Liquid crystals and the Heilmann-Lieb model

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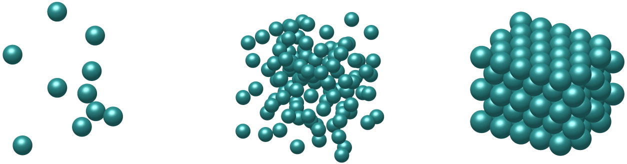
joint with **Elliott H. Lieb**

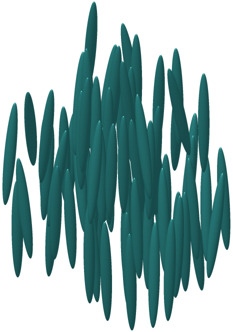
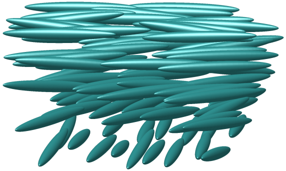
arXiv: `1709.05297` `http://ian.jauslin.org`

Gas-liquid-crystal



Liquid crystals

- Orientational order and positional disorder.

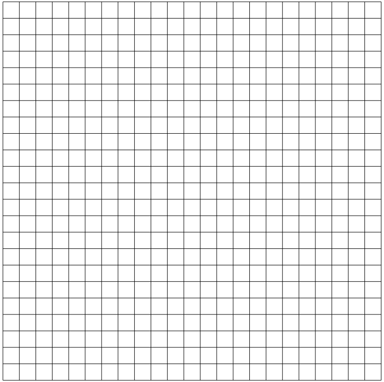
 

History

- [Onsager, 1949]: mean field model for hard needles in $\mathbb{R}^3$.
- [Heilmann, Lieb, 1979]: interacting dimers.

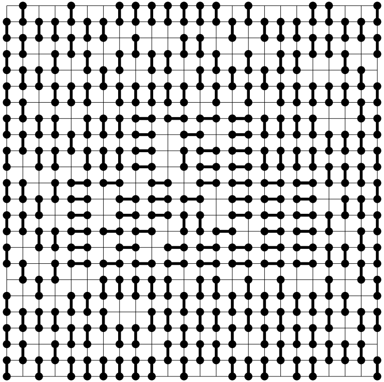
Heilmann-Lieb model

[Heilmann, Lieb, 1979]



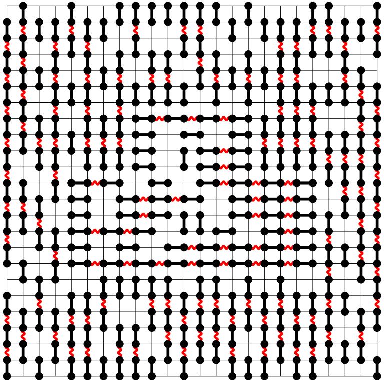
Heilmann-Lieb model

[Heilmann, Lieb, 1979]



Heilmann-Lieb model

[Heilmann, Lieb, 1979]



Heilmann-Lieb conjecture

- [Heilmann, Lieb, 1979]: proved orientational order using reflection positivity.
- HL Conjecture: absence of positional order.

History

- [Onsager, 1949]: mean field model for hard needles in $\mathbb{R}^3$.
- [Heilmann, Lieb, 1979]: interacting dimers.
- [Bricmont, Kuroda, Lebowitz, 1984]: hard needles in $\mathbb{R}^2$ with a *finite* number of orientations.
- [Ioffe, Velenik, Zahradník, 2006]: hard rods in $\mathbb{Z}^2$ (variable length).
- [Disertori, Giuliani, 2013]: hard rods in $\mathbb{Z}^2$.

Heilmann-Lieb conjecture

- [Heilmann, Lieb, 1979]: proved orientational order using reflection positivity.
- HL Conjecture: absence of positional order.
- [Alberici, 2016]: different fugacities for horizontal and vertical dimers.
- [Papanikolaou, Charrier, Fradkin, 2014]: numerics.

Heilmann-Lieb model

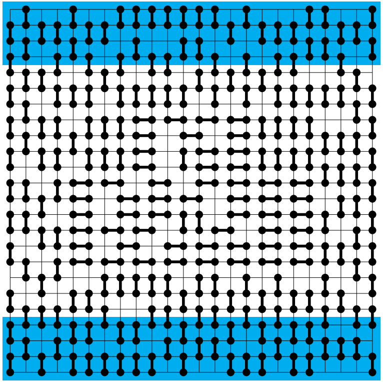
- Grand-canonical Gibbs measure:

$$\langle A\rangle_{\mathrm{v}} := \lim_{\Lambda\to\mathbb{Z}^2} \frac{1}{\Xi_{\mathrm{v}}(\Lambda)} \sum_{\underline{\delta}\in\Omega_{\mathrm{v}}(\Lambda)} A(\underline{\delta}) z^{|\underline{\delta}|} \prod_{\delta\neq\delta'\in\underline{\delta}} e^{\frac{1}{2}J\mathbb{1}_{\delta\sim\delta'}}$$

  - Λ: finite box.
  - $\Omega_{\mathrm{v}}(\Lambda)$: non-overlapping dimer configurations satisfying the boundary condition.
  - $z \geqslant 0$: fugacity.
  - $J \geqslant 0$: interaction strength.
  - $\mathbb{1}_{\delta\sim\delta'}$ indicator that dimers are adjacent and aligned.

