# Solution of time-dependent Schrödinger equation for field emission

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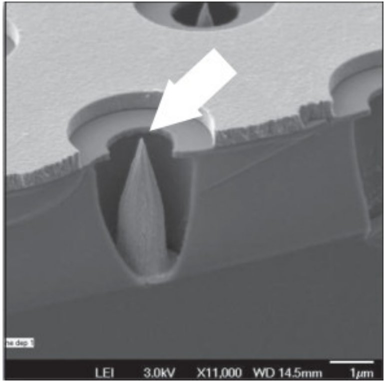
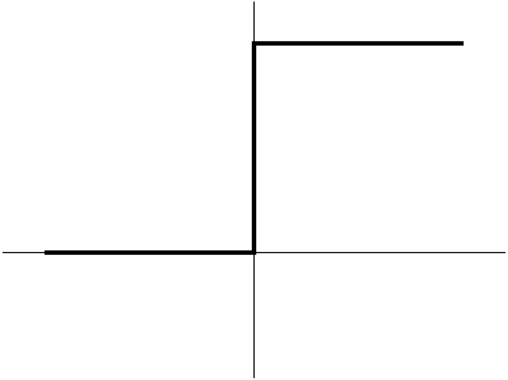
arXiv:1808.00936 http://ian.jauslin.org

# Field emission



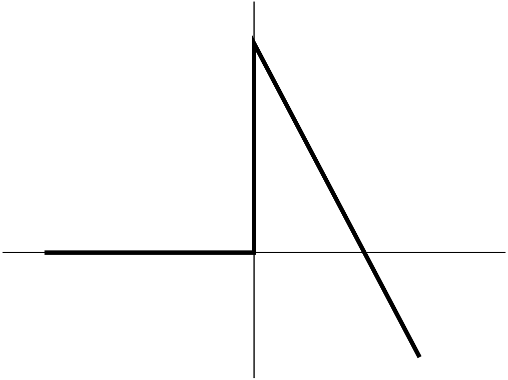
# Emitter

$$V(x) = U\Theta(x)$$



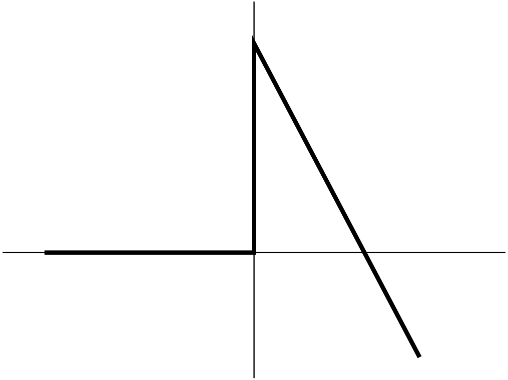
# Semi-classical model

$$V_t(x) = \Theta(x)(U - E_t x), \quad E_t = e_0 + e_1 \cos(\omega t)$$



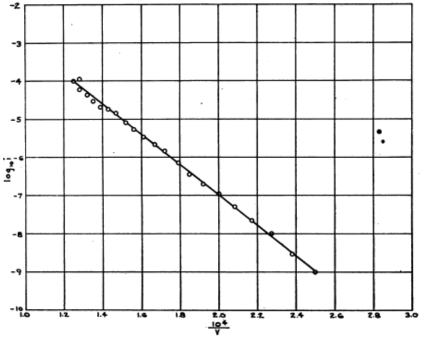
# Triangular barrier

$$V(x) = \Theta(x)(U - Ex)$$



# Field emission through a triangular barrier

- [Millikan, Lauritsen, 1928]: experimental plot of the logarithm of the current against $1/E$



# Field emission through a triangular barrier

- [Fowler, Nordheim, 1928]: predicted that the current is, for small $E$,

$$J \approx Ce^{-\frac{a}{E}}$$

- ([Rokhlenko, 2011]: studied the range of applicability of the approximation, and found more accurate approximations for larger fields.)

# Fowler-Nordheim equation

- Schrödinger equation

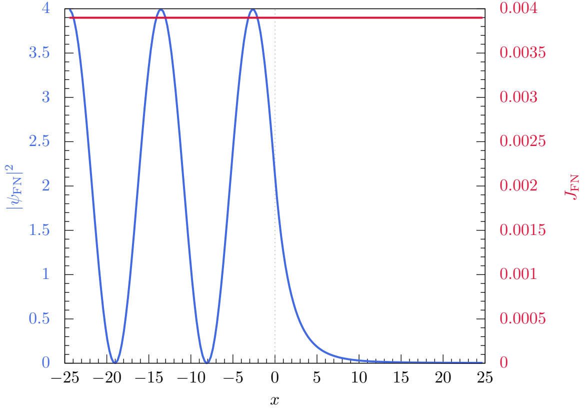
$$i\partial_t \psi = -\Delta\psi + \Theta(x)(U - Ex)\psi$$

- Fowler-Nordheim: quasi-stationary solution: $\psi_{\mathrm{FN}}(x,t) = e^{-ik^2 t}\varphi_{\mathrm{FN}}(x)$

$$\varphi_{\mathrm{FN}}(x) = \begin{cases} e^{ikx} + R_E e^{-ikx} & x < 0 \\ T_E \mathrm{Ai}(e^{-\frac{i\pi}{3}}(E^{\frac{1}{3}}x - E^{-\frac{2}{3}}(U - k^2)) & x > 0 \end{cases}$$

$R_E$ and $T_E$ are chosen so that $\varphi_{\mathrm{FN}}$ and $\partial\varphi_{\mathrm{FN}}$ are continuous at $x = 0$.

