Liquid crystals and the Heilmann-Lieb model

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arXiv: 1709.05297

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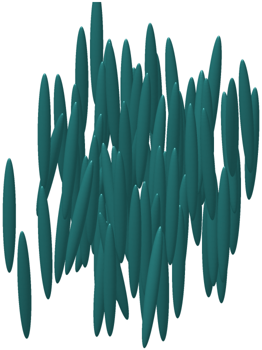
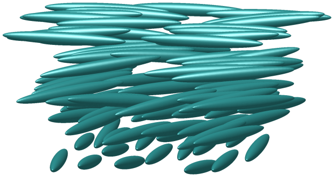
Liquid crystals

- Orientational order and positional disorder.



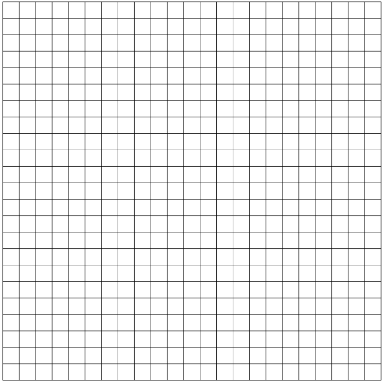


History

- [Onsager, 1949]: mean field model for hard needles in $\mathbb{R}^3$.
- [Heilmann, Lieb, 1979]: interacting dimers.

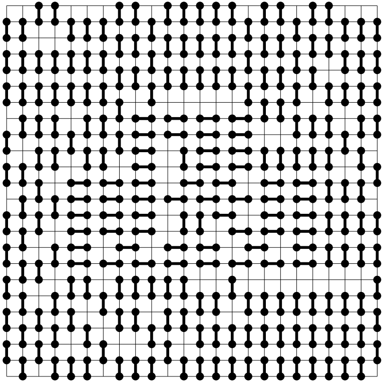
Heilmann-Lieb model

[Heilmann, Lieb, 1979]



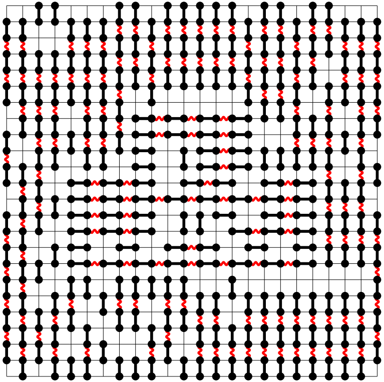
Heilmann-Lieb model

[Heilmann, Lieb, 1979]



Heilmann-Lieb model

[Heilmann, Lieb, 1979]



History

- [Onsager, 1949]: mean field model for hard needles in $\mathbb{R}^3$.
- [Heilmann, Lieb, 1979]: interacting dimers.
- [Bricmont, Kuroda, Lebowitz, 1984]: hard needles in $\mathbb{R}^2$.
- [Ioffe, Velenik, Zahradník, 2006]: hard rods in $\mathbb{Z}^2$ (variable length).
- [Disertori, Giuliani, 2013]: hard rods in $\mathbb{Z}^2$.

Heilmann-Lieb conjecture

- [Alberici, 2016]: different fugacities for horizontal and vertical dimers.

- [Papanikolaou, Charrier, Fradkin, 2014]: numerics.

Heilmann-Lieb model

- Grand-canonical Gibbs measure:

$$\langle A\rangle_{\mathrm{v}} := \lim_{\Lambda\to\mathbb{Z}^2} \frac{1}{\Xi_{\Lambda,\mathrm{v}}(z)} \sum_{\underline{\delta}\in\Omega_{\mathrm{v}}(\Lambda)} A(\underline{\delta}) z^{|\underline{\delta}|} \prod_{\delta\neq\delta'\in\underline{\delta}} e^{\frac{1}{2}J\mathbb{1}_{\delta\sim\delta'}}$$

  - ▶ Λ: finite box.
  - ▶ $\Omega_{\mathrm{v}}(\Lambda)$: non-overlapping dimer configurations satisfying the boundary condition.
  - ▶ $z \geqslant 0$: fugacity.
  - ▶ $J \geqslant 0$: interaction strength.
  - ▶ $\mathbb{1}_{\delta\sim\delta'}$ indicator that dimers are adjacent and aligned.

Theorem

For $1 \ll z \ll J$, $\|(x,y)\|_{\mathrm{HL}} := J|x| + e^{-\frac{3}{2}J} z^{-\frac{1}{2}} |y|$,

- Given two vertical edges $e_{\mathrm{v}}, f_{\mathrm{v}}$, $\langle \mathbb{1}_{e_{\mathrm{v}}} \rangle_{\mathrm{v}}$ is *independent* of e_{v} and

$$\langle \mathbb{1}_{e_{\mathrm{v}}} \rangle_{\mathrm{v}} = \tfrac{1}{2}(1 + O(e^{-\frac{1}{2}J} z^{-\frac{1}{2}}))$$

$$\langle \mathbb{1}_{e_{\mathrm{v}}} \mathbb{1}_{f_{\mathrm{v}}} \rangle_{\mathrm{v}} - \langle \mathbb{1}_{e_{\mathrm{v}}} \rangle_{\mathrm{v}} \langle \mathbb{1}_{f_{\mathrm{v}}} \rangle_{\mathrm{v}} = O(e^{-c\ \mathrm{dist}_{\mathrm{HL}}(e_{\mathrm{v}}, f_{\mathrm{v}})})$$

- Given two horizontal edges $e_{\mathrm{h}}, f_{\mathrm{h}}$, $\langle \mathbb{1}_{e_{\mathrm{h}}} \rangle_{\mathrm{v}}$ is *independent* of e_{h} and

$$\langle \mathbb{1}_{e_{\mathrm{h}}} \rangle_{\mathrm{v}} = O(e^{-3J})$$

$$\langle \mathbb{1}_{e_{\mathrm{h}}} \mathbb{1}_{f_{\mathrm{v}}} \rangle_{\mathrm{v}} - \langle \mathbb{1}_{e_{\mathrm{h}}} \rangle_{\mathrm{v}} \langle \mathbb{1}_{f_{\mathrm{h}}} \rangle_{\mathrm{v}} = O(e^{-3J-c\ \mathrm{dist}_{\mathrm{HL}}(e_{\mathrm{v}}, f_{\mathrm{v}})})$$

1D system

- [Heilmann, Lieb, 1979]: mostly vertical dimers.
- *Only* vertical dimers: integrable.
- Given two vertical edges $e_\mathrm{v}, f_\mathrm{v}$, $\left\langle \mathbb{1}_{e_\mathrm{v}} \right\rangle_\mathrm{v}$ is *independent* of e_v and

$$\left\langle \mathbb{1}_{e_\mathrm{v}} \right\rangle_\mathrm{v} = \tfrac{1}{2}\left(1 + O\left(e^{-\frac{1}{2}J} z^{-\frac{1}{2}}\right)\right)$$

$$\left\langle \mathbb{1}_{e_\mathrm{v}} \mathbb{1}_{f_\mathrm{v}} \right\rangle_\mathrm{v} - \left\langle \mathbb{1}_{e_\mathrm{v}} \right\rangle_\mathrm{v} \left\langle \mathbb{1}_{f_\mathrm{v}} \right\rangle_\mathrm{v} = O\left(e^{-c\ \mathrm{dist}_{1\mathrm{D}}(e_\mathrm{v}, f_\mathrm{v})}\right)$$

with $\|(x,y)\|_{1\mathrm{D}} := e^{-\frac{3}{2}J} z^{-\frac{1}{2}} |y|$.

Peierls argument