

**Field electron emission
and the Fowler-Nordheim equation**

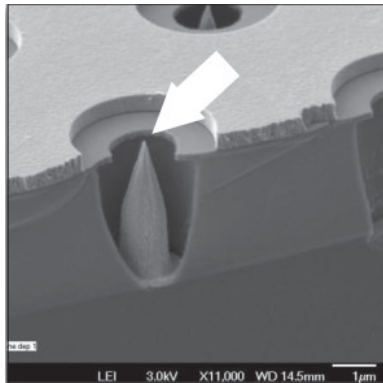
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arXiv:1808.00936

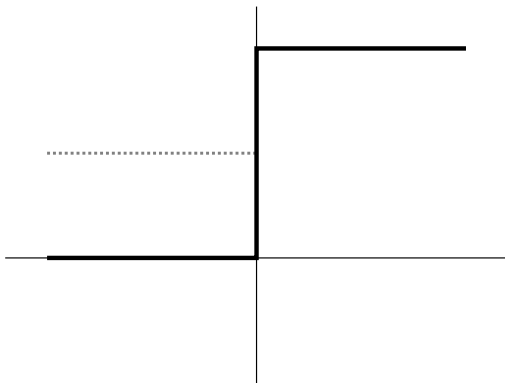
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Field emission



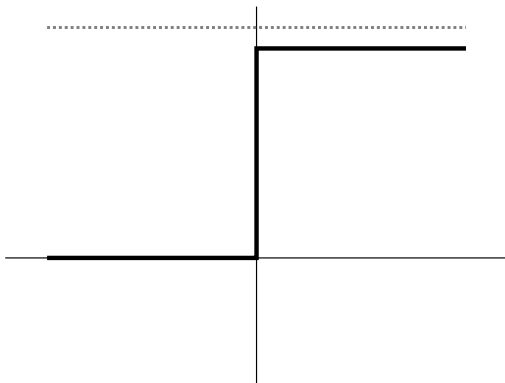
Field emission

$$V(x) = U\Theta(x), \quad E_F = k_F^2 < U$$



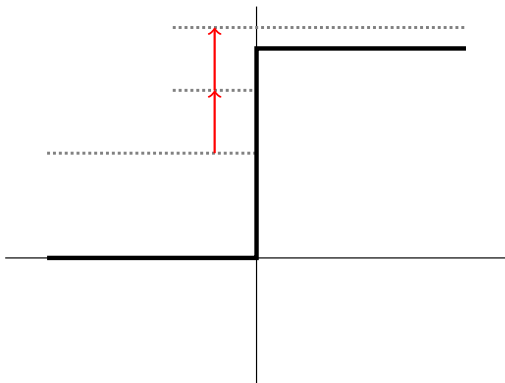
Thermal emission

$$V(x) = U\Theta(x), \quad E_F = k_F^2 > U$$



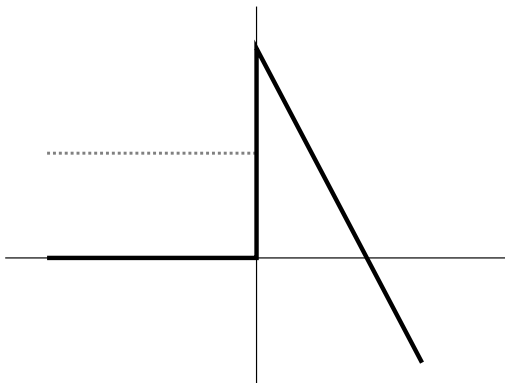
Photonic emission

$$V_t(x) = \Theta(x)(U - E_t x), \quad E_t = \epsilon\omega \cos(\omega t)$$



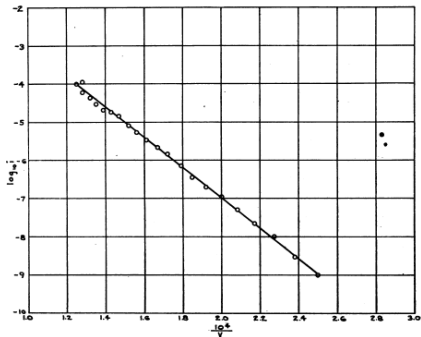
Field emission

$$V(x) = \Theta(x)(U - Ex)$$



Field emission

- [Millikan, Lauritsen, 1928]: experimental plot of the logarithm of the current against $1/E$



Field emission through a triangular barrier

- [Fowler, Nordheim, 1928]: predicted that the current is, for small E ,

$$J \approx CE^2 e^{-\frac{\alpha}{E}}$$

- ([Rokhlenko, 2011]: studied the range of applicability of the approximation, and found more accurate approximations for larger fields.)

Fowler-Nordheim equation

- Schrödinger equation

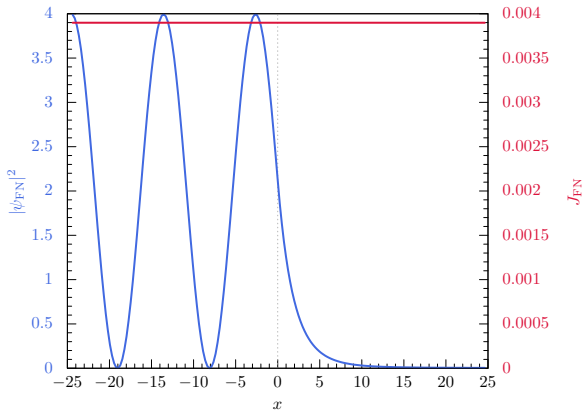
$$i\partial_t\psi = -\Delta\psi + \Theta(x)(U - Ex)\psi$$

- Fowler-Nordheim: stationary solution: $\psi_{\text{FN}}(x, t) = e^{-ik^2t}\varphi_{\text{FN}}(x)$

$$\varphi_{\text{FN}}(x) = \begin{cases} e^{ikx} + R_E e^{-ikx} & x < 0 \\ T_E \text{Ai}(e^{-\frac{i\pi}{3}}(E^{\frac{1}{3}}x - E^{-\frac{2}{3}}(U - k^2))) & x > 0 \end{cases}$$

R_E and T_E are chosen so that φ_{FN} and $\partial\varphi_{\text{FN}}$ are continuous at $x = 0$.

