Non-perturbative renormalization group
in a hierarchical Kondo model

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Kondo model

• [P. Anderson, 1960], [J. Kondo, 1964]:

\[ H = H_0 + V \quad \text{on } \mathcal{H} = \mathcal{F}_L \otimes \mathbb{C}^2 \]

- \( H_0 \): kinetic term of the electrons

\[ H_0 := \sum_x \sum_{\alpha=\uparrow,\downarrow} c_{\alpha}^\dagger (x) \left( \left( -\frac{\Delta}{2} - 1 \right) c_{\alpha} \right) (x) \otimes 1 \]

- \( V \): interaction with the impurity

\[ V = -\lambda_0 \sum_{j=1,2,3} \sum_{\alpha_1,\alpha_2} c_{\alpha_1}^\dagger (0) \sigma_{\alpha_1,\alpha_2}^j c_{\alpha_2} (0) \otimes \tau^j \]
Kondo effect: magnetic susceptibility

- Magnetic susceptibility: response to a magnetic field $h$:
  \[ \chi(h, \beta) := \partial_h m(h, \beta). \]
  ($m(h, \beta)$: magnetization).

- Isolated impurity:
  \[ \chi^{(0)}(0, \beta) = \beta \xrightarrow{\beta \to \infty} \infty \]

- Chain of electrons: Pauli paramagnetism:
  \[ \lim_{\beta \to \infty} \lim_{L \to \infty} \frac{1}{L} \chi_e(0, \beta) < \infty. \]
Kondo effect: magnetic susceptibility

- Turn on the interaction: $\lambda_0 \neq 0$. Impurity susceptibility $\chi^{(\lambda_0)}(h, \beta)$.

- Ferromagnetic interaction ($\lambda_0 > 0$):

$$\lim_{\beta \to \infty} \chi^{(\lambda_0)}(0, \beta) = \infty.$$ 

- Anti-ferromagnetic interaction ($\lambda_0 < 0$):

$$\lim_{\beta \to \infty} \chi^{(\lambda_0)}(0, \beta) < \infty.$$ 

- *Non-perturbative* effect: the qualitative behavior changes as soon as $\lambda_0 \neq 0$. 


Previous results

• [J. Kondo, 1964]: third order Born approximation.

• [P. Anderson, 1970], [K. Wilson, 1975]: renormalization group approach
  ▶ Sequence of effective Hamiltonians at varying energy scales.
  ▶ For anti-ferromagnetic interactions, the effective Hamiltonians go to a non-trivial fixed point.

• Remark: [N. Andrei, 1980]: the Kondo model (suitably linearized) is exactly solvable via Bethe Ansatz (which breaks down under any perturbation of the model).
Current results

• Hierarchical Kondo model: idealization of the Kondo model that has the same scaling properties.

• It is exactly solvable: the map relating the effective theories at different scales is explicit (no perturbative expansions).

• For $\lambda_0 < 0$, the flow tends to a non-trivial fixed point, and $\chi^{(\lambda_0)} < \infty$ in the $\beta \to \infty$ limit (Kondo effect).
Field theory for the Kondo model

- By introducing an extra dimension (*imaginary time*), the partition function $Z := \text{Tr}(e^{-\beta H})$ can be expressed as the *Gaussian* average over a *Grassmann* algebra:

$$Z = \text{Tr} \left< e^{-\int_0^\beta dt \mathcal{V}(t)} \right>$$

- Potential:

$$\mathcal{V}(t) = -\lambda_0 \sum_{j=1,2,3} \psi^+_{\alpha_1}(t) \sigma^j_{\alpha_1,\alpha_2} \psi^-_{\alpha_2}(t) \tau^j$$

with $\{\psi^\pm_{\alpha}(t), \psi^\pm_{\alpha'}(t')\} = 0$.

- $\langle \cdot \rangle$ is defined by its second moment $\langle \psi^-_{\alpha_1}(t_1) \psi^+_{\alpha_2}(t_2) \rangle$. 
Hierarchical model

• Replace $\psi_{\alpha}^\pm(t)$ in $V(t)$ by

$$
\psi_{\alpha}^\pm(t) := \sum_{m \leq 0} \psi_{\alpha}^{[m]}^\pm(t)
$$

where $\psi_{\alpha}^{[m]}^\pm(t)$ is constant over the “time” intervals $\Delta^{[m]}_{i,\pm}$:

• There are 4 fields in each $\Delta^{[m]}_{i,\pm}$.
• Moments:

$$
\left\langle \psi_{\alpha}^{[m]} - (\Delta^{[m]}_{i,\mp})\psi_{\alpha}^{[m]} + (\Delta^{[m]}_{i,\pm}) \right\rangle = \pm 2^m
$$
• Moments:

\[ \langle \psi_{\alpha}^-[m] - (\Delta_{i,\mp}^{[m]}) \psi_{\alpha}^+[m] + (\Delta_{i,\pm}^{[m]}) \rangle = \pm 2^m \]

