Non-perturbative renormalization group in a hierarchical Kondo model

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Kondo model

- [P. Anderson, 1960], [J. Kondo, 1964]:

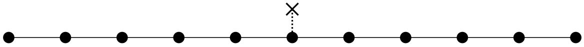
$$H = H_0 + V \quad \text{on } \mathcal{H} = \mathcal{F}_L \otimes \mathbb{C}^2$$

  - H_0: kinetic term of the *electrons*

$$H_0 := \sum_x \sum_{\alpha=\uparrow,\downarrow} c_\alpha^\dagger(x) \left(\left(-\frac{\Delta}{2} - 1 \right) c_\alpha \right)(x) \otimes \mathbb{1}$$

  - V: interaction with the *impurity*

$$V = -\lambda_0 \sum_{j=1,2,3} \sum_{\alpha_1,\alpha_2} c_{\alpha_1}^\dagger(0) \sigma^j_{\alpha_1,\alpha_2} c_{\alpha_2}(0) \otimes \tau^j$$



Kondo effect: magnetic susceptibility

- Magnetic susceptibility: response to a magnetic field h:

$$\chi(h,\beta) := \partial_h m(h,\beta).$$

  ($m(h,\beta)$: magnetization).
- Isolated impurity:

$$\chi^{(0)}(0,\beta) = \beta \underset{\beta\to\infty}{\longrightarrow} \infty$$

- Chain of electrons: Pauli paramagnetism:

$$\lim_{\beta\to\infty} \lim_{L\to\infty} \frac{1}{L}\chi_e(0,\beta) < \infty.$$

Kondo effect: magnetic susceptibility

- Turn on the interaction: $\lambda_0 \neq 0$. Impurity susceptibility $\chi^{(\lambda_0)}(h, \beta)$.
- Ferromagnetic interaction ($\lambda_0 > 0$):

$$\lim_{\beta \to \infty} \chi^{(\lambda_0)}(0, \beta) = \infty.$$

- Anti-ferromagnetic interaction ($\lambda_0 < 0$):

$$\lim_{\beta \to \infty} \chi^{(\lambda_0)}(0, \beta) < \infty.$$

- *Non-perturbative* effect: the qualitative behavior changes as soon as $\lambda_0 \neq 0$.

Previous results

- [J. Kondo, 1964]: third order Born approximation.

- [P. Anderson, 1970], [K. Wilson, 1975]: renormalization group approach

  - Sequence of effective Hamiltonians at varying energy scales.

  - For anti-ferromagnetic interactions, the effective Hamiltonians go to a *non-trivial fixed point*.

- `Remark`: [N. Andrei, 1980]: the Kondo model (suitably linearized) is exactly solvable via Bethe Ansatz (which breaks down under any perturbation of the model).

Current results

- Hierarchical Kondo model: idealization of the Kondo model that has the same scaling properties.

- It is *exactly solvable*: the map relating the effective theories at different scales is *explicit* (no perturbative expansions).

- For $\lambda_0 < 0$, the flow tends to a non-trivial fixed point, and $\chi^{(\lambda_0)} < \infty$ in the $\beta \to \infty$ limit (Kondo effect).

Field theory for the Kondo model

- By introducing an extra dimension (*imaginary time*), the partition function $Z := \mathrm{Tr}(e^{-\beta H})$ can be expressed as the *Gaussian* average over a *Grassmann* algebra:

$$Z = \mathrm{Tr}\left\langle e^{-\int_0^\beta dt\ \mathcal{V}(t)} \right\rangle$$

- Potential:

$$\mathcal{V}(t) = -\lambda_0 \sum_{\substack{j=1,2,3 \\ \alpha_1,\alpha_2,\alpha_3,\alpha_4}} \psi^+_{\alpha_1}(t)\sigma^j_{\alpha_1,\alpha_2}\psi^-_{\alpha_2}(t)\tau^j$$

with $\{\psi^\pm_\alpha(t), \psi^\pm_{\alpha'}(t')\} = 0$.

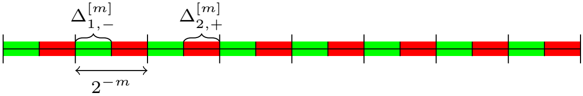
- $\langle\cdot\rangle$ is defined by its second moment $\left\langle \psi^-_{\alpha_1}(t_1)\psi^+_{\alpha_2}(t_2)\right\rangle$.

