A Pfaffian formula for monomer-dimer partition functions

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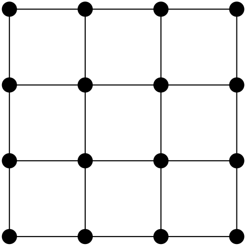
joint with **A. Giuliani** and **E.H. Lieb**

arXiv: 1510.05027 http://ian.jauslin.org/

Monomer-dimer system



- Monomer: occupies a single vertex.
- Dimer: occupies an edge and its end-vertices.
- Monomer-Dimer (MD) covering: every vertex is occupied exactly once.

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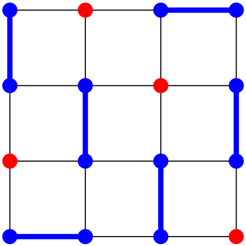

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Partition function

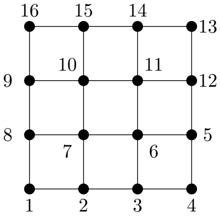
- Weights: edges d_e, vertices ℓ_v (for simplicity, assume they are ≥ 0).
- Partition function:

$$\Xi(\boldsymbol{\ell}, \mathbf{d}) = \sum_{\text{MD coverings}} \prod_{\substack{e: \\ \text{occupied} \\ \text{by dimer}}} d_e \prod_{\substack{v: \\ \text{occupied} \\ \text{by monomer}}} \ell_v.$$

Kasteleyn's theorem

- If there are **no monomers** (i.e. $\ell_v = 0$ for all v), then Ξ counts pairs of neighboring vertices:

$$\Xi(\mathbf{0}, \mathbf{d}) = \frac{1}{n!2^n} \sum_{\pi \in \mathcal{S}_{2n}} \prod_{i=1}^{n} d_{(\pi(2i-1),\pi(2i))}.$$



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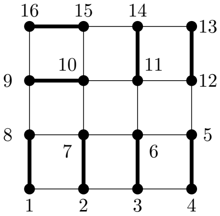
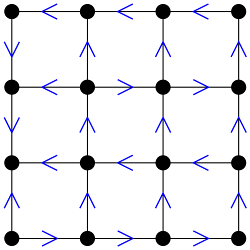


Kasteleyn's theorem

- Assume, in addition, that the graph is **planar**.
- [Kasteleyn, 1963]: Direct the graph in such a way that, for every face, moving along the boundary of the face in the counterclockwise direction, the number of arrows going against the motion is odd.



Kasteleyn's theorem

- Recall

$$\Xi(\mathbf{0}, \mathbf{d}) = \frac{1}{n!2^n} \sum_{\pi \in \mathcal{S}_{2n}} \prod_{i=1}^{n} d_{(\pi(2i-1),\pi(2i))}.$$

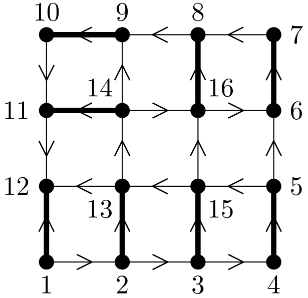
- Kasteleyn's theorem: if $s_{i,j} := +1$ if $i \to j$ and -1 if $j \to i$, then

$$\Xi(\mathbf{0}, \mathbf{d}) = \left| \frac{1}{n!2^n} \sum_{\pi \in \mathcal{S}_{2n}} (-1)^{\pi} \prod_{i=1}^{n} d_{(\pi(2i-1),\pi(2i))} s_{\pi(2i-1),\pi(2i)} \right|.$$

- In other words, $(-1)^{\pi} \prod_{i=1}^{n} s_{\pi(2i-1),\pi(2i)}$ is independent of π.

Kasteleyn's theorem

- $(-1)^{\pi} \prod_{i=1}^{n} s_{\pi(2i-1),\pi(2i)}$:



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- Recall

$$\Xi(\mathbf{0},\mathbf{d}) = \left| \frac{1}{n!2^n} \sum_{\pi \in \mathcal{S}_{2n}} (-1)^\pi \prod_{i=1}^{n} d_{(\pi(2i-1),\pi(2i))} s_{\pi(2i-1),\pi(2i)} \right|.$$

- Introducing an antisymmetric matrix a with entries $a_{i,j} := d_{(i,j)} s_{i,j}$ for $i < j$,

$$\Xi(\mathbf{0},\mathbf{d}) = |\mathrm{pf}(a)|$$

with

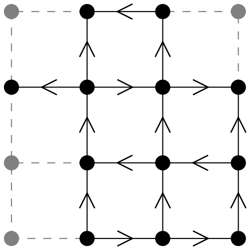
$$\mathrm{pf}(a) = \frac{1}{n!2^n} \sum_{\pi \in \mathcal{S}_{2n}} (-1)^\pi \prod_{i=1}^{n} a_{\pi(2i-1),\pi(2i)}.$$

- Determinantal relation: $\mathrm{pf(a)}^2 = \det(a)$.

Boundary monomers

- Assume the monomers are on the **boundary** of the graph.

- If the monomers are fixed, the MD partition function reduces to a pure dimer partition function on a sub-graph.



Boundary monomers

- By Kasteleyn's theorem

$$\Xi(\boldsymbol{\ell}, \mathbf{d}) = \sum_{\mathcal{M}\text{: monomers}} |\mathrm{pf}([a]_{\mathcal{M}})| \prod_{v \in \mathcal{M}} \ell_v$$

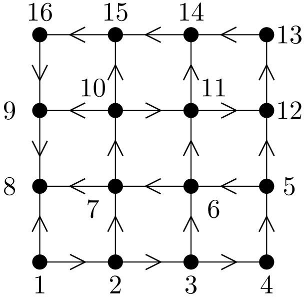
  where $[a]_{\mathcal{M}}$ is obtained from a by removing the lines and columns corresponding to vertices in $\mathcal{M}$.

