The renormalization group in the weak- and strong-coupling regimes

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Outline

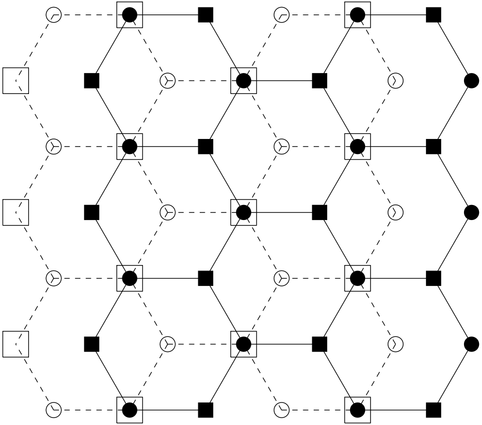
- Weak coupling: **Bilayer graphene**

- Strong coupling: **Hierarchical Kondo model**

Bilayer graphene

joint with **A. Giuliani** arXiv:1507.06024

Bilayer graphene



Bilayer graphene

- Hamiltonian
$$H = H_0 + UH_I$$
  - H_0: Kinetic term for the electrons (hoppings between atoms).
  - UH_I: Short-range screened Coulomb interaction between electrons (of *strength* U):

$$UH_I = U \sum_{x,y} v(|x-y|) \left(a_x^\dagger a_x - \frac{1}{2} \right) \left(a_y^\dagger a_y - \frac{1}{2} \right)$$

- Non-interacting case ($U = 0$): *integrable.*
- Assume $|U| \ll 1$: perturb.

Non-interacting Hamiltonian

- In Fourier space:

$$H_0 = \sum_k A_k^\dagger \hat{H}_0(k) A_k$$

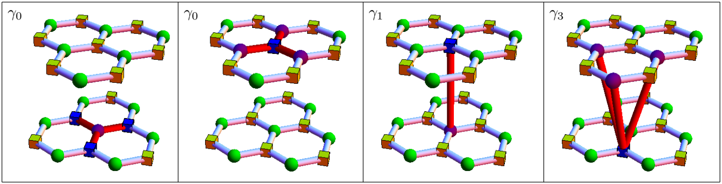
$A_k := (a_{k,1}, a_{k,2}, a_{k,3}, a_{k,4})$, where 1,2,3,4 is the *valley index*,

$$\hat{H}_0(k) = -\begin{pmatrix} 0 & \gamma_1 & 0 & \gamma_0 \Omega^*(k) \\ \gamma_1 & 0 & \gamma_0 \Omega(k) & 0 \\ 0 & \gamma_0 \Omega^*(k) & 0 & \gamma_3 \Omega(k) e^{3ik_x} \\ \gamma_0 \Omega(k) & 0 & \gamma_3 \Omega^*(k) e^{-3ik_x} & 0 \end{pmatrix}$$

with $\Omega(k_1, k_2) := 1 + 2e^{\frac{3}{2}ik_1}\cos(\frac{\sqrt{3}}{2}k_2)$.

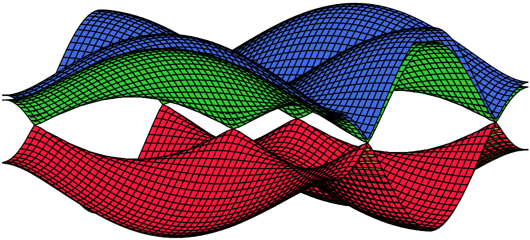
Non-interacting Hamiltonian

- Hopping strengths: $\gamma_0 = 1$, $\gamma_1 = 0.1$, $\gamma_3 = 0.034$



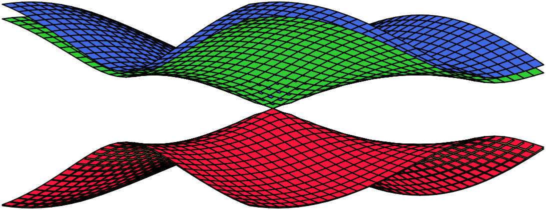
Non-interacting Hamiltonian

- Eigenvalues of $\hat{H}_0(k)$ (*bands*)



