An effective equation to study Bose gasses at all densities

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arXiv:1912.04987 http://ian.jauslin.org

Repulsive Bose gas

- Potential: $v(r) \geqslant 0$ and $v \in L_1(\mathbb{R}^3)$, Hamiltonian:

$$H_N := -\frac{1}{2}\sum_{i=1}^{N}\Delta_i + \sum_{1\leqslant i<j\leqslant N} v(|x_i - x_j|)$$

- Ground state: ψ_0, energy E_0.
- Observables in the thermodynamic limit: ground state energy per particle and condensate fraction:

$$e_0 := \lim_{\substack{V,N\to\infty \\ \frac{N}{V}=\rho}} \frac{E_0}{N}, \quad \eta_0 := \lim_{\substack{V,N\to\infty \\ \frac{N}{V}=\rho}} \frac{1}{N}\left\langle\psi_0\right| \sum_{i=1}^{N}\int\frac{dx_i}{V}\left|\psi_0\right\rangle.$$

Effective theories

- Bogolubov theory: approximation scheme that makes H integrable.
- Low density predictions [Lee, Huang, Yang, 1957]:
  - Energy:

$$e_0 = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)$$

  - Condensate fraction:

$$1 - \eta_0 \sim \frac{8\sqrt{\rho a^3}}{3\sqrt{\pi}}$$

- At high density: Hartree mean-field theory.

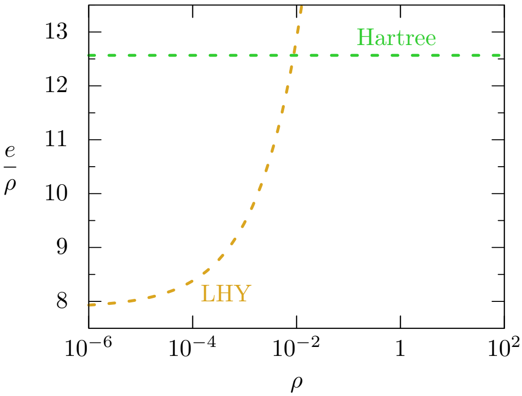
Mathematical results

- Energy asymptotics: proved: [Lieb, Yngvason, 1998], [Yau, Yin, 2009], [Fournais, Solovej, 2019]
- Condensate fraction: still open in the thermodynamic limit.
- At high density, proved in [Lieb, 1963].

$$e_0 \sim \frac{\rho}{2} \int v$$

Energy as a function of density

For $v(x) = e^{-|x|}$:



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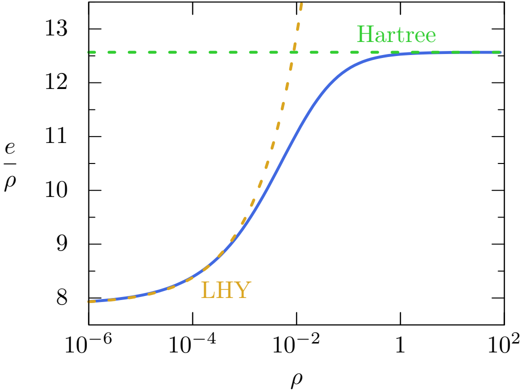


Big equation

- Solve for

$$u(x_2 - x_1) := 1 - \frac{\int \frac{dx_3}{V} \cdots \frac{dx_N}{V}\ \psi_0(x_1, \cdots, x_N)}{\int \frac{dx_1}{V} \cdots \frac{dx_N}{V}\ \psi_0(x_1, \cdots, x_N)} \tag{1}$$

- "Big" equation:

$$-\Delta u(x) = (1 - u(x))\left(v(x) - 2\rho u * S(x) + \rho^2 u * u * S(x)\right)$$

$$S(y) := (1 - u(y))v(y)$$

Simple equation

- Further approximate $S(x) \approx \frac{2e}{\rho}\delta(x)$ and $u \ll 1$.
- Simple equation:

$$-\Delta u(x) = (1 - u(x))v(x) - 4eu(x) + 2e\rho\ u * u(x)$$

$$e = \frac{\rho}{2} \int dx\ (1 - u(x))v(x)$$

- **Theorem 1**: If $v(x) \geqslant 0$ and $v \in L_1 \cap L_2(\mathbb{R}^3)$, then the simple equation has an integrable solution (proved constructively), with $0 \leqslant u \leqslant 1$.

