A simple equation to study interacting Bose gasses

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Lieb's simple equation (1963)

- [Lieb, 1963]: $x \in \mathbb{R}^d$

$$(-\Delta + v(x) + 4e)u(x) = v(x) + 2e\rho\ u * u(x)$$
$$e = \frac{\rho}{2} \int dx\ (1 - u(x))v(x)$$

- with

$$\rho > 0, \quad v(x) \geqslant 0, \quad v \in L_1 \cap L_{\frac{d}{2}+\epsilon}(\mathbb{R}^d)$$

- and

$$u \in L_1(\mathbb{R}^d), \quad u * u(x) := \int dy\ u(x-y)u(y)$$

Interacting Bose gas

- State: symmetric wave functions in a finite box of volume V with periodic boundary conditions:

$$\psi(x_1, \cdots, x_N), \quad x_i \in \Lambda_d := V^{\frac{1}{d}} \mathbb{T}^d$$

- Probability distribution: $|\psi(x_1, \cdots, x_N)|^2$
- N-particle Hamiltonian:

$$H_N := -\frac{1}{2} \sum_{i=1}^{N} \Delta_i + \sum_{1 \leqslant i < j \leqslant N} v(x_i - x_j)$$

with $v(x - y) \geqslant 0$ and $v \in L_1 \cap L_{\frac{d}{2}+\epsilon}(\mathbb{R}^d)$.

Interacting Bose gas

- Ground state:

$$H_N \psi_0 = E_0 \psi_0, \quad E_0 = \min \operatorname{spec}(H_N)$$

- Compute the ground state-energy per particle in the thermodynamic limit:

$$e_0 := \lim_{\substack{V,N\to\infty \\ \frac{N}{V}=\rho}} \frac{E_0}{N}$$

Asymptotics for the Bose gas

- **Theorem** [Lieb, 1963]: if $\hat{v}(k) := \int dx\ e^{ikx} v(x) \geqslant 0$, then

$$\frac{e_0}{\rho} \xrightarrow[\rho\to\infty]{} \frac{1}{2}\int dx\ v(x)$$

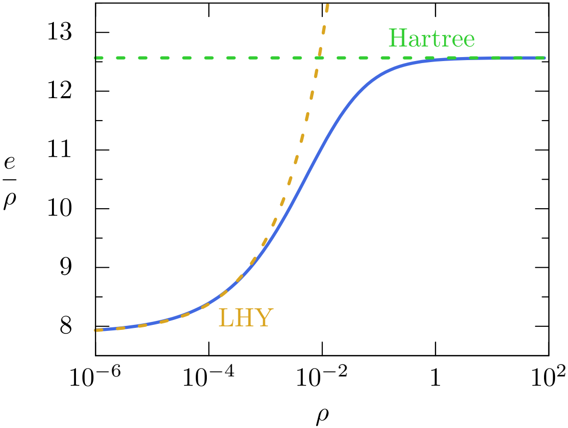
- **Theorem** [Lieb, Yngvason, 1998]: in 3 dimensions (a: scattering length)

$$\frac{e_0}{\rho} \xrightarrow[\rho\to 0]{} 2\pi a$$

[Lee, Huang, Yang, 1957], [Yau, Yin, 2009], [Fournais, Solovej, 2019]:

$$e_0 = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)$$

Simple equation for $v(x) = e^{-|x|}$ in 3 dimensions



Main Theorem

- If $v(x) \geqslant 0$ and $v \in L_1 \cap L_{\frac{d}{2}+\epsilon}(\mathbb{R}^d)$, then Lieb's simple equation

$$(-\Delta + 4e + v)u = v + 2e\rho u * u, \quad e = \frac{\rho}{2}\int dx\ (1 - u(x))v(x)$$

has an integrable solution (proved constructively), with $0 \leqslant u \leqslant 1$.