Hierarchical model

- Replace $\psi_\alpha^\pm(t)$ in $\mathcal{V}(t)$ by

$$\psi_\alpha^\pm(t) := \sum_{m\leqslant 0} \psi_\alpha^{[m]\pm}(t)$$

where $\psi_\alpha^{[m]\pm}(t)$ is *constant* over the "time" intervals $\Delta_{i,\pm}^{[m]}$:



- There are 4 fields in each $\Delta_{i,\pm}^{[m]}$.
- Moments:

$$\left\langle \psi_\alpha^{[m]-}(\Delta_{i,\mp}^{[m]})\psi_\alpha^{[m]+}(\Delta_{i,\pm}^{[m]}) \right\rangle = \pm 2^m$$

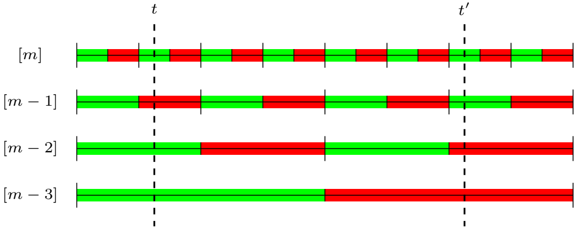
Full propagator

- Moments:

$$\left\langle \psi_\alpha^{[m]-}(\Delta_{i,\mp}^{[m]})\psi_\alpha^{[m]+}(\Delta_{i,\pm}^{[m]})\right\rangle = \pm 2^m$$

- Full propagator:

$$\left\langle \psi_\alpha^{-}(t)\psi_\alpha^{+}(t')\right\rangle = 2^{m_{t,t'}}\,\mathrm{sign}(t-t')$$



Comparison with the Kondo model

- Hierarchical model:

$$\left\langle \psi_\alpha^-(t)\psi_\alpha^+(t')\right\rangle = 2^{m_{t,t'}}\,\mathrm{sign}(t-t')$$

- For the (non-hierarchical) Kondo model:

$$\left\langle \psi_\alpha^-(t)\psi_\alpha^+(t')\right\rangle \approx \sum_m 2^m g_\psi^{[0]}(2^m(t-t'))$$

where $g_\psi^{[0]}$ is odd and decays faster than any power.

Hierarchical beta function

- Compute Z by $\mathcal{V}^{[0]}(t) := \mathcal{V}(t)$

$$e^{-\int dt\ \mathcal{V}^{[m-1]}(t)} := \left\langle e^{-\int dt\ \mathcal{V}^{[m]}(t)} \right\rangle_m$$

- Effective potential:

$$\int dt\ \mathcal{V}^{[m]}(t) = \sum_{i=1}^{2^{-m}} \mathcal{V}_{i,-}^{[m]} + \mathcal{V}_{i,+}^{[m]}$$

- Iteration

$$\left\langle e^{-\int dt\ \mathcal{V}^{[m]}(t)} \right\rangle_m = \prod_{i=1}^{2^{-m}} \left\langle e^{-(\mathcal{V}_{i,-}^{[m]}+\mathcal{V}_{i,+}^{[m]})} \right\rangle_m$$

- By anti-commutation of the fields, $e^{-(\mathcal{V}_{i,-}^{[m]}+\mathcal{V}_{i,+}^{[m]})}$ is a polynomial in the fields of order $\leqslant 8$.

Hierarchical beta function

- $\mathcal{V}^{[m]}$ is parametrized by 2 real numbers (*running coupling constants*) $\ell_0^{[m]}, \ell_1^{[m]}$:

$$\frac{e^{-\int dt\ \mathcal{V}^{[m]}(t)}}{C^{[m]}} = 1 + \frac{\ell_0^{[m]}}{2} \int dt \sum_{\substack{j=1,2,3 \\ \alpha_1,\alpha_2}} \psi_{\alpha_1}^{[\leq m]+}(t)\sigma^j_{\alpha_1,\alpha_2}\psi_{\alpha_2}^{[\leq m]-}(t)\tau^j$$

$$+\frac{\ell_1^{[m]}}{2} \int dt \left(\sum_{\substack{j=1,2,3 \\ \alpha_1,\alpha_2}} \psi_{\alpha_1}^{[\leq m]+}(t)\sigma^j_{\alpha_1,\alpha_2}\psi_{\alpha_2}^{[\leq m]-}(t) \right)^2$$

