Time-evolution of electron emission from a metal surface

Ian Jauslin

joint with Ovidiu Costin, Rodica Costin, and Joel L. Lebowitz

Field emission
Field emission

\[ V(x) = U \Theta(x), \quad E_F = k_F^2 < U \]
Thermal emission

\[ V(x) = U \Theta(x), \quad k^2 > U \]
Photonic emission

\[ V_t(x) = \Theta(x)(U - E_t x), \quad E_t = 2\epsilon\omega \cos(\omega t) \]
Field emission

\[ V(x) = \Theta(x)(U - Ex) \]
Field emission

- [Millikan, Lauritsen, 1928]: experimental plot of the logarithm of the current against $1/E$
Field emission through a triangular barrier

- [Fowler, Nordheim, 1928]: predicted that the current is, for small $E$,

\[ J \approx CE^2 e^{-\frac{a}{E}} \]

- ([Rokhlenko, 2011]: studied the range of applicability of the approximation, and found more accurate approximations for larger fields.)
Fowler-Nordheim equation

- Schrödinger equation

\[ i \partial_t \psi = -\Delta \psi + \Theta(x)(U - Ex)\psi \]

- Fowler-Nordheim: stationary solution: \( \psi_{FN}(x, t) = e^{-ik^2t} \varphi_{FN}(x) \)

\[ \varphi_{FN}(x) = \begin{cases} 
  e^{ikx} + R_E e^{-ikx} & x < 0 \\
  T_E \text{Ai}(e^{-\frac{i\pi}{3}}(E^\frac{1}{3} x - E^{-\frac{2}{3}}(U - k^2))) & x > 0 
\end{cases} \]

\( R_E \) and \( T_E \) are chosen so that \( \varphi_{FN} \) and \( \partial \varphi_{FN} \) are continuous at \( x = 0 \).
Fowler-Nordheim equation
Initial value problem

• Initial condition:

\[ \psi(x, 0) = \begin{cases} 
  e^{ikx} + R_0 e^{-ikx} & x < 0 \\
  T_0 e^{-\sqrt{U-k^2}x} & x > 0
\end{cases} \]

\( R_0 \) and \( T_0 \) ensure that \( \psi \) and \( \partial \psi \) are continuous.

• Theorem: \( \psi(x, t) \) behaves asymptotically like \( \psi_{\text{FN}} \):

\[ \psi(x, t) = \psi_{\text{FN}}(x, t) + \left( \frac{t}{\tau_E(x)} \right)^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}). \]
• Photoelectric effect: early observations: [Hertz, 1887], [Hallwachs, 1888], [Lenard, 1900].

• When a metal is irradiated with ultra-violet light, electrons are ionized, with kinetic energies at integer multiples of $\hbar \omega$.

• [Einstein, 1905]: interpretation: quanta of light (photons) of energy $\hbar \omega$ are absorbed by the electrons, whose kinetic energy is raised by $n\hbar \omega$, and can escape the metal.
Photoemission

- Time dependent potential:

\[ V_t(x) = \Theta(x)(U - 2\epsilon\omega \cos(\omega t)x) \]

- Magnetic gauge:

\[ \Psi(x, t) := \psi(x, t)e^{-ix\Theta(x)A(t)}, \quad A(t) := \int_0^t ds \ 2\epsilon\omega \cos(\omega s) = 2\epsilon \sin(\omega t) \]

satisfies

\[ i\partial_t \Psi(x, t) = \left( (-i\nabla + \Theta(x)A(t))^2 + \Theta(x)U \right) \Psi(x, t) \]
Periodic solution

- [Faisal, Kamiński, Saczuk, 2005]

\[
\Psi_{\text{FKS}}(x, t) = \begin{cases} 
  e^{ikx} \exp(-ik^2t) + \Psi_R(x, t) & \text{if } x < 0 \\
  \Psi_T(x, t) & \text{if } x > 0 
\end{cases}
\]

\[
\Psi_R(x, t) = \sum_{M \in \mathbb{Z}} R_M \exp(-iq_M x) \exp(-iq_M^2 t), \quad q_M = \sqrt{k^2 + M\omega}
\]

\[
\Psi_T(x, t) = \sum_{M \in \mathbb{Z}} T_M \exp(i p_M x) \exp \left( -iUt - i \int_0^t d\tau (p_M - A(\tau))^2 \right)
\]

\[
p_M = \sqrt{k^2 - U + M\omega - 2\epsilon^2}
\]

- \(\Psi(x, t), (-i\nabla + \Theta(x)A(t))\Psi(x, t)\) are continuous.
Initial value problem

• Initial condition:

\[ \Psi(x, 0) = \begin{cases} 
  e^{ikx} + R_0 e^{-ikx} & x < 0 \\
  T_0 e^{-\sqrt{U-k^2}x} & x > 0 
\end{cases} \]

\( R_0 \) and \( T_0 \) ensure that \( \Psi \) and \( \partial \Psi \) are continuous.

• Conjecture (in progress): \( \Psi(x, t) \) behaves asymptotically like \( \Psi_{\text{FKS}} \):

\[ \psi(x, t) = \psi_{\text{FKS}}(x, t) + \left( \frac{t}{\tau_{\text{FKS}}(x)} \right)^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}). \]
Idea of the proof: field emission

- Laplace transform:

\[ \hat{\psi}_p(x) := \int_0^\infty dt \ e^{-pt} \psi(x,t) \]

- Schrödinger equation:

\[ (-\Delta + \Theta(x)V(x) - ip)\psi_p(x) = -i\psi(x,0), \quad V(x) := U - Ex \]
For simplicity, $R_0 \equiv T_0 \equiv 0$.

Solution:

$$\hat{\psi}_p(x) = \begin{cases} 
  c(p)e^{\sqrt{-ip}x} - \frac{ie^{ikx}}{-ip + k^2} & \text{if } x < 0 \\
  d(p)\varphi_p(x) & \text{if } x > 0 
\end{cases}$$

with

$$(-\Delta + V(x) - ip)\varphi_p(x) = 0$$

$$\varphi_p(x) = \text{Ai} \left( e^{-\frac{i\pi}{3}} \left( E^\frac{1}{3}x - E^{-\frac{2}{3}}(U - ip) \right) \right)$$
Solution in Laplace space

- $c$ and $d$ ensure that $\hat{\psi}_p(x)$ and $\partial \hat{\psi}_p(x)$ are continuous at $x = 0$:

$$c(p) = \frac{i(ik\varphi_p(0) - \partial \varphi_p(0))}{(-ip + k^2)(\sqrt{-ip}\varphi_p(0) - \partial \varphi_p(0))}$$

$$d(p) = -\frac{i}{(\sqrt{-ip} + ik)(\sqrt{-ip}\varphi_p(0) - \partial \varphi_p(0))}.$$
Poles in Laplace plane
Asymptotic behavior

• As \( t \to \infty \):

\[
\psi(x, t) = \psi_{FN}(x, t) + \left( \frac{t}{\tau_E(x)} \right)^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}).
\]

• If \( k < 0 \) (reflected wave), then there is no pole on the imaginary axis, so there is no contribution as \( t \to \infty \).

• Similarly, the transmitted wave in the initial condition does not contribute.