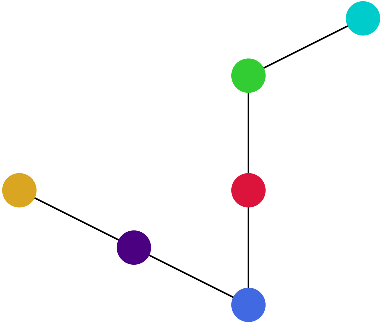
Dimers, Spins and Loops
Transfer Matrices, the TL Algebra and Emerging Fermions

Ian Jauslin

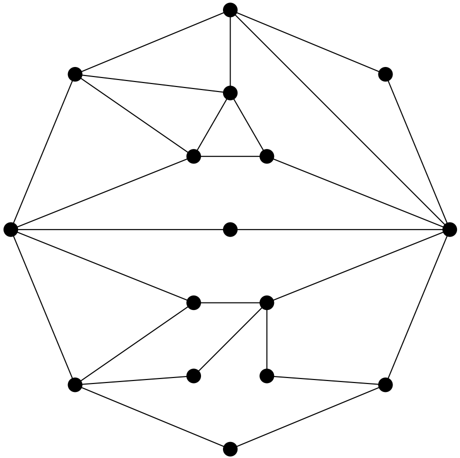
http://ian.jauslin.org

Outline

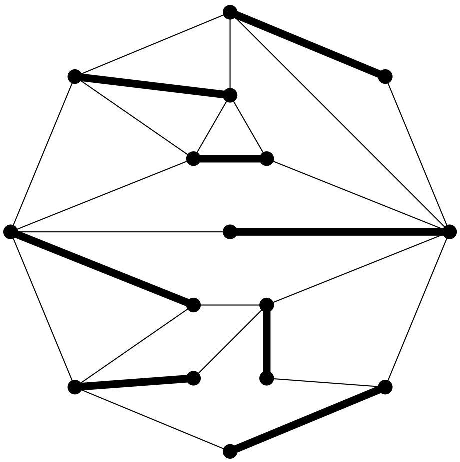


- [Li67] Dimers
- [SML64] 2D Ising
- [TL71] Temperley-Lieb algebras
- [HL72] Monomers and Dimers
- [HL79] Interacting Dimers
- [Li89] Hubbard model

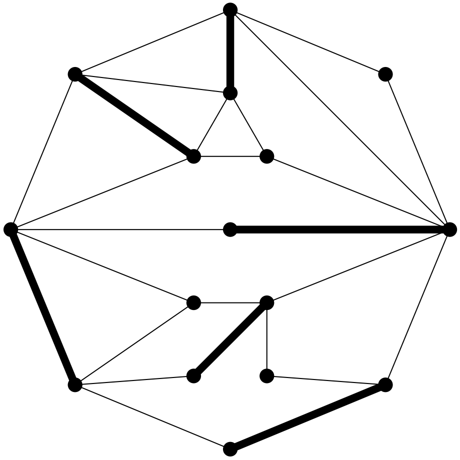
Dimers



Dimer covering



Monomer-Dimer configuration



Counting dimer coverings

- On planar graphs: [Kasteleyn, 1963], [Temperley, Fisher, 1961]: Pfaffian formula

$$Z \equiv \text{number of coverings} = \text{determinant}.$$

- **E.H. Lieb**, *Solution of the Dimer Problem by the Transfer Matrix Method*, Journal of Mathematical Physics, 1967: on $M \times N$ discrete torus:

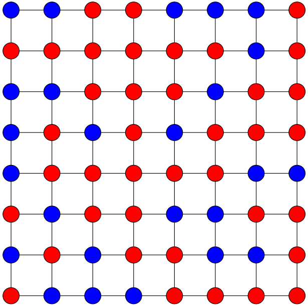
$$\lim_{M,N\to\infty} \frac{1}{MN} \log Z = \frac{1}{2\pi} \int_0^\pi dq \;\log\left(\sin q + \sqrt{1+\sin^2 q}\right).$$

- Using a Transfer Matrix and Emergent Fermions.

