Field electron emission and the Fowler-Nordheim equation

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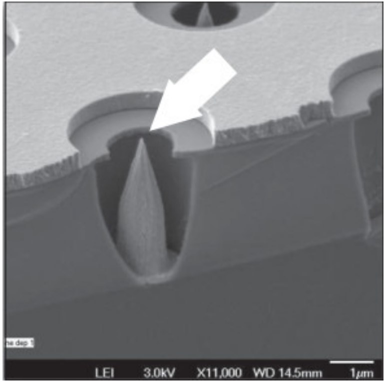
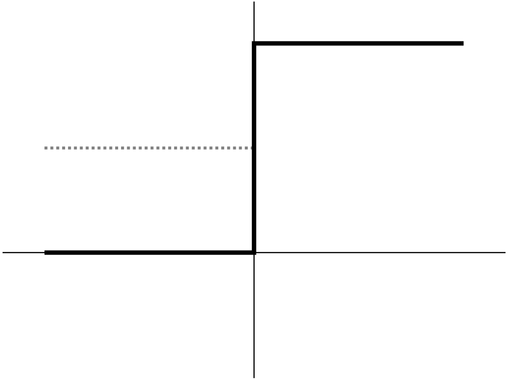
Field emission



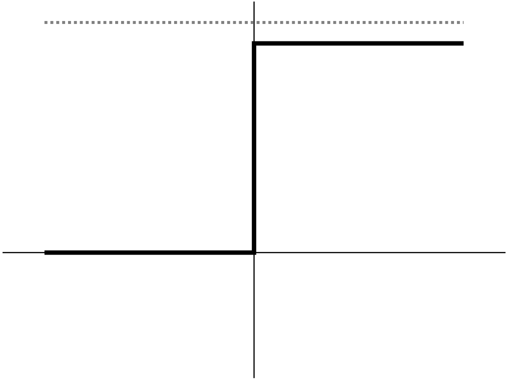
Field emission

$$V(x) = U\Theta(x), \quad E_{\mathrm{F}} = k_{\mathrm{F}}^2 < U$$



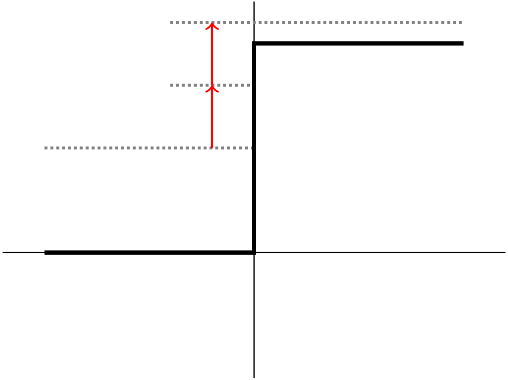
Thermal emission

$$V(x) = U\Theta(x), \quad E_{\mathrm{F}} = k_{\mathrm{F}}^2 > U$$



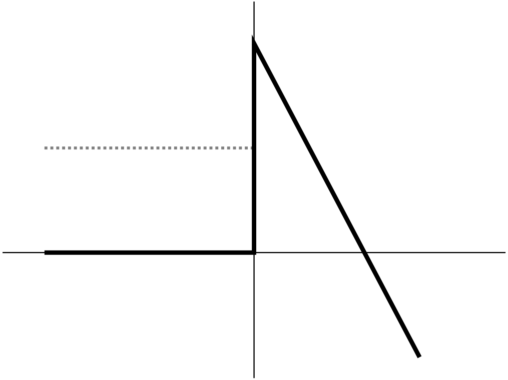
Photonic emission

$$V_t(x) = \Theta(x)(U - E_t x), \quad E_t = 2\epsilon\omega\cos(\omega t)$$



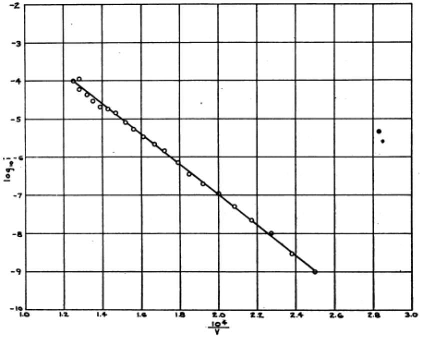
Field emission

$$V(x) = \Theta(x)(U - Ex)$$



Field emission

- [Millikan, Lauritsen, 1928]: experimental plot of the logarithm of the current against $1/E$



Field emission through a triangular barrier

- [Fowler, Nordheim, 1928]: predicted that the current is, for small E,

$$J \approx CE^2 e^{-\frac{a}{E}}$$

- ([Rokhlenko, 2011]: studied the range of applicability of the approximation, and found more accurate approximations for larger fields.)