Energy for the simple equation

- **Theorem 2**:

$$\frac{e}{\rho} \xrightarrow[\rho\to\infty]{} \frac{1}{2}\int dx\ v(x)$$

(note that there is no condition that $\hat{v} \geqslant 0$). This coincides with the Hartree energy.

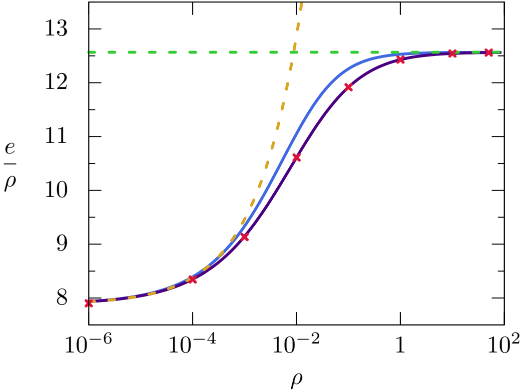
- **Theorem 3**:

$$e = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)$$

This coincides with the Lee-Huang-Yang prediction.

Energy

$v(x) = e^{-|x|}$, Blue: simple equation; purple: big equation; red: Monte Carlo



Condensate fraction

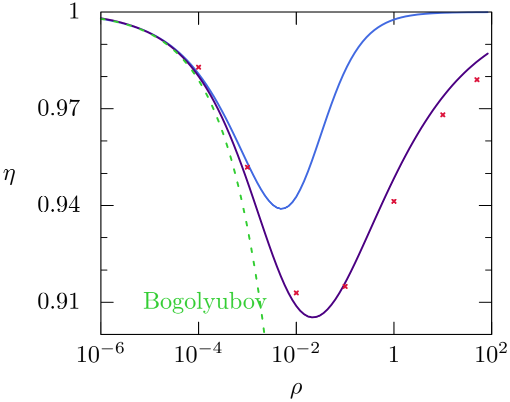
- **Theorem 4**: For the simple equation, as $\rho \to 0$

$$1 - \eta \sim \frac{8\sqrt{\rho a^3}}{3\sqrt{\pi}}$$

which coincides with Bogolubov's prediction.

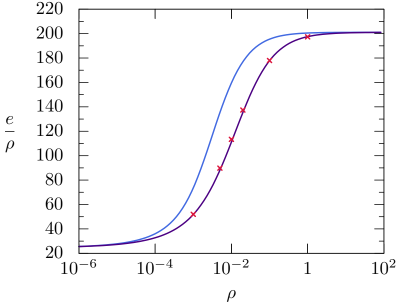
Condensate fraction

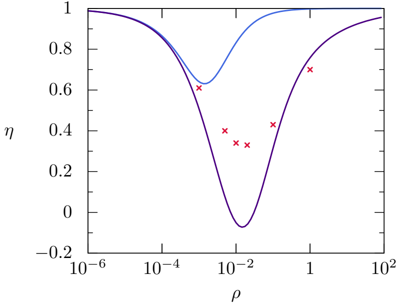
$v(x)=e^{-|x|}$, Blue: simple equation; purple: big equation; red: Monte Carlo



Limitations of the simple and big equations

- Only works at high densities for $\hat{v} \geqslant 0$.
- Less accurate for large potentials: for $v(x) = 16e^{-|x|}$,





Conclusions and outlooks

- Two effective equations: the big equation and the simple equation, which are non-linear 1-particle equations.
- Reproduce the known results for both small and large densities.
- Their derivation is different from Bogolubov theory, so they may give new insights onto studying the Bose gas in these asymptotic regimes.
- The big equation is quantitatively accurate at intermediate densities.
- This opens up the possibility of studying the physics of the Bose gas at intermediate densities.

Open problems

- Analysis of the simple equation: Monotonicity of $e(\rho)$, and convexity of $\rho e(\rho)$. (So far, we have proofs for small and large ρ.). Similarly, prove that $0 \leqslant \eta \leqslant 1$. (We have a proof for small ρ.)
- Analysis of the big equation: everything is still open.
- Relate these equations back to the many-body Bose gas.