- [Lieb, 1968]: if $A_{i,j} = a_{i,j} + (-1)^{i+j} \ell_i \ell_j$ for $i < j$ and $A_{j,i} = -A_{i,j}$, then

$$\mathrm{pf}(A) = \sum_{\mathcal{M}} \mathrm{pf}([a]_{\mathcal{M}}) \prod_{v \in \mathcal{M}} \ell_v.$$

Boundary monomers

- Question: is the sign of $\mathrm{pf}([a]_{\mathcal{M}})$ independent of $\mathcal{M}$?
- In general, no:



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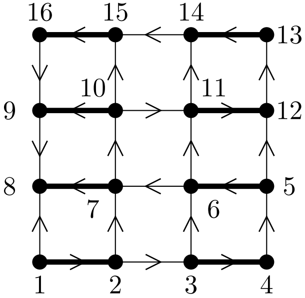

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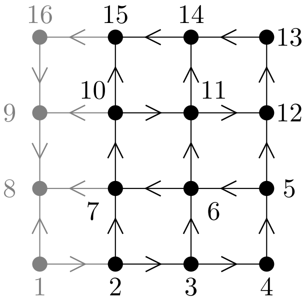

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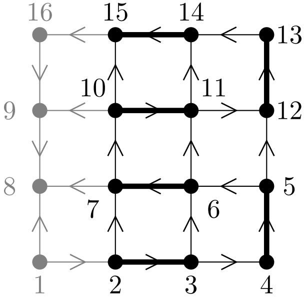
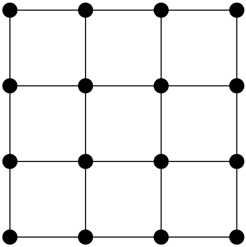


Boundary monomers

- The vertices must be labeled and the edges directed correctly.



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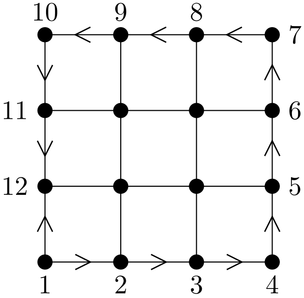

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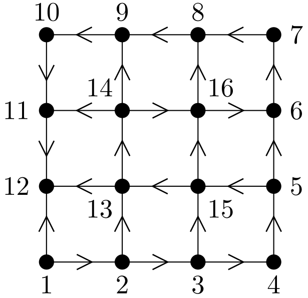


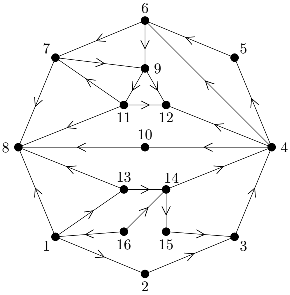
Main theorem

Every **planar** graph can be labeled and directed in such a way that the **boundary** monomer-dimer partition function is

$$\Xi(\boldsymbol{\ell}, \mathbf{d}) = \mathrm{pf}(A)$$

with $A_{i,j} = d_{(i,j)} s_{i,j} + (-1)^{i+j} \ell_i \ell_j$ for $i < j$.

Sketch of the proof



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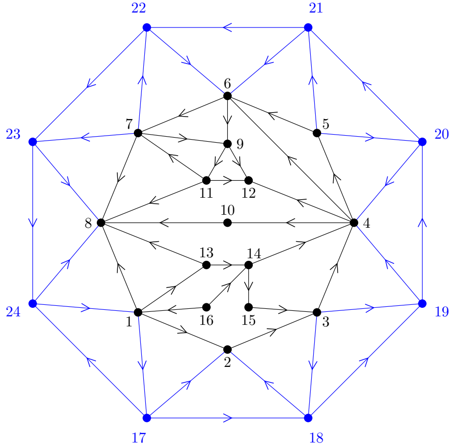

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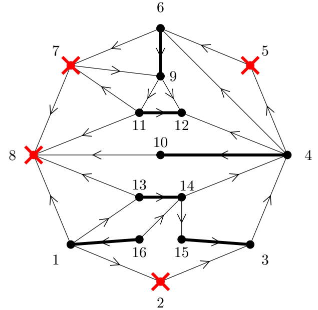

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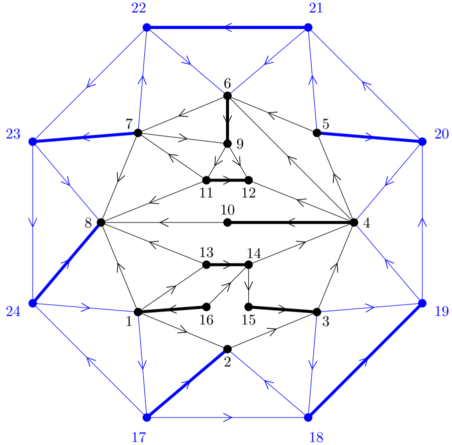

Boundary monomer correlations

- Monomer correlations at close packing:

$$M_n(i_1, \cdots, i_{2n}) = \frac{1}{\Xi(\mathbf{0}, \mathbf{d})} \left. \frac{\partial^{2n} \Xi(\boldsymbol{\ell}, \mathbf{d})}{\partial \ell_{i_1} \cdots \partial \ell_{i_{2n}}} \right|_{\boldsymbol{\ell}=\mathbf{0}} .$$

- Fermionic Wick rule:

$$M_n(i_1, \cdots, i_{2n}) = \frac{1}{n! 2^n} \sum_{\pi \in \mathcal{S}_{2n}} (-1)^\pi \prod_{j=1}^{n} M_1(i_{\pi(2j-1)}, i_{\pi(2j)}).$$