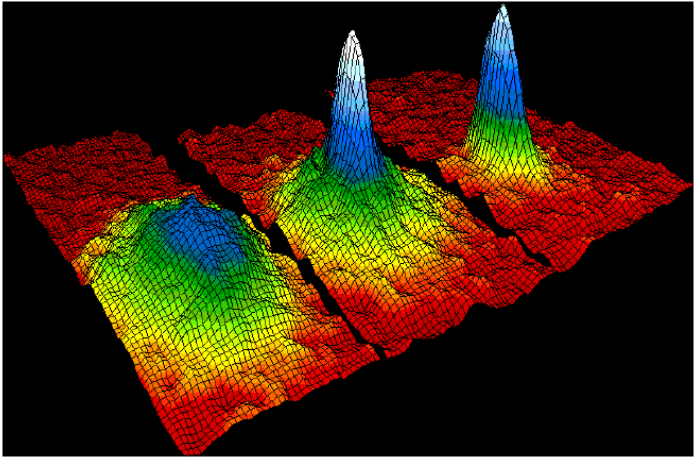
Lieb's simplified approach to interacting Bose gases

Ian Jauslin

joint with **Eric Carlen**, **Elliott H. Lieb**

http://ian.jauslin.org

Bose gas



Interacting Bose gas

- State: symmetric wave functions in a finite box of volume V with periodic boundary conditions:

$$\psi(x_1, \cdots, x_N), \quad x_i \in \Lambda_d := V^{\frac{1}{d}} \mathbb{T}^d$$

- N-particle Hamiltonian:

$$H_N := -\frac{1}{2} \sum_{i=1}^{N} \Delta_i + \sum_{1 \leqslant i < j \leqslant N} v(|x_i - x_j|)$$

with a repulsive, integrable interaction: $v(|x-y|) \geqslant 0$, $\int dx\ v(x) < \infty$.

Interacting Bose gas

- Ground state:

$$H_N\psi_0 = E_0\psi_0, \quad E_0 = \min \operatorname{spec}(H_N)$$

- Compute the ground state-energy per particle in the thermodynamic limit:

$$e_0 := \lim_{\substack{V,N\to\infty \\ \frac{N}{V}=\rho}} \frac{E_0}{N}$$

Energy

- Integrate $H_N\psi_0 = E_0\psi_0$:

$$\frac{E_0}{N} = \frac{N-1}{2V}\int dx\ v(|x-y|)g_2(x,y)$$

- g_n: marginal of ψ_0

$$g_n(x_1,\cdots,x_n) := \frac{V^n\int dx_{n+1}\cdots dx_N\ \psi_0(x_1,\cdots,x_N)}{\int dx_1\cdots dx_N\ \psi_0(x_1,\cdots,x_N)}$$

- $\psi_0 \geqslant 0$, so it can be thought of as a probability distribution

Hierarchy

- Equation for g_2: integrate $H_N\psi_0 = E_0\psi_0$ with respect to $x_3, \cdots, x_N$:

$$-\frac{1}{2}(\Delta_x + \Delta_y)g_2(x,y) + \frac{N-2}{V}\int dz\ (v(|x-z|) + v(|y-z|))g_3(x,y,z)$$

$$+v(|x-y|)g_2(x,y) + \frac{(N-2)(N-3)}{2V^2}\int dzdt\ v(|z-t|)g_4(x,y,z,t) = E_0 g_2(x,y)$$

- Factorization assumption:

$$g_3(x_1,x_2,x_3) = g_2(x_1,x_2)g_2(x_1,x_3)g_2(x_2,x_3)$$

$$g_4(x_1,x_2,x_3,x_4) = \prod_{i<j}(g_2(x_i,x_j) + O(V^{-1}))$$

Lieb's simple equation

- In the thermodynamic limit, after making a few additional assumptions, [Lieb, 1963]:

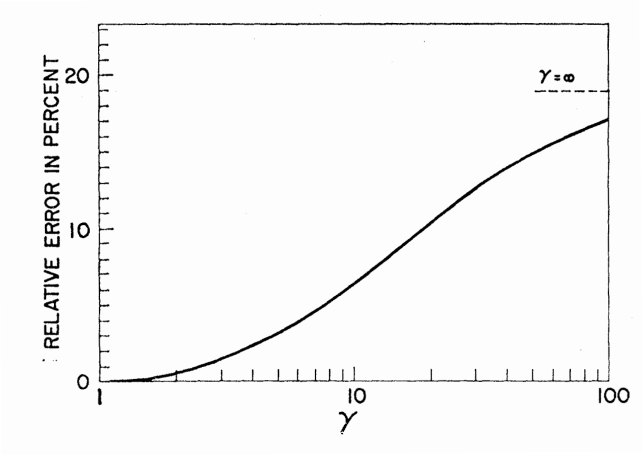
$$(-\Delta + v(|x|) + 4e_0)u(|x|) = v(|x|) + 2e_0\rho\ u * u(|x|)$$

$$e_0 = \frac{\rho}{2}\int dx\ (1 - u(|x|))v(|x|)$$

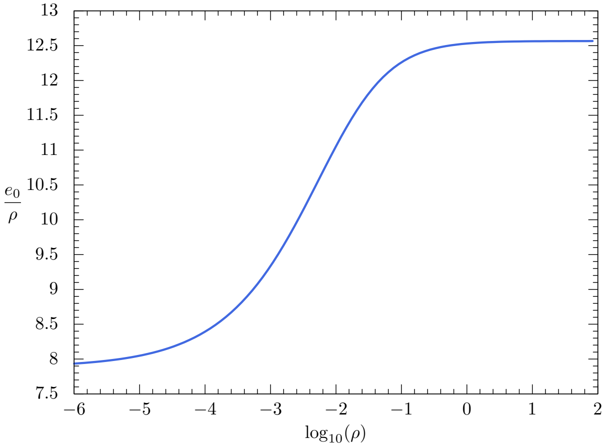
- with $\rho := \frac{N}{V}$

$$g_2(x, y) = 1 - u(|x - y|), \quad u * u(|x|) := \int dy\ u(|x - y|)u(|y|)$$

One dimension



Numerical solution for $v(x) = e^{-|x|}$ in 3 dimensions



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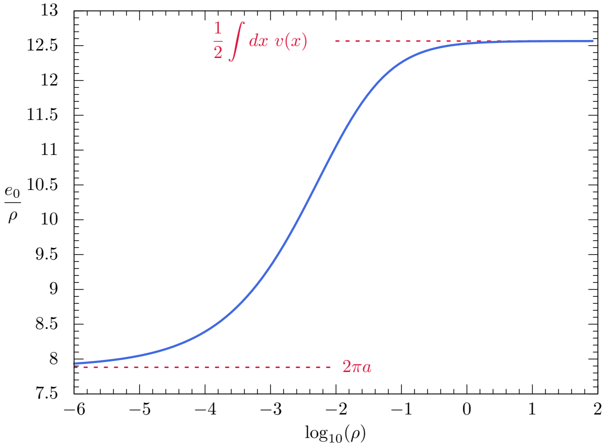

Asymptotics

- **Theorem** [Lieb, 1963]: if $\hat{v}(k) := \int dx\ e^{ikx} v(x) \geqslant 0$, then

$$\frac{e_0}{\rho} \underset{\rho\to\infty}{\longrightarrow} \frac{1}{2} \int dx\ v(x)$$

- **Theorem** [Lieb, Yngvason, 1998]: in 3 dimensions

$$\frac{e_0}{\rho} \underset{\rho\to 0}{\longrightarrow} 2\pi a$$

- **Theorem** [Lieb, Yngvason, 2001]: in 2 dimensions

$$\frac{e_0}{\rho} \underset{\rho\to 0}{\longrightarrow} \frac{2\pi}{|\log(\rho a^2)|}$$

Lee-Huang-Yang formula

- In 3 dimensions, [Lee, Huang, Yang, 1957] conjecture:

$$e_0 = 2\pi\rho a \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + o(\sqrt{\rho})\right)$$

- This was recently proved by [Fournais, Solovej, 2019].

Results

- **Theorem**: For Lieb's simplified equation,

$$\frac{e_0}{\rho} \underset{\rho\to\infty}{\longrightarrow} \frac{1}{2}\int dx\ v(x)$$

and, in 3 dimensions,

$$e_0 = 2\pi\rho a\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho a^3} + O(\rho)\right)$$

in 2 dimensions

$$\frac{e_0}{\rho} \underset{\rho\to\infty}{\longrightarrow} \frac{2\pi}{|\log(\rho a)|}.$$