Boundary condition

- Fix length $\ell_0 := e^{\frac{3}{2}J}\sqrt{z}$,



Theorem

For $1 \ll z \ll J$, $\|(x,y)\|_{\mathrm{HL}} := J|x| + e^{-\frac{3}{2}J} z^{-\frac{1}{2}} |y|$,

- Given two vertical edges $e_{\mathrm{v}}, f_{\mathrm{v}}$, $\langle \mathbb{1}_{e_{\mathrm{v}}} \rangle_{\mathrm{v}}$ is *independent* of e_{v} and

$$\langle \mathbb{1}_{e_{\mathrm{v}}} \rangle_{\mathrm{v}} = \frac{1}{2}(1 + O(e^{-\frac{1}{2}J} z^{-\frac{1}{2}}))$$

$$\langle \mathbb{1}_{e_{\mathrm{v}}} \mathbb{1}_{f_{\mathrm{v}}} \rangle_{\mathrm{v}} - \langle \mathbb{1}_{e_{\mathrm{v}}} \rangle_{\mathrm{v}} \langle \mathbb{1}_{f_{\mathrm{v}}} \rangle_{\mathrm{v}} = O(e^{-c\ \mathrm{dist}_{\mathrm{HL}}(e_{\mathrm{v}}, f_{\mathrm{v}})})$$

- Given two horizontal edges $e_{\mathrm{h}}, f_{\mathrm{h}}$, $\langle \mathbb{1}_{e_{\mathrm{h}}} \rangle_{\mathrm{v}}$ is *independent* of e_{h} and

$$\langle \mathbb{1}_{e_{\mathrm{h}}} \rangle_{\mathrm{v}} = O(e^{-3J})$$

$$\langle \mathbb{1}_{e_{\mathrm{h}}} \mathbb{1}_{f_{\mathrm{h}}} \rangle_{\mathrm{v}} - \langle \mathbb{1}_{e_{\mathrm{h}}} \rangle_{\mathrm{v}} \langle \mathbb{1}_{f_{\mathrm{h}}} \rangle_{\mathrm{v}} = O(e^{-6J-c\ \mathrm{dist}_{\mathrm{HL}}(e_{\mathrm{h}}, f_{\mathrm{h}})})$$

1D system

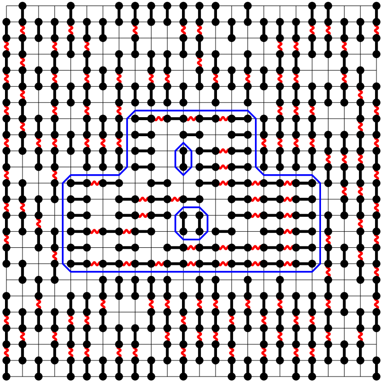
- *Only* vertical dimers: integrable.
- Given two vertical edges $e_{\mathrm{v}}, f_{\mathrm{v}}$, $\langle \mathbb{1}_{e_{\mathrm{v}}} \rangle_{\mathrm{v}}$ is *independent* of e_{v} and

$$\langle \mathbb{1}_{e_{\mathrm{v}}} \rangle_{\mathrm{v}} = \tfrac{1}{2}(1 + O(e^{-\frac{1}{2}J} z^{-\frac{1}{2}}))$$

$$\langle \mathbb{1}_{e_{\mathrm{v}}} \mathbb{1}_{f_{\mathrm{v}}} \rangle_{\mathrm{v}} - \langle \mathbb{1}_{e_{\mathrm{v}}} \rangle_{\mathrm{v}} \langle \mathbb{1}_{f_{\mathrm{v}}} \rangle_{\mathrm{v}} = O(e^{-c\ \mathrm{dist}_{1\mathrm{D}}(e_{\mathrm{v}}, f_{\mathrm{v}})})$$

with $\|(x,y)\|_{1\mathrm{D}} := e^{-\frac{3}{2}J} z^{-\frac{1}{2}} |y|$.

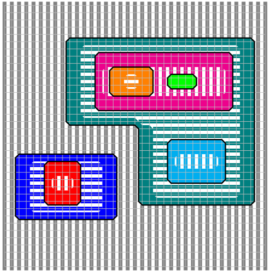
Loop model



- Weight of a loop of length $|l|$: $e^{-\frac{1}{2}J|l|}$.