2D Ising

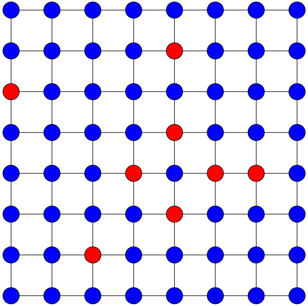
- *Spin* on every $x \in \mathbb{Z}^2$. Random configuration with probability

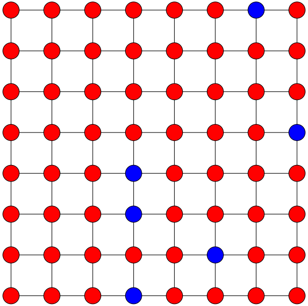
$$\frac{1}{Z(T)} e^{\frac{1}{T} \sum_{\langle i,j \rangle} \sigma_i \sigma_j}.$$



2D Ising

- At $T \ll 1$, two phases:





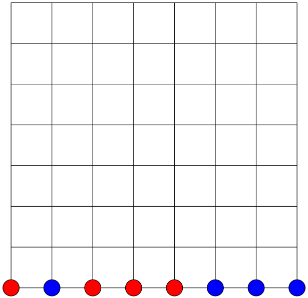
2D Ising

- Free energy:

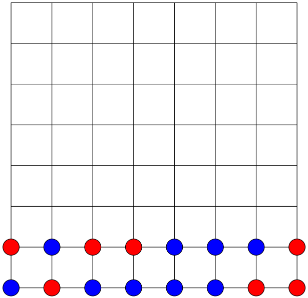
$$f(T) = -T \lim_{M,N \to \infty} \frac{1}{MN} \log Z(T).$$

- Exact solution: [Onsager, 1944]: first example of a microscopic model with a phase transition.
- **T.D. Schultz, D.C. Mattis, E.H. Lieb**, *Two-Dimensional Ising Model as a Soluble Problem of Many Fermions*, Reviews of Modern Physics, 1964.

2D Ising - Transfer Matrix



2D Ising - Transfer Matrix



2D Ising - Transfer Matrix

- Transfer matrix: V, is a $2^M \times 2^M$ real symmetric matrix, and

$$Z(T) = \mathrm{Tr}(V^N).$$

- The free energy

$$f(T) = -T \lim_{N,M\to\infty} \frac{1}{NM} \log Z(T) = -\lim_{M\to\infty} \frac{T}{M} \log \lambda_M$$

where λ_M is the *largest* eigenvalue of V.

2D Ising - Emergent Fermions

- To diagonalize V: turn spins into Fermions using a *Jordan-Wigner* transformation.
- Fermions: particle excitations.
- *Non-interacting* Fermions:

$$V = (2\sinh(2JT^{-1}))^{\frac{M}{2}} e^{-\sum_q \epsilon_q (c_q^\dagger c_q - \frac{1}{2})}.$$

- Remark: in the *ice model* (*cf* Duminil-Copin), Fermions *interact*.
- Remark: the Ising model with weak nearest neighbor interactions is mapped to a weakly interacting Fermion model [Giuliani, Greenblatt, Mastropietro, 2012].

Temperley-Lieb algebras

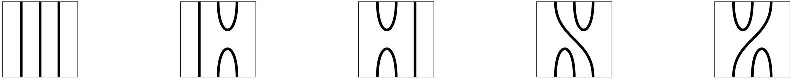
- **H.N.V. Temperley, E.H. Lieb**, *Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem*, Proceedings of the Royal Society of London A, 1971.
- Compute *Whitney* polynomial on a square lattice

$$W(x,y) = \sum_G x^{l_G - s_G} y^{s_G}.$$

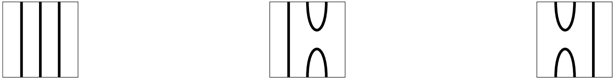
  where l_G is the number of lines and s_G the number of cycles.
- Related to counting the number of connected components in a random graph, and to the number of ways of coloring the $\mathbb{Z}^2$ lattice.
- Transfer Matrix technique: difficult because the setting is non-Markovian.

