sum_{i=1}^{N} P_i$$

- Projection onto condensate wavefunction: $P_i$.
- Condensate fraction:

$$\eta_0 := \frac{1}{N}\langle\psi_0|\sum_{i=1}^{N} P_i|\psi_0\rangle = -\frac{1}{N}\partial_\mu \langle\psi_0| H_N(\mu)|\psi_0\rangle|_{\mu_0} \equiv -\partial_\mu e_0(\mu)|_{\mu=0}$$

# Condensate fraction

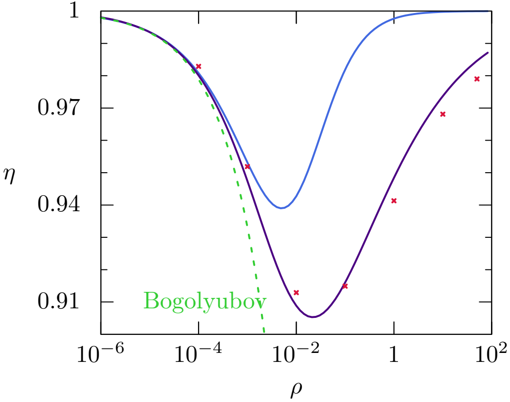
- **Theorem 4**: For the simple equation, as $\rho \to 0$

$$1 - \eta \sim \frac{8\sqrt{\rho a^3}}{3\sqrt{\pi}}$$

which coincides with Bogolubov's prediction.

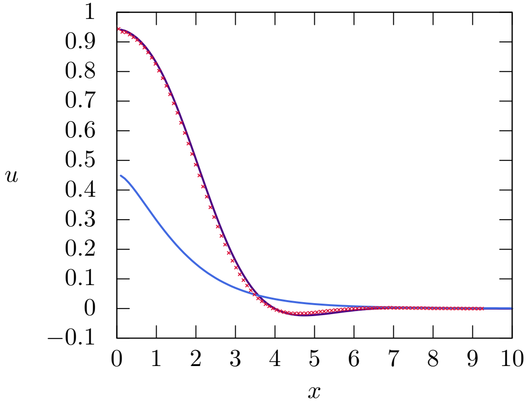
# Condensate fraction

$v(x) = e^{-|x|}$, Blue: simple equation; purple: big equation; red: Monte Carlo



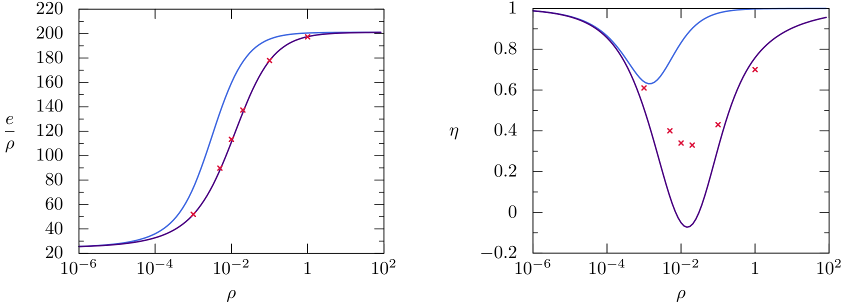
# Linear correlation function

$v(x) = 16e^{-|x|}$, Blue: simple equation; purple: big equation; red: Jastrow



## Limitations of the simple and big equations

- Only works at high densities for $\hat{v} \geqslant 0$.
- Less accurate for large potentials: for $v(x) = 16e^{-|x|}$,



## Conclusions and outlooks

- Two effective equations: the big equation and the simple equation, which are non-linear 1-particle equations.
- Reproduce the known results for both small and large densities.
- Their derivation is different from Bogolubov theory, so they may give new insights onto studying the Bose gas in these asymptotic regimes.
- The big equation is quantitatively accurate at intermediate densities.
- This opens up the possibility of studying the physics of the Bose gas at intermediate densities.

# Open problems

- Analysis of the simple equation: Monotonicity of $e(\rho)$, and convexity of $\rho e(\rho)$. (So far, we have proofs for small and large $\rho$.). Similarly, prove that $0 \leqslant \eta \leqslant 1$. (We have a proof for small $\rho$.)
- Analysis of the big equation: everything is still open.
- Relate these equations back to the many-body Bose gas.