• Full propagator:

\[ \langle \psi_{\alpha}^- (t) \psi_{\alpha}^+ (t') \rangle = 2^{m_{t,t'}} \text{sign}(t - t') \]
Comparison with the Kondo model

• Hierarchical model:

\[
\langle \psi_\alpha^-(t) \psi_\alpha^+(t') \rangle = 2^{m_{t,t'}} \text{sign}(t - t')
\]

• For the (non-hierarchical) Kondo model:

\[
\langle \psi_\alpha^-(t) \psi_\alpha^+(t') \rangle \approx \sum_m 2^m g_\psi^{[0]}(2^m (t - t'))
\]

where \( g_\psi^{[0]} \) is odd and decays faster than any power.
Hierarchical beta function

- Compute $Z$ by $\mathcal{V}^{[0]}(t) := \mathcal{V}(t)$

\[
e^{-\int dt \ \mathcal{V}^{[m-1]}(t)} := \left\langle e^{-\int dt \ \mathcal{V}^{[m]}(t)} \right\rangle_m
\]

- Effective potential:

\[
\int dt \ \mathcal{V}^{[m]}(t) = \sum_{i=1}^{2^{-m}} \mathcal{V}_{i,-}^{[m]} + \mathcal{V}_{i,+}^{[m]}
\]

- Iteration

\[
\left\langle e^{-\int dt \ \mathcal{V}^{[m]}(t)} \right\rangle_m = \prod_{i=1}^{2^{-m}} \left\langle e^{-(\mathcal{V}_{i,-}^{[m]} + \mathcal{V}_{i,+}^{[m]})} \right\rangle_m
\]

- By anti-commutation of the fields, $e^{-(\mathcal{V}_{i,-}^{[m]} + \mathcal{V}_{i,+}^{[m]})}$ is a polynomial in the fields of order $\leq 8$. 

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Hierarchical beta function

- $\mathcal{V}^{[m]}$ is parametrized by 2 real numbers (running coupling constants) $\ell^{[m]}_0, \ell^{[m]}_1$:

$$
e^{-\int dt \frac{\mathcal{V}^{[m]}(t)}{C^{[m]}}} = 1 + \frac{\ell^{[m]}_0}{2} \int dt \sum_{j=1,2,3}^{\alpha_1,\alpha_2} \psi_{\alpha_1}^{[\leq m]} + (t) \sigma_{\alpha_1,\alpha_2}^j \psi_{\alpha_2}^{[\leq m]} - (t) \tau^j$$

$$+ \frac{\ell^{[m]}_1}{2} \int dt \left( \sum_{j=1,2,3}^{\alpha_1,\alpha_2} \psi_{\alpha_1}^{[\leq m]} + (t) \sigma_{\alpha_1,\alpha_2}^j \psi_{\alpha_2}^{[\leq m]} - (t) \right)^2$$
Hierarchical beta function

- Beta function (exact)

\[
C^{[m]} = 1 + \frac{3}{2} (\ell_0^{[m]})^2 + 9 (\ell_1^{[m]})^2
\]

\[
\ell_0^{[m-1]} = \frac{1}{C^{[m]}} \left( \ell_0^{[m]} + 3 \ell_0^{[m]} \ell_1^{[m]} - (\ell_0^{[m]})^2 \right)
\]

\[
\ell_1^{[m-1]} = \frac{1}{C^{[m]}} \left( \frac{1}{2} \ell_1^{[m]} + \frac{1}{8} (\ell_0^{[m]})^2 \right)
\]
Fixed points: 0 (stable), $\ell^*$ (marginal in $\ell_0$ and stable in $\ell_1$)
Susceptibility

• Add magnetic field $h$ on the impurity.

• New term in the potential:

$$-h \sum_{j \in \{1,2,3\}} \omega_j \tau^j$$

• 6 running coupling constants.

• The susceptibility can be computed by deriving $C^m$ with respect to $h$. 
Kondo effect

- Fix $h = 0$.

- At $0$, the susceptibility diverges as $\beta$.

- At $\ell^*$, the susceptibility remains finite in the $\beta \to \infty$ limit.
\[ \lambda_0 = -0.28 \]
\( \lambda_0 = -0.02 \)
• $\lambda_0 = -0.005$
Open questions

• Magnetic field on the chain as well. This requires defining the hierarchical model to reflect the \( x \)-dependence of \( \psi(x, t) \).

• Rigorous renormalization group analysis for the Kondo model (non-hierarchical).

• The exact solvability of the hierarchical Kondo model is merely a consequence of the fermionic nature of the system. Other fermionic hierarchical models can be studied to investigate other non-perturbative phenomena, e.g. high-\( T_c \) superconductivity.
Epilogue: meankondo

- The computation in the $\hbar$-dependent case requires computing many Feynman diagrams ($\approx 100$).

- Software to perform the computation: meankondo.

- meankondo can be configured to study any fermionic hierarchical model.

http://ian.jauslin.org/software/meankondo/