Temperley-Lieb algebras

- In each row, keep track of who is connected to whom.
- Graphical representation of the transfer matrix:



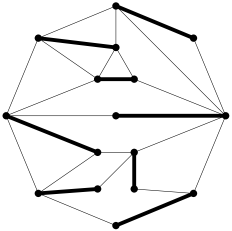
- Algebra generated by

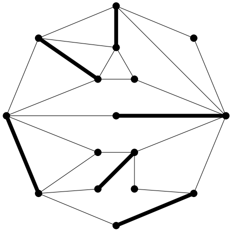


- Applications to knot theory, the Jones polynomial, braids, 2D Ising, quantum groups...

Counting dimer coverings

- **E.H. Lieb**, *Solution of the Dimer Problem by the Transfer Matrix Method*, Journal of Mathematical Physics, 1967.
- Similar approach to Schultz-Mattis-Lieb: Transfer Matrix/Fermions.
- What if there are monomers?





Monomer-Dimer

- **O.J. Heilmann, E.H. Lieb**, *Theory of monomer-dimer systems*, Communications in Mathematical Physics, 1972.
- Random configuration of dimers: (z: *monomer fugacity*)

$$\frac{z^{\#\text{monomers}}}{\Xi_G(z)}.$$

- Free energy:

$$f(z) := -\lim_{\text{Vol}\to\infty} \frac{1}{\text{Vol}} \log(\Xi_G(z)).$$

- There is a *phase transition* when f is singular.
- Roots of $\Xi_G(z)$: *Lee-Yang* zeros.

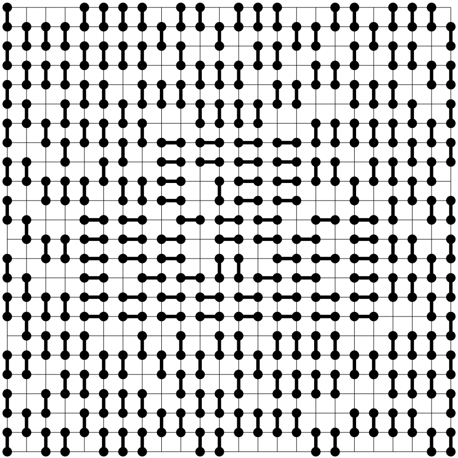
Monomer-Dimer

- Recurrence relation:

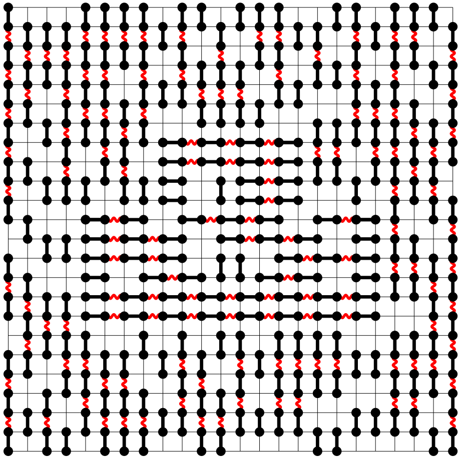
$$\Xi_G(z) = z\Xi_{G\setminus\{i\}} + \sum_{j:(i,j)\in G} \Xi_{G\setminus\{i,j\}}(z).$$

- The Lee-Yang zeros lie in a bounded subset of the imaginary axis.
- This result was recently used to solve the Kadison-Singer problem [Marcus, Spielman, Srivastava, 2014].
- No phase transitions in the monomer-dimer model!

Dimers as particles



Interacting dimers



Heilmann-Lieb model

- **O.J. Heilmann, E.H. Lieb**, *Lattice models for liquid crystals*, Journal of Statistical Physics, 1979.
- Long range orientational order: dimers are either mostly vertical or mostly horizontal (if the interaction is strong enough).
- There is a phase transition!
- Argument uses *reflection positivity* and a *chessboard estimate*.