Fowler-Nordheim equation

- Schrödinger equation

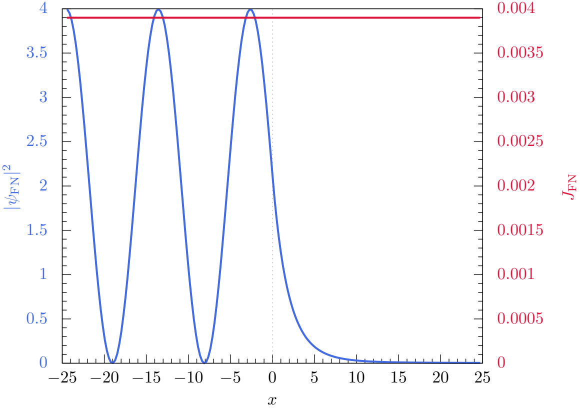
$$i\partial_t\psi = -\Delta\psi + \Theta(x)(U - Ex)\psi$$

- Fowler-Nordheim: stationary solution: $\psi_{\mathrm{FN}}(x,t) = e^{-ik^2t}\varphi_{\mathrm{FN}}(x)$

$$\varphi_{\mathrm{FN}}(x) = \begin{cases} e^{ikx} + R_E e^{-ikx} & x < 0 \\ T_E \mathrm{Ai}(e^{-\frac{i\pi}{3}}(E^{\frac{1}{3}}x - E^{-\frac{2}{3}}(U - k^2)) & x > 0 \end{cases}$$

R_E and T_E are chosen so that φ_{FN} and $\partial\varphi_{\mathrm{FN}}$ are continuous at $x = 0$.

Fowler-Nordheim equation



Initial value problem

- Initial condition:

$$\psi(x,0) = \begin{cases} e^{ikx} + R_0 e^{-ikx} & x < 0 \\ T_0 e^{-\sqrt{U-k^2}x} & x > 0 \end{cases}$$

  R_0 and T_0 ensure that ψ and $\partial\psi$ are continuous.

- Behaves asymptotically like ψ_{FN}:

$$\psi(x,t)e^{ik^2 t} \underset{t\to\infty}{\longrightarrow} \varphi_{\mathrm{FN}}(x)$$

Initial value problem

- Laplace transform:

$$\hat{\psi}_p(x) := \int_0^\infty dt\ e^{-pt}\psi(x,t)$$

- Schrödinger equation:

$$(-\Delta + \Theta(x)V(x) - ip)\psi_p(x) = -i\psi(x,0), \quad V(x) := U - Ex$$

Solution in Laplace space

- For simplicity, $R_0 \equiv T_0 \equiv 0$.
- Solution:

$$\hat{\psi}_p(x) = \begin{cases} c(p)e^{\sqrt{-ip}x} - \dfrac{ie^{ikx}}{-ip+k^2} & \text{if } x < 0 \\ d(p)\varphi_p(x) & \text{if } x > 0 \end{cases}$$

with

$$(-\Delta + V(x) - ip)\varphi_p(x) = 0$$

$$\varphi_p(x) = \mathrm{Ai}\left(e^{-\frac{i\pi}{3}}\left(E^{\frac{1}{3}}x - E^{-\frac{2}{3}}(U - ip)\right)\right)$$

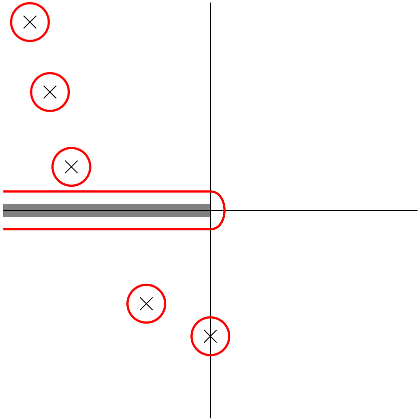
Solution in Laplace space

- c and d ensure that $\hat{\psi}_p(x)$ and $\partial\hat{\psi}_p(x)$ are continuous at $x = 0$:

$$c(p) = \frac{i(ik\varphi_p(0) - \partial\varphi_p(0))}{(-ip + k^2)(\sqrt{-ip}\varphi_p(0) - \partial\varphi_p(0))}$$

$$d(p) = -\frac{i}{(\sqrt{-ip} + ik)(\sqrt{-ip}\varphi_p(0) - \partial\varphi_p(0))}.$$

Poles in Laplace plane



Asymptotic behavior

- As $t \to \infty$:

$$\psi(x,t) = \psi_{\mathrm{FN}}(x,t) + \left(\frac{t}{\tau_E(x)}\right)^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}).$$

- If $k < 0$ (reflected wave), then there is no pole on the imaginary axis, so there is no contribution as $t \to \infty$.
- Similarly, the transmitted wave in the initial condition does not contribute.