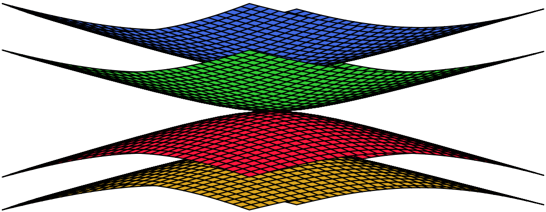
Non-interacting Hamiltonian

- For $|k| \gg \gamma_1$ (*irrelevant, superrenormalizable* regime)



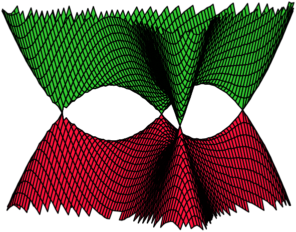
Non-interacting Hamiltonian

- For $\gamma_1^2 \ll |k| \ll \gamma_1$ (*marginal* regime)



Non-interacting Hamiltonian

- For $|k| \ll \gamma_1^2$ (*irrelevant, superrenormalizable* regime)



Main result

If $|U|$ and γ_1 and γ_3 are small enough, then the specific ground state energy

$$e_0 := -\lim_{\beta\to\infty}\lim_{|\Lambda|\to\infty}\frac{1}{\beta|\Lambda|}\log(\mathrm{Tr}(e^{-\beta H}))$$

and the two-point correlation functions are *analytic* functions of U.

- In other words, if $|U|$ is small enough, then the qualitative behavior of the system is similar to that at $U = 0$ (*weak coupling*).

Hierarchical Kondo model

joint with **G. Benfatto** and **G. Gallavotti**

doi:10.1007/s10955-015-1378-7 and doi:10.1007/s10955-015-1370-2

Kondo model

- [P. Anderson, 1960], [J. Kondo, 1964]:

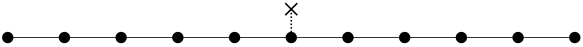
$$H = H_0 + V \quad \text{on } \mathcal{H} = \mathcal{F}_L \otimes \mathbb{C}^2$$

  - H_0: kinetic term of the *electrons*

$$H_0 := \sum_x \sum_{\alpha=\uparrow,\downarrow} a_\alpha^\dagger(x) \left(-\frac{\Delta}{2} - 1 \right) a_\alpha(x)$$

  - V: interaction with the *impurity*

$$V = -\lambda_0 \sum_{j=1,2,3} \sum_{\alpha_1,\alpha_2} a_{\alpha_1}^\dagger(0) \sigma_{\alpha_1,\alpha_2}^j a_{\alpha_2}(0) \otimes \tau^j$$



Kondo effect: magnetic susceptibility

- Non-interacting magnetic susceptibility
  - Isolated impurity: $\chi^{(0)}(0,\beta) \underset{\beta\to\infty}{\longrightarrow} \infty$
  - Chain of electrons: $\lim_{\beta\to\infty} \lim_{L\to\infty} \frac{1}{L}\chi_e(0,\beta) < \infty$.
- Anti-ferromagnetic interaction: $\lambda_0 < 0$:

$$\lim_{\beta\to\infty} \chi^{(\lambda_0)}(0,\beta) < \infty.$$

- *Strong-coupling* effect: the qualitative behavior changes as soon as $\lambda_0 \neq 0$.

Previous results

- [J. Kondo, 1964]: third order Born approximation.

- [P. Anderson, 1970], [K. Wilson, 1975]: renormalization group approach

  - Sequence of effective Hamiltonians at varying energy scales.

  - For anti-ferromagnetic interactions, the effective Hamiltonians go to a *non-trivial fixed point*.