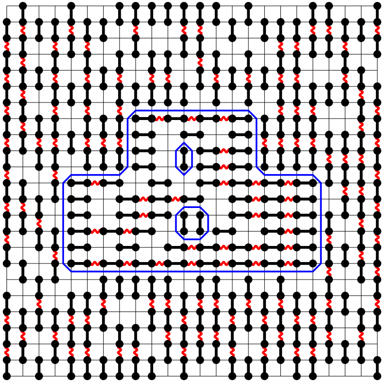
Difficulty: loops interact

- Correlated dimers induce an interaction between loops, which decays exponentially with a rate $e^{-\frac{3}{2}J}z^{-\frac{1}{2}}$.

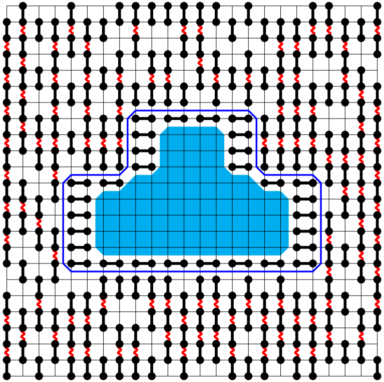


- Vertical-to-horizontal boundaries and horizontal-to-vertical ones have different geometries.

Pirogov-Sinai

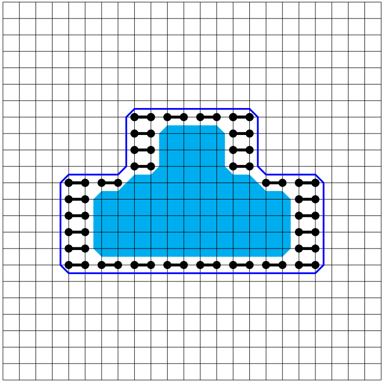


Pirogov-Sinai



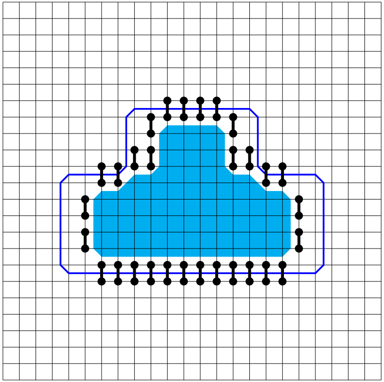
$$\Xi_{\mathrm{v}}(\Lambda) = \Xi_{\mathrm{v}}(\mathrm{Out})\ \Xi_{\mathrm{h}}(\mathrm{In})$$

Pirogov-Sinai



$$\Xi_{\mathrm{v}}(\Lambda) = \Xi_{\mathrm{v}}(\mathrm{Out})\ \Xi_{\mathrm{h}}(\mathrm{In})$$

Pirogov-Sinai



$$\Xi_{\mathrm{v}}(\Lambda) = \Xi_{\mathrm{v}}(\mathrm{Out})\ \Xi_{\mathrm{v}}(\mathrm{In})\eta_{\mathrm{h,v}}(\mathrm{In}), \quad \eta_{\mathrm{h,v}}(\mathrm{In}) = \frac{\Xi_{\mathrm{h}}(\mathrm{In})}{\Xi_{\mathrm{v}}(\mathrm{In})}$$

Pirogov-Sinai

- Boundary term:

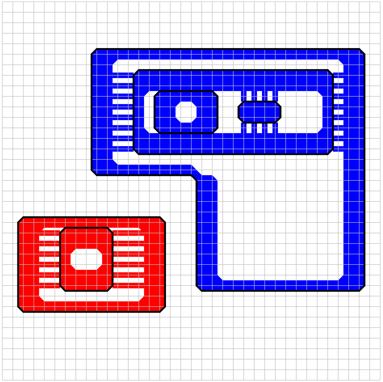
$$\eta_{\mathrm{h,v}}(\mathrm{In}) \leqslant e^{c|\partial \mathrm{In}|}$$

- Energy gain:

$$e^{-\frac{1}{2}J|\partial \mathrm{In}|}e^{c|\partial \mathrm{In}|} \ll 1$$

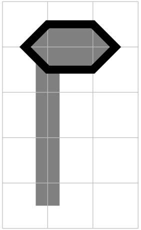
Entropy of contours

- Loops at a distance $< e^{\frac{3}{2}J} z^{\frac{1}{2}} \equiv \ell_0$ form a single object: a *contour*.



Entropy of contours

- Weight of a contour:
  - $e^{-\frac{1}{2}J|l|}$ for each loop l.
  - $e^{-\ell_0^{-1}|\sigma|}$ for each segment σ.
- Each new loop in a contour contributes $e^{-3J}\ell_0 \equiv e^{-\frac{3}{2}J}z^{\frac{1}{2}} \ll 1$



Parameter regime