Hubbard model

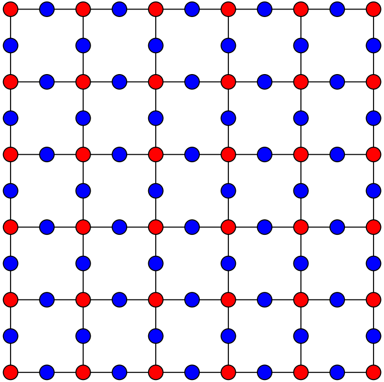
- **E.H. Lieb**, *Two Theorems on the Hubbard Model*, Physical Review Letters, 1989.
- Electrons on a graph, with a local interaction.
- If the interaction is repulsive, the graph is bipartite, and the number of electrons is equal to the number of vertices, then the spin of the ground state is

$$S = \frac{1}{2}||B| - |A||$$

  where $|B|$ and $|A|$ are the numbers of vertices on the B- and A-subgraphs.

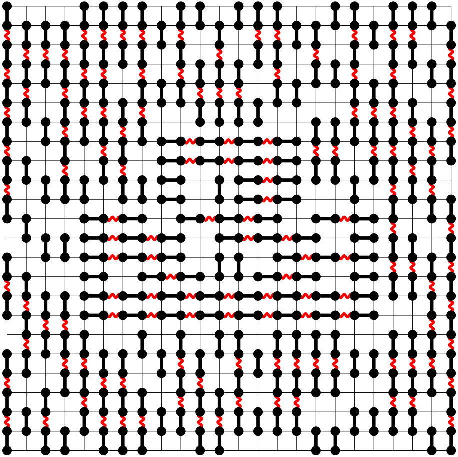
- Uses reflection positivity in *spin space*.

Lieb lattice

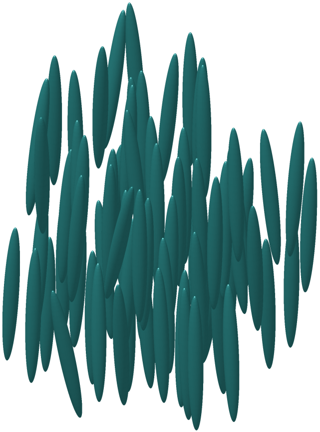


$$|B| = 2|A|$$

Heilmann-Lieb model



Liquid crystals



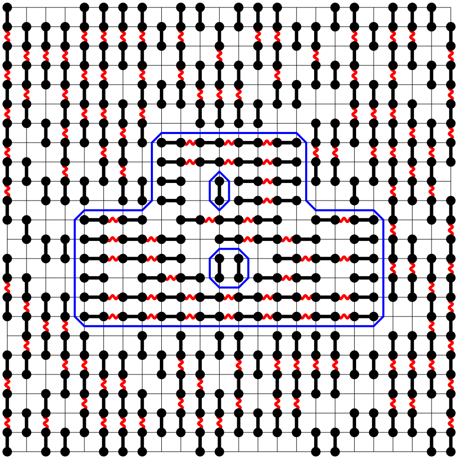
Liquid crystals

- Orientational order *and* positional disorder.
- Heilmann-Lieb: orientational order.
- Conjecture: positional disorder.
- Previous results:
  - [Bricmont, Kuroda, Lebowitz, 1984]: hard needles in $\mathbb{R}^2$ with a *finite* number of orientations.
  - [Ioffe, Velenik, Zahradník, 2006]: hard rods in $\mathbb{Z}^2$ (variable length).
  - [Disertori, Giuliani, 2013]: hard rods in $\mathbb{Z}^2$.

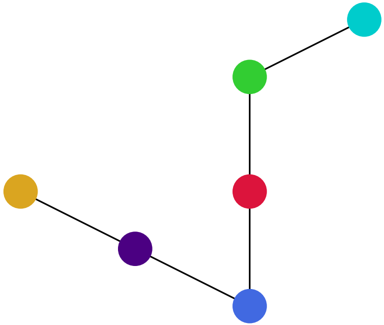
Nematic phase in the Heilmann-Lieb model

- Proof of positional disorder: [Jauslin, Lieb, 2017] (uses Pirogov-Sinai theory).
- Correlations between the positions of the dimers decay exponentially.
- The rate of the decay is strongly anisotropic: in a vertical phase, the correlation length is very large in the vertical direction, and small in the horizontal.

Pirogov-Sinai theory



Summary



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Macbeth - act V scene 8

[...] Before my body
I throw my warlike shield. Lay on, Macduff,
And damn'd be him that first cries, 'Hold, enough!'