# Fowler-Nordheim equation



# Initial value problem

- Initial condition:

$$\psi(x,0) = \begin{cases} e^{ikx} + R_0 e^{-ikx} & x < 0 \\ T_0 e^{-\sqrt{U-k^2}x} & x > 0 \end{cases}$$

  $R_0$ and $T_0$ ensure that $\psi$ and $\partial\psi$ are continuous.

- Behaves asymptotically like $\psi_{\mathrm{FN}}$:

$$\psi(x,t)e^{ik^2t} \underset{t\to\infty}{\longrightarrow} \varphi_{\mathrm{FN}}(x)$$

# Initial value problem

- Laplace transform:

$$\hat{\psi}_p(x) := \int_0^\infty dt\ e^{-pt}\psi(x,t)$$

- Schrödinger equation:

$$(-\Delta + \Theta(x)V(x) - ip)\psi_p(x) = -i\psi(x,0), \quad V(x) := U - Ex$$

# Solution in Laplace space

- For simplicity, $R_0 \equiv T_0 \equiv 0$.
- Solution:

$$\hat{\psi}_p(x) = \begin{cases} C_1(p) e^{\sqrt{-ip}x} - \dfrac{i e^{ikx}}{-ip + k^2} & \text{if } x < 0 \\ C_2(p) \varphi_p(x) & \text{if } x > 0 \end{cases}$$

with

$$(-\Delta + V(x) - ip)\varphi_p(x) = 0$$

$$\varphi_p(x) = \mathrm{Ai}\left(e^{-\frac{i\pi}{3}} \left(E^{\frac{1}{3}} x - E^{-\frac{2}{3}}(U - ip)\right)\right)$$

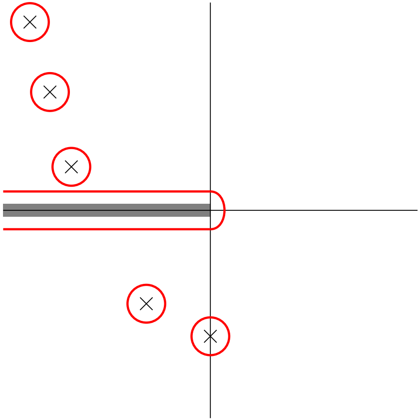
# Solution in Laplace space

- $C_1$ and $C_2$ ensure that $\hat{\psi}_p(x)$ and $\partial\hat{\psi}_p(x)$ are continuous at $x = 0$:

$$C_1(p) = \frac{i(ik\varphi_p(0) - \partial\varphi_p(0))}{(-ip + k^2)(\sqrt{-ip}\varphi_p(0) - \partial\varphi_p(0))}$$

$$C_2(p) = -\frac{i}{(\sqrt{-ip} + ik)(\sqrt{-ip}\varphi_p(0) - \partial\varphi_p(0))}.$$

# Poles in Laplace plane



# Asymptotic behavior

- As $t \to \infty$:

$$\psi(x,t) = \psi_{\mathrm{FN}}(x,t) + \left(\frac{t}{\tau_E(x)}\right)^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}).$$

- If $k < 0$ (reflected wave), then there is no pole on the imaginary axis, so there is no contribution as $t \to \infty$.
- Similarly, the transmitted wave in the initial condition does not contribute.

# Laser field

- Time dependent potential:

$$V_t(x) = \Theta(x)(U - \epsilon\omega\cos(\omega t)x)$$

- Magnetic gauge:

$$\Psi(x,t) :=:= \psi(x,t)e^{-ix\Theta(x)A(t)}, \quad A(t) := \int_0^t ds\ \epsilon\omega\cos(\omega s) = \epsilon\sin(\omega t)$$

satisfies

$$i\partial_t\Psi(x,t) = \left((-i\nabla + \Theta(x)A(t))^2 + \Theta(x)U\right)\Psi(x,t)$$

# Periodic solution

- A solution:

$$\Psi(x,t) = \begin{cases} \Psi_I(x,t) + \Psi_R(x,t) & \text{if } x < 0 \\ \Psi_T(x,t) & \text{if } x > 0 \end{cases}$$

$$\Psi_I(x,t) = e^{ikx} \exp\left(-ik^2 t\right)$$

$$\Psi_R(x,t) = \sum_{M\in\mathbb{Z}} R_M e^{iq_M x} \exp\left(-iq_M^2 t\right)$$

$$\Psi_T(x,t) = \sum_{M\in\mathbb{Z}} T_M e^{ip_M x} \exp\left(-iUt - i\int_0^t d\tau\ (p_M + A(\tau))^2\right)$$

# Periodic solution

- Choose $q_M$ and $p_M$ to make the solution periodic (up to the phase $e^{ik^2t}$):

$$q_M = \pm\sqrt{k^2 + M\omega}, \quad p_M = \pm\sqrt{k^2 - U + M\omega - U_V}$$

and

$$U_V := \frac{\omega}{2\pi}\int_0^{\frac{2\pi}{\omega}} d\tau \; A^2(\tau) = \frac{\epsilon^2}{2}.$$