- For $d = 3$,

$$e = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right), \quad \frac{e}{\rho} \underset{\rho\to\infty}{\longrightarrow} \frac{1}{2}\int dx\ v(x).$$

- For $d = 3$, if $v(x) \equiv v(|x|)$ is radially symmetric and decays exponentially,

$$u(|x|) \underset{|x|\to\infty}{\sim} \frac{\alpha}{|x|^4}.$$

Existence of a solution (sketch)

- Simple equation: fixed $\rho > 0$,

$$(-\Delta + 4e + v)u = v + 2e\rho u * u, \quad e = \frac{\rho}{2}\int dx\ (1 - u(x))v(x)$$

- Change the point of view: fix $e > 0$, and compute ρ and u.
- Iteration: $u_0 = 0$,

$$(-\Delta + 4e + v)u_n = v + 2e\rho_{n-1}u_{n-1} * u_{n-1}, \quad \rho_n := \frac{2e}{\int dx\ (1 - u_n(x))v(x)}.$$

Existence of a solution (sketch)

$$u_n = (-\Delta + 4e + v)^{-1}\left(v + 2e\rho_{n-1}u_{n-1} * u_{n-1}\right), \quad \rho_n := \frac{2e}{\int dx\ (1-u_n(x))v(x)}.$$

- $u_n(x)$ is an increasing sequence: since $v \geqslant 0$,
  - $(-\Delta + 4e + v)^{-1}$ is positivity preserving.
  - ρ_n is an increasing function of u_n.

Existence of a solution (sketch)

$$-\Delta u_n = (1-u_n)v - 4eu_n + 2e\rho_{n-1}u_{n-1} * u_{n-1}, \quad \frac{2e}{\rho_n} = \int dx\ (1-u_n(x))v(x).$$

- $\int dx\ u_n(x) < \frac{1}{\rho_n}$: integrating,

$$0 = \frac{2e}{\rho_n} - 4e\int u_n + 2e\rho_{n-1}\left(\int u_{n-1}\right)^2 < \frac{2e}{\rho_n} - 2e\int u_n$$

(since $\int u_{n-1} < \frac{1}{\rho_{n-1}}$ and $\int u_{n-1} \leqslant \int u_n$).

Existence of a solution (sketch)

$$-\Delta u_n = (1-u_n)v - 4eu + 2e\rho_{n-1}u_{n-1} * u_{n-1}, \quad \frac{2e}{\rho_n} = \int dx\ (1-u_n(x))v(x).$$

- $u_n(x) \leqslant 1$: since $v \geqslant 0$,
  - for $x \in \Sigma := \{x :\ u_n(x) > 1\}$,

    $$-\Delta u_n < -4e + 2e\rho_{n-1}u_{n-1} * u_{n-1} \leqslant -2e < 0$$

    (since $u_{n-1} * u_{n-1} \leqslant \|u_{n-1}\|_\infty \|u_{n-1}\|_1 < \frac{1}{\rho_{n-1}}$).
  - Therefore u_n is subharmonic on Σ, so it reaches its maximum on $\partial\Sigma$. But, for $x \in \partial\Sigma$, $u_n(x) = 1$, so $u_n(x) \leqslant 1$ in Σ, so $\Sigma = \emptyset$.

Uniqueness

- In addition, one can prove that the limiting u is the unique non-negative integrable solution of the simple equation, for every fixed e.
- In addition, we prove that $e \mapsto \rho(e)$ is continuous, and $\rho(0) = 0$ and $\rho(\infty) = \infty$, which allows us to compute solutions for the problem at fixed ρ.
- In order to show uniqueness of the solution of the simple equation for fixed ρ, one would have to show that $e \mapsto \rho(e)$ is monotone.
- Nevertheless, all non-negative integrable solutions are obtained by taking the limit of the sequence u_n with the appropriate e.

Asymptotics (sketch)

$$-\Delta u = (1-u)v - 4eu + 2e\rho u * u, \quad e = \frac{\rho}{2}\int dx\ (1-u(x))v(x).$$

- When ρ is small, e is small as well, so the solution u is *not too far from* the solution of the scattering equation

$$(-\Delta + v)\varphi = v.$$

- The energy of φ is

$$\frac{\rho}{2}\int dx\ (1-\varphi(x))v(x) = 2\pi\rho a$$

which yields the first term in the expansion.