- `Remark`: [N. Andrei, 1980]: the Kondo model (suitably linearized) is exactly solvable via Bethe Ansatz.

Current results

- Hierarchical Kondo model: idealization of the Kondo model that shares its scaling properties.

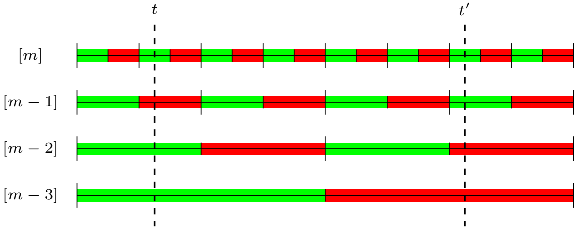
- It is *exactly solvable*: the map relating the effective theories at different scales is *explicit* (no perturbative expansions).

- For $\lambda_0 < 0$, the flow tends to a non-trivial fixed point, and $\chi^{(\lambda_0)} < \infty$ in the $\beta \to \infty$ limit (Kondo effect).

Hierarchical model

- Imaginary time: $\psi^{\pm}_{\alpha}(t) := e^{tH_0} a^{\pm}_{\alpha}(0) e^{-tH_0}$.
- Scale decomposition

$$\psi^{\pm}_{\alpha}(t) := \sum_{m \leqslant 0} \psi^{[m]\pm}_{\alpha}(t)$$

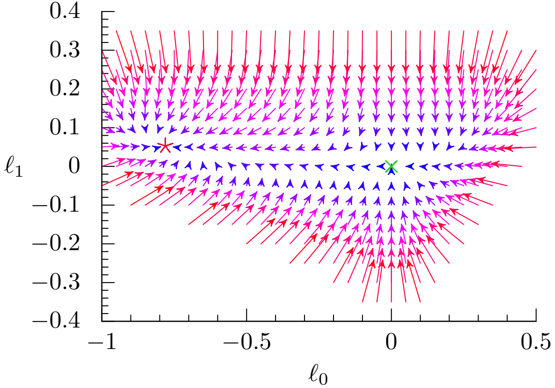


Hierarchical beta function

- Beta function (*exact*)

$$
\begin{aligned}
C^{[m]} &= 1 + \frac{3}{2}(\ell_0^{[m]})^2 + 9(\ell_1^{[m]})^2 \\
\ell_0^{[m-1]} &= \frac{1}{C^{[m]}}\Big(\ell_0^{[m]} + 3\ell_0^{[m]}\ell_1^{[m]} - (\ell_0^{[m]})^2\Big) \\
\ell_1^{[m-1]} &= \frac{1}{C^{[m]}}\Big(\frac{1}{2}\ell_1^{[m]} + \frac{1}{8}(\ell_0^{[m]})^2\Big)
\end{aligned}
$$

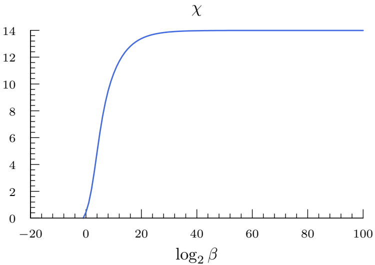
Flow



Fixed points: 0 (stable), $\boldsymbol{\ell}^*$ (marginal in ℓ_0 and stable in ℓ_1)

Kondo effect

- Fix $h = 0$.
- At 0, the susceptibility diverges as β.
- At $\boldsymbol{\ell}^*$, the susceptibility remains finite in the $\beta \to \infty$ limit.



Conclusion and perspectives

- Two examples: bilayer graphene and the hierarchical Kondo model, which can be studied via *constructive*, *rigorous* implementations of the renormalization group technique.
- Bilayer graphene: weak coupling.
- Hierarchical Kondo model: strong coupling (non-trivial fixed point).
- Extensions: full Kondo model, BCS theory, high-T_c superconductivity...