Lieb’s simplified approach to interacting Bose gases

Ian Jauslin

joint with Eric Carlen, Elliott H. Lieb

http://ian.jauslin.org
Bose gas
Interacting Bose gas

- State: symmetric wave functions in a finite box of volume $V$ with periodic boundary conditions:

$$\psi(x_1, \cdots, x_N), \quad x_i \in \Lambda_d := V^{\frac{1}{d}} \mathbb{T}^d$$

- $N$-particle Hamiltonian:

$$H_N := -\frac{1}{2} \sum_{i=1}^{N} \Delta_i + \sum_{1 \leq i < j \leq N} v(|x_i - x_j|)$$

with a repulsive, integrable interaction: $v(|x - y|) \geq 0$, $\int dx \ v(x) < \infty$. 

2/12
Interacting Bose gas

- **Ground state:**
  \[ H_N \psi_0 = E_0 \psi_0, \quad E_0 = \min \text{spec}(H_N) \]

- Compute the ground state-energy per particle in the thermodynamic limit:
  \[ e_0 := \lim_{V,N \to \infty} \frac{E_0}{N} \]
  \[ \frac{N}{V} = \rho \]
Energy

- Integrate $H_N \psi_0 = E_0 \psi_0$:

\[
\frac{E_0}{N} = \frac{N - 1}{2V} \int dx\ v(|x - y|)g_2(x, y)
\]

- $g_n$: marginal of $\psi_0$

\[
g_n(x_1, \cdots, x_n) := \frac{V^n \int dx_{n+1} \cdots dx_N\ \psi_0(x_1, \cdots, x_N)}{\int dx_1 \cdots dx_N\ \psi_0(x_1, \cdots, x_N)}
\]

- $\psi_0 \geq 0$, so it can be thought of as a probability distribution
Hierarchy

- Equation for $g_2$: integrate $H_N \psi_0 = E_0 \psi_0$ with respect to $x_3, \cdots, x_N$:

\[-\frac{1}{2}(\Delta_x + \Delta_y)g_2(x, y) + \frac{N - 2}{V} \int dz \left( v(|x - z|) + v(|y - z|) \right)g_3(x, y, z)\]

\[+ v(|x - y|)g_2(x, y) + \frac{(N - 2)(N - 3)}{2V^2} \int dzdt \ v(|z - t|)g_4(x, y, z, t) = E_0 g_2(x, y)\]

- Factorization assumption:

\[g_3(x_1, x_2, x_3) = g_2(x_1, x_2)g_2(x_1, x_3)g_2(x_2, x_3)\]

\[g_4(x_1, x_2, x_3, x_4) = \prod_{i<j}(g_2(x_i, x_j) + O(V^{-1}))\]
• In the thermodynamic limit, after making a few additional assumptions, [Lieb, 1963]:

\[
(-\Delta + v(|x|) + 4e_0)u(|x|) = v(|x|) + 2e_0 \rho \ u \ast u(|x|)
\]

\[
e_0 = \frac{\rho}{2} \int dx \ (1 - u(|x|))v(|x|)
\]

• with \( \rho := \frac{N}{V} \)

\[
g_2(x, y) = 1 - u(|x - y|), \quad u \ast u(|x|) := \int dy \ u(|x - y|)u(|y|)
\]
One dimension
Numerical solution for $v(x) = e^{-|x|}$ in 3 dimensions
Numerical solution for $v(x) = e^{-|x|}$ in 3 dimensions
Asymptotics

- **Theorem** [Lieb, 1963]: if \( \hat{v}(k) := \int dx \, e^{ikx} v(x) \geq 0 \), then
  \[
  \frac{e_0}{\rho} \xrightarrow{\rho \to \infty} \frac{1}{2} \int dx \, v(x)
  \]

- **Theorem** [Lieb, Yngvason, 1998]: in 3 dimensions
  \[
  \frac{e_0}{\rho} \xrightarrow{\rho \to 0} 2\pi a
  \]

- **Theorem** [Lieb, Yngvason, 2001]: in 2 dimensions
  \[
  \frac{e_0}{\rho} \xrightarrow{\rho \to 0} \frac{2\pi}{|\log(\rho a^2)|}
  \]
• In 3 dimensions, [Lee, Huang, Yang, 1957] conjecture:

\[ e_0 = 2\pi \rho a \left( 1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3} + o(\sqrt{\rho}) \right) \]

• This was recently proved by [Fournais, Solovej, 2019].
Results

• **Theorem**: For Lieb’s simplified equation,

\[
\frac{e_0}{\rho} \xrightarrow{\rho \to \infty} \frac{1}{2} \int dx \ v(x)
\]

and, in 3 dimensions,

\[
e_0 = 2\pi \rho a \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\rho a^3} + O(\rho)\right)
\]

in 2 dimensions

\[
\lim_{\rho \to \infty} \frac{2\pi}{\rho} \xrightarrow{\rho \to \infty} \left| \log(\rho a) \right|.
\]