High density phases
of hard-core lattice particle systems

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Gas-liquid-crystal
Hard-core lattice particle (HCLP) systems
Non-sliding HCLPs

- There exist a finite number $\tau$ of tilings which are periodic and isometric to each other.
Non-sliding HCLPs

• Defects are localized: for every connected particle configuration $X$ that is not the subset of a close packing and every $Y \supset X$, there is empty space in $Y$ neighboring $X$. 
Observables

- Gibbs measure:

\[
\langle A \rangle_\nu := \lim_{\Lambda \to \Lambda_\infty} \frac{1}{\Xi_{\Lambda,\nu}(z)} \sum_{X \subset \Lambda} A(X) z^{|X|} \mathcal{B}_\nu(X) \prod_{x \neq x' \in X} \varphi(x, x')
\]

- \( \Lambda \): finite subset of lattice \( \Lambda_\infty \).
- \( z \geq 0 \): fugacity.
- \( \varphi(x, x') \): hard-core interaction.
- \( \mathcal{B}_\nu \): boundary condition: favors the \( \nu \)-th tiling.

- Pressure:

\[
p(z) := \lim_{\Lambda \to \Lambda_\infty} \frac{1}{|\Lambda|} \log \Xi_{\Lambda,\nu}(z).
\]
Theorem

• $p(z) - \rho_m \log z$ and $\langle 1_{x_1} \cdots 1_{x_n} \rangle_\nu$ are analytic functions of $1/z$ for large values of $z$.

• There are $\tau$ distinct Gibbs states:

$$\langle 1_x \rangle_\nu = \begin{cases} 1 + O(y) & \text{if } x \in \mathcal{L}_\nu \\ O(y) & \text{if not.} \end{cases}$$
High-fugacity expansion

\[ p(y) = -\rho_m \log y + \sum_{k=1}^{\infty} c_k y^k \]

- [Gaunt, Fisher, 1965]: diamonds: for \( k \leq 9 \).
- [Joyce, 1988]: hexagons (integrable, [Baxter, 1980]).
- [Eisenberg, Baram, 2005]: crosses: for \( k \leq 6 \).
- For sliding models, the high-fugacity expansion is ill-defined.
Liquid crystals

- Orientational order and positional disorder.
Heilmann-Lieb model

[Heilmann, Lieb, 1979]
Heilmann-Lieb model

- Gibbs measure:

\[
\langle A \rangle_v := \lim_{\Lambda \to \mathbb{Z}^2} \frac{1}{\Xi_{\Lambda,v}(z)} \sum_{\delta \in \Omega_v(\Lambda)} A(\delta) z^{\vert \delta \vert} \prod_{\delta \neq \delta' \in \delta} e^{\frac{1}{2} J 1_{\delta \sim \delta'}}
\]

- \( \Lambda \): finite box.
- \( \Omega_v(\Lambda) \): non-overlapping dimer configurations satisfying the boundary condition.
- \( z \geq 0 \): fugacity.
- \( J \geq 0 \): interaction strength.
- \( 1_{\delta \sim \delta'} \) indicator that dimers are adjacent and aligned.
Heilmann-Lieb conjecture

- [Heilmann, Lieb, 1979]: proved orientational order using reflection positivity.
- HL Conjecture: absence of positional order.
- [Ioffe, Velenik, Zahradník, 2006], [Disertori, Giuliani, 2013]: nematic liquid crystal phase in systems of hard rods on $\mathbb{Z}^2$.
- [Alberici, 2016]: different fugacities for horizontal and vertical dimers.
- [Papanikolaou, Charrier, Fradkin, 2014]: numerics.
Theorem

For $1 \ll z \ll J$, $\|(x, y)\|_{HL} := J|x| + e^{-\frac{3}{2}J}z^{-\frac{1}{2}}|y|$, 

- Given two vertical edges $e_v, f_v$, $\langle 1_{e_v} \rangle_v$ is independent of $e_v$ and

\[
\langle 1_{e_v} \rangle_v = \frac{1}{2}(1 + O(e^{-\frac{1}{2}J}z^{-\frac{1}{2}}))
\]

\[
\langle 1_{e_v} 1_{f_v} \rangle_v - \langle 1_{e_v} \rangle_v \langle 1_{f_v} \rangle_v = O(e^{-c \text{ dist}_{HL}(e_v, f_v)})
\]

- Given two horizontal edges $e_h, f_h$, $\langle 1_{e_h} \rangle_v$ is independent of $e_h$ and

\[
\langle 1_{e_h} \rangle_v = O(e^{-3J})
\]

\[
\langle 1_{e_h} 1_{f_v} \rangle_v - \langle 1_{e_h} \rangle_v \langle 1_{f_h} \rangle_v = O(e^{-3J-c \text{ dist}_{HL}(e_v, f_v)})
\]