Hierarchical beta function

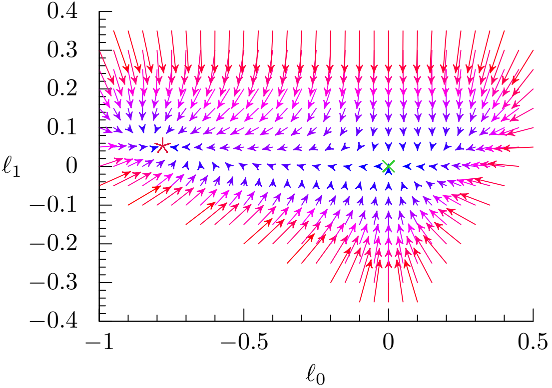
- Beta function (*exact*)

$$C^{[m]} = 1 + \frac{3}{2}(\ell_0^{[m]})^2 + 9(\ell_1^{[m]})^2$$

$$\ell_0^{[m-1]} = \frac{1}{C^{[m]}}\Big(\ell_0^{[m]} + 3\ell_0^{[m]}\ell_1^{[m]} - (\ell_0^{[m]})^2\Big)$$

$$\ell_1^{[m-1]} = \frac{1}{C^{[m]}}\Big(\frac{1}{2}\ell_1^{[m]} + \frac{1}{8}(\ell_0^{[m]})^2\Big)$$

Flow



Fixed points: 0 (stable), $\boldsymbol{\ell}^*$ (marginal in ℓ_0 and stable in ℓ_1)

Susceptibility

- Add magnetic field h on the impurity.
- New term in the potential:

$$-h \sum_{j\in\{1,2,3\}} \boldsymbol{\omega}_j \tau^j$$

- 6 running coupling constants.
- The susceptibility can be computed by deriving $C^{[m]}$ with respect to h.

Kondo effect

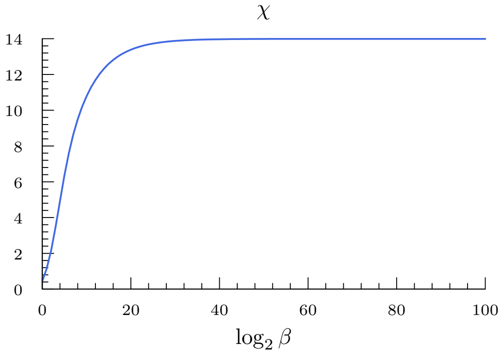
- Fix $h = 0$.

- At 0, the susceptibility diverges as β.

- At $\boldsymbol{\ell}^*$, the susceptibility remains finite in the $\beta \to \infty$ limit.

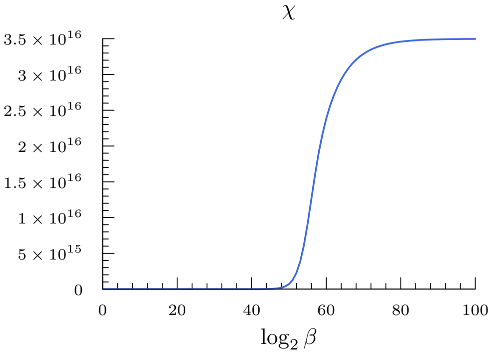
Susceptibility

- $\lambda_0 = -0.28$



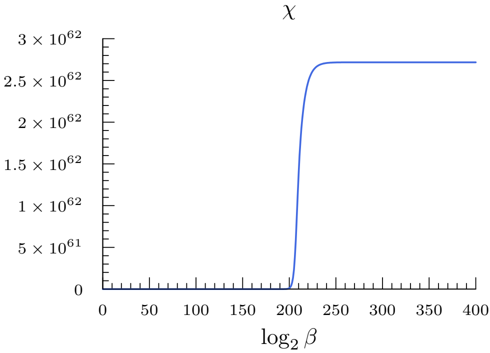
Susceptibility

- $\lambda_0 = -0.02$



Susceptibility

- $\lambda_0 = -0.005$



Open questions

- Magnetic field on the chain as well. This requires defining the hierarchical model to reflect the x-dependence of $\psi(x,t)$.
- Rigorous renormalization group analysis for the Kondo model (non-hierarchical).
- The exact solvability of the hierarchical Kondo model is merely a consequence of the fermionic nature of the system. Other fermionic hierarchical models can be studied to investigate other non-perturbative phenomena, e.g. high-T_c superconductivity.

Epilogue: `meankondo`

- The computation in the h-dependent case requires computing many Feynman diagrams (≈ 100).

- Software to perform the computation: `meankondo`.

- `meankondo` can be configured to study any fermionic hierarchical model.

`http://ian.jauslin.org/software/meankondo/`