Crystalline ordering
in hard-core lattice particle systems

Ian Jauslin

joint with Joel L. Lebowitz

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http://ian.jauslin.org
Hard-core lattice particle (HCLP) systems
Non-sliding HCLPs

- There exist a finite number $\tau$ of tilings $\{\mathcal{L}_1, \cdots, \mathcal{L}_\tau\}$ which are periodic and isometric to each other.
Non-sliding HCLPs

- Defects are **localized**: for every connected particle configuration $X$ that is *not* the subset of a close packing and every $Y \supset X$, there is empty space in $Y$ neighboring $X$. 
Example of a sliding HCLP

- $2 \times 2$ squares:
Gibbs measure

- Gibbs measure:
  \[
  \langle A \rangle_\nu := \lim_{\Lambda \to \Lambda_\infty} \frac{1}{\Xi_{\Lambda,\nu}(z)} \sum_{X \subset \Lambda} A(X) z^{|X|} \mathcal{B}_\nu(X) \prod_{x \neq x' \in X} \varphi(x, x')
  \]
  - $\Lambda$: finite subset of lattice $\Lambda_\infty$.
  - $z \geq 0$: fugacity.
  - $\varphi(x, x')$: hard-core interaction.
  - $\mathcal{B}_\nu$: boundary condition: favors the $\nu$-th tiling.

- Pressure:
  \[
  p(z) := \lim_{\Lambda \to \Lambda_\infty} \frac{1}{|\Lambda|} \log \Xi_{\Lambda,\nu}(z).
  \]
Theorem

- \( p(z) - \rho_m \log z \) and \( \langle 1_{x_1} \cdots 1_{x_n} \rangle_\nu \) are \textbf{analytic} functions of \( 1/z \) for large values of \( z \).

- There are \( \tau \) distinct Gibbs states:

\[
\langle 1_x \rangle_\nu = \begin{cases} 
1 + O(z^{-1}) & \text{if } x \in \mathcal{L}_\nu \\
O(z^{-1}) & \text{if not.}
\end{cases}
\]
Low-fugacity (Mayer) expansion

- Partition function: $Z_\Lambda(n)$: number of configurations with $n$ particles:

  \[
  \Xi_\Lambda(z) = \sum_{n=0}^{\infty} z^n Z_\Lambda(n)
  \]

- Formally,

  \[
  \frac{1}{|\Lambda|} \log \Xi_\Lambda(z) = \sum_{k=1}^{\infty} b_k(\Lambda) z^k
  \]

  where, if $Z_\Lambda(k_i)$ denotes the number of configurations with $k_i$ particles, then

  \[
  b_k(\Lambda) := \frac{1}{|\Lambda|} \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} \sum_{k_1,\ldots,k_j \geq 1, k_1+\cdots+k_j=k} Z_\Lambda(k_1) \cdots Z_\Lambda(k_j)
  \]
High-fugacity expansion

- Partition function: $Z_\Lambda(n)$: number of configurations with $n$ particles:

$$\Xi_\Lambda(z) = \sum_{n=0}^{N_{\text{max}}} z^n Z_\Lambda(n)$$

- Inverse fugacity $y \equiv z^{-1}$:

$$\Xi_\Lambda(z) = z^{N_{\text{max}}} \sum_{n=0}^{N_{\text{max}}} y^n Q_\Lambda(n)$$

with $Q_\Lambda(n) \equiv Z_\Lambda(N_{\text{max}} - n)$. 
Formally,

\[
\frac{1}{|\Lambda|} \log \Xi_\Lambda = \rho_m \log z + \sum_{k=1}^{\infty} c_k(\Lambda) y^k
\]

where \( \rho_m = \frac{N_{\text{max}}}{|\Lambda|} \),

\[
c_k(\Lambda) := \frac{1}{|\Lambda|} \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j \tau^j} \sum_{k_1, \ldots, k_j \geq 1 \atop k_1 + \cdots + k_j = k} Q_\Lambda(k_1) \cdots Q_\Lambda(k_j)
\]
High-fugacity expansion
High-fugacity expansion

• [Gaunt, Fisher, 1965]: diamonds: $c_k(\Lambda) \rightarrow c_k$ for $k \leq 9$.

• [Joyce, 1988]: hexagons (integrable, [Baxter, 1980]).

• [Eisenberg, Baram, 2005]: crosses: $c_k(\Lambda) \rightarrow c_k$ for $k \leq 6$.

• Cannot be done systematically: there exist counter-examples: e.g. hard $2 \times 2$ squares on \( \mathbb{Z}^2 \):

\[
c_1(\Lambda) \propto \sqrt{|\Lambda|}
\]
Holes interact

- Total volume of holes: $\in \rho_m^{-1}\mathbb{N}$. 
Non-sliding condition

- Distinct defects are decorrelated.
Gaunt-Fisher configurations

- Group together empty space and neighboring particles.
Defect model

- Map particle system to a model of defects:

\[
\Xi_{\Lambda, \nu}(z) = z^{\rho_m |\Lambda|} \sum_{\gamma \subset \mathcal{C}_\nu(\Lambda)} \left( \prod_{\gamma \neq \gamma' \in \gamma} \Phi(\gamma, \gamma') \right) \prod_{\gamma \in \gamma} \zeta^{(z)}_\nu(\gamma)
\]

- \(\Phi\): hard-core repulsion of defects.
- \(\zeta^{(z)}_\nu(\gamma)\): activity of defect.

- The activity of a defect is exponentially small: \(\exists \epsilon \ll 1\)

\[
\zeta^{(z)}_\nu(\gamma) < \epsilon |\gamma|
\]

- Low-fugacity expansion for defects.
• Peierls argument: in order to have a particle at $x$ that is not compatible with the $\nu$-th perfect packing, it must be part of or surrounded by a defect.

• Note: a naive Peierls argument requires the partition function to be independent from the boundary condition. This is not necessarily the case here, and we need elements from Pirogov-Sinai theory.
Lee-Yang zeros

- Lee-Yang zeros: roots of $\Xi_\Lambda(z)$ $\iff$ singularities of $p_\Lambda(z)$.
- Whenever the high fugacity expansion has a radius of convergence $\tilde{R}$, there are no Lee-Yang zeros outside of a disc of radius $\tilde{R}^{-1}$. 

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