Asymptotics (sketch)

$$-\Delta u = (1-u)v - 4eu + 2e\rho u * u, \quad \frac{2e}{\rho} = \int dx\ (1-u(x))v(x).$$

- We work in Fourier space:

$$\rho\hat{u}(k) = \frac{k^2}{4e} + 1 - \sqrt{\left(\frac{k^2}{4e}+1\right)^2 - \frac{\rho}{2e}\hat{S}(k)}$$

  where $\hat{S}$ is the Fourier transform of $(1-u)v$.

- Small e is related to small k. We approximate $\hat{S}(k)$ by $\hat{S}(0) = \frac{2e}{\rho}$, and control the error terms.

Decay (sketch)

$$u = (-\Delta + 4e)^{-1}\left((1-u)v + 2e\rho u * u\right), \quad e = \frac{\rho}{2}\int dx\ (1-u(x))v(x)$$

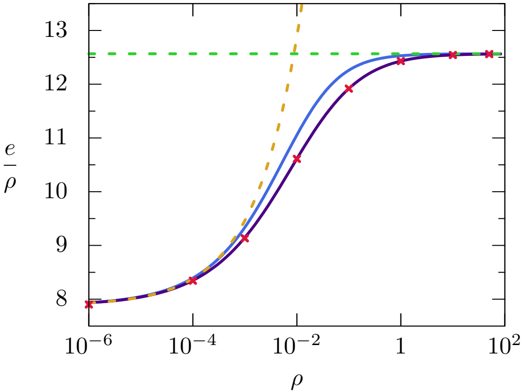
- $(-\Delta + 4e)^{-1}$ has an exponentially decaying kernel, so u cannot decay faster than $2e\rho u * u$.
- This is true for algebraically decaying functions: if $u \sim \alpha|x|^{-n}$ with $n > 3$, then
$$u * u \sim \frac{2\alpha \int u}{|x|^n}.$$
- But why $|x|^{-4}$?

Decay (sketch)

$$u = (-\Delta + 4e)^{-1}\left((1-u)v + 2e\rho u * u\right)$$

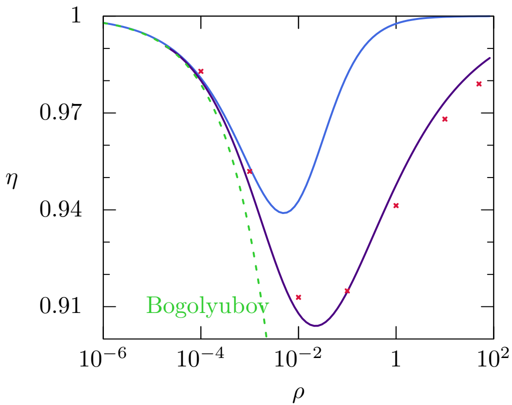
- $w := 2e\rho(-\Delta + 4e)^{-1}u$ satisfies
$$w = 2e\rho(-\Delta + 4e)^{-2}(1-u)v + w * w \geqslant w * w, \quad \int w = \frac{1}{2}.$$
- **Theorem** [Carlen, Jauslin, Lieb, Loss, 2020]: for $0 \leqslant \alpha < 1$,
$$\int dx\ |x| w(x) = \infty, \quad \int dx\ |x|^{\alpha} w(x) < \infty.$$
Furthermore, $w \geqslant 0$.

Full equation



Monte Carlo computation courtesy of M. Holzmann

Condensate fraction



Monte Carlo computation courtesy of M. Holzmann

Conclusion

- Simple equation: correct asymptotics for the ground state energy at both high and low densities.
- Condensate fraction seems right at low densities.
- Intriguing non-linear PDE.
- Proved existence, asymptotics, and decay rate.
- Full equation: does even better for the energy and condensate fraction.

Open problems and conjectures

- Monotonicity of $e \mapsto \rho(e)$, and concavity of $e \mapsto \frac{1}{\rho(e)}$ (would imply uniqueness). (So far, we have proofs for small and large ρ.)
- Condensate fraction: prove that $0 \leqslant \eta \leqslant 1$. (Again, we have a proof for small and large ρ.)
- Other equations: interpolate between full equation and simple equation.
- Potentials which are not $\geqslant 0$?