Laser field

- Time dependent potential:

$$V_t(x) = \Theta(x)(U - 2\epsilon\omega\cos(\omega t)x)$$

- Magnetic gauge:

$$\Psi(x,t) := \psi(x,t)e^{-ix\Theta(x)A(t)}, \quad A(t) := \int_0^t ds\ 2\epsilon\omega\cos(\omega s) = 2\epsilon\sin(\omega t)$$

satisfies

$$i\partial_t\Psi(x,t) = \left((-i\nabla + \Theta(x)A(t))^2 + \Theta(x)U\right)\Psi(x,t)$$

Periodic solution

- A solution:

$$\Psi(x,t) = \begin{cases} \Psi_I(x,t) + \Psi_R(x,t) & \text{if } x < 0 \\ \Psi_T(x,t) & \text{if } x > 0 \end{cases}$$

$$\Psi_I(x,t) = e^{ikx} \exp\left(-ik^2 t\right)$$

$$\Psi_R(x,t) = \sum_{M \in \mathbb{Z}} R_M e^{-iq_M x} \exp\left(-iq_M^2 t\right)$$

$$\Psi_T(x,t) = \sum_{M \in \mathbb{Z}} T_M e^{-ip_M x} \exp\left(-iUt - i \int_0^t d\tau \; (p_M + A(\tau))^2\right)$$

Periodic solution

- Choose q_M and p_M to make the solution periodic (up to the phase e^{ik^2t}):

$$q_M = \sqrt{k^2 + M\omega}, \quad p_M = \pm\sqrt{k^2 - U + M\omega - U_V}$$

and

$$U_V := \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} d\tau \ A^2(\tau) = 2\epsilon^2.$$

- The coefficients R_M and T_M are chosen such that

$$\Psi(x,t), \quad (-i\nabla + \Theta(x)A(t))\Psi(x,t)$$

are continuous at $x = 0$.

Initial value problem

- In Laplace space:

$$\hat{\Psi}_p(x) := \int_0^\infty dt\ e^{-pt}\Psi(x,t)$$

the equation is discrete:

$$\mathfrak{f}_n^{(\sigma)}(x) := \hat{\Psi}_{-ik^2-i\sigma-in\omega}(x), \quad \mathcal{R}e(\sigma) \in [-\tfrac{\omega}{2}, \tfrac{\omega}{2})$$

$$\left(-\Delta - k^2 - \sigma - n\omega + \Theta(x)\left(U + 2\epsilon^2\right)\right)\mathfrak{f}_n^{(\sigma)}(x) - \Theta(x)2\epsilon\nabla(\mathfrak{f}_{n+1}^{(\sigma)}(x) - \mathfrak{f}_{n-1}^{(\sigma)}(x))$$

$$-\Theta(x)\epsilon^2(\mathfrak{f}_{n+2}^{(\sigma)}(x) + \mathfrak{f}_{n-2}^{(\sigma)}(x)) = -i\psi(x,0)$$

Initial value problem

- This system of ODEs is *integrable* for $x < 0$ and $x > 0$, so we have closed form expressions for a family of solutions $\mathfrak{f}_n^{(\sigma)}(x)$, parametrized by two sequences $c_n^{(\sigma)}$ and $d_n^{(\sigma)}$:

$$\mathfrak{f}_n^{(\sigma)}(x) = \begin{cases} c_n^{(\sigma)} e^{-ix\sqrt{k^2+\sigma+n\omega}} + \dfrac{ie^{ikx}}{\sigma + n\omega} & , \ x < 0 \\ \dfrac{\omega}{2\pi} \displaystyle\sum_{m\in\mathbb{Z}} d_m^{(\sigma)} e^{-\kappa_m^{(\sigma)} x} \int_0^{\frac{2\pi}{\omega}} dt \ e^{-i(n-m)\omega t} e^{\frac{i\epsilon^2}{\omega}\sin(2\omega t) + \kappa_m^{(\sigma)} \frac{4\epsilon}{\omega}\cos(\omega t)}, & x > 0 \end{cases}$$

with

$$\kappa_m^{(\sigma)} := \sqrt{U + 2\epsilon^2 - k^2 - \sigma - m\omega}$$

Initial value problem

- The sequences c_n and d_n are determined by the continuity condition at $x = 0$:

$$\sum_{m\in\mathbb{Z}} G^{(\sigma)}_{n,m} d^{(\sigma)}_m = v^{(\sigma)}_n, \quad c^{(\sigma)}_n = \sum_{m\in\mathbb{Z}} H^{(\sigma)}_{n,m} d^{(\sigma)}_m + w^{(\sigma)}_n.$$

- The long-time behavior of Ψ depends on the singularities of $\hat{\Psi}_p$ with $p \in i\mathbb{R}$.

Work in progress

- If $G^{(\sigma)}$ is invertible, then the imaginary poles of $\hat{\Psi}$ are at $-i(k^2 + \omega\mathbb{Z})$, and $\Psi(x,t)$ converges to the periodic solution as $t \to \infty$.
- We have made progress in proving this using a RAGE-like theorem and studying the spectrum of the Floquet operator.
- Numerical challenge: good approximation of the inverse of $G^{(\sigma)}$.