Fowler-Nordheim equation



Initial value problem

- Initial condition:

$$\psi(x, 0) = \begin{cases} e^{ikx} + R_0 e^{-ikx} & x < 0 \\ T_0 e^{-\sqrt{U-k^2}x} & x > 0 \end{cases}$$

R_0 and T_0 ensure that ψ and $\partial\psi$ are continuous.

- Behaves asymptotically like ψ_{FN} :

$$\psi(x, t)e^{ik^2t} \xrightarrow[t \rightarrow \infty]{} \varphi_{\text{FN}}(x)$$

Initial value problem

- Laplace transform:

$$\hat{\psi}_p(x) := \int_0^\infty dt e^{-pt} \psi(x, t)$$

- Schrödinger equation:

$$(-\Delta + \Theta(x)V(x) - ip)\psi_p(x) = -i\psi(x, 0), \quad V(x) := U - Ex$$

Solution in Laplace space

- For simplicity, $R_0 \equiv T_0 \equiv 0$.
- Solution:

$$\hat{\psi}_p(x) = \begin{cases} C_1(p)e^{\sqrt{-ip}x} - \frac{ie^{ikx}}{-ip + k^2} & \text{if } x < 0 \\ C_2(p)\varphi_p(x) & \text{if } x > 0 \end{cases}$$

with

$$\begin{aligned} (-\Delta + V(x) - ip)\varphi_p(x) &= 0 \\ \varphi_p(x) &= \text{Ai} \left(e^{-\frac{i\pi}{3}} \left(E^{\frac{1}{3}}x - E^{-\frac{2}{3}}(U - ip) \right) \right) \end{aligned}$$

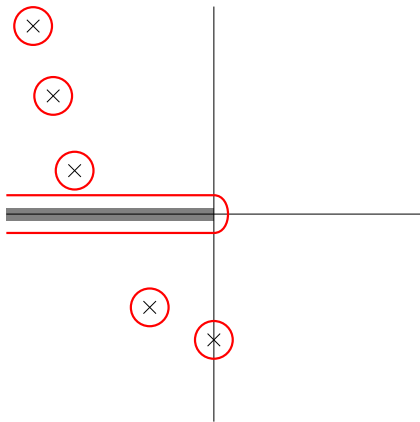
Solution in Laplace space

- C_1 and C_2 ensure that $\hat{\psi}_p(x)$ and $\partial\hat{\psi}_p(x)$ are continuous at $x = 0$:

$$C_1(p) = \frac{i(ik\varphi_p(0) - \partial\varphi_p(0))}{(-ip + k^2)(\sqrt{-ip}\varphi_p(0) - \partial\varphi_p(0))}$$

$$C_2(p) = -\frac{i}{(\sqrt{-ip} + ik)(\sqrt{-ip}\varphi_p(0) - \partial\varphi_p(0))}.$$

Poles in Laplace plane



Asymptotic behavior

- As $t \rightarrow \infty$:

$$\psi(x, t) = \psi_{\text{FN}}(x, t) + \left(\frac{t}{\tau_E(x)} \right)^{-\frac{3}{2}} + O(t^{-\frac{5}{2}}).$$

- If $k < 0$ (reflected wave), then there is no pole on the imaginary axis, so there is no contribution as $t \rightarrow \infty$.
- Similarly, the transmitted wave in the initial condition does not contribute.

Laser field

- Time dependent potential:

$$V_t(x) = \Theta(x)(U - \epsilon\omega \cos(\omega t)x)$$

- Magnetic gauge:

$$\Psi(x, t) := \psi(x, t)e^{-ix\Theta(x)A(t)}, \quad A(t) := \int_0^t ds \epsilon\omega \cos(\omega s) = \epsilon \sin(\omega t)$$

satisfies

$$i\partial_t \Psi(x, t) = ((-i\nabla + \Theta(x)A(t))^2 + \Theta(x)U) \Psi(x, t)$$

Periodic solution

- A solution:

$$\Psi(x, t) = \begin{cases} \Psi_I(x, t) + \Psi_R(x, t) & \text{if } x < 0 \\ \Psi_T(x, t) & \text{if } x > 0 \end{cases}$$

$$\Psi_I(x, t) = e^{ikx} \exp(-ik^2t)$$

$$\Psi_R(x, t) = \sum_{M \in \mathbb{Z}} R_M e^{iq_M x} \exp(-iq_M^2 t)$$

$$\Psi_T(x, t) = \sum_{M \in \mathbb{Z}} T_M e^{ip_M x} \exp\left(-iUt - i \int_0^t d\tau (p_M + A(\tau))^2\right)$$

Periodic solution

- Choose q_M and p_M to make the solution periodic (up to the phase e^{ik^2t}):

$$q_M = \pm\sqrt{k^2 + M\omega}, \quad p_M = \pm\sqrt{k^2 - U + M\omega - U_V}$$

and

$$U_V := \frac{\omega}{2\pi} \int_0^{\frac{2\pi}{\omega}} d\tau A^2(\tau) = \frac{\epsilon^2}{2}.$$