Non-perturbative renormalization group in a hierarchical Kondo model

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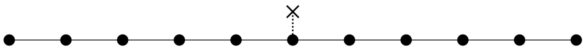
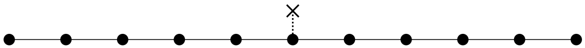
Kondo model

- s-d model: [P. Anderson, 1960] [J. Kondo, 1964]:
  - 1D chain of non-interacting spin-1/2 fermions: *electrons*.
  - lone spin-1/2 fermion: *impurity*.
  - the impurity interacts with the electron at 0.



Kondo Hamiltonian



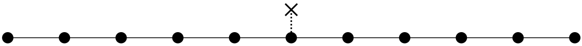
$$H = H_0 + V$$

- H_0: kinetic term of the *electrons*

$$H_0 := \sum_x \sum_{\alpha=\uparrow,\downarrow} c_\alpha^\dagger(x) \left(-\frac{\Delta}{2} - 1 \right) c_\alpha(x)$$

  - $c_\alpha(x)$: fermionic annihilation operator
  - α: spin
  - x: site

Kondo Hamiltonian



$$H = H_0 + V$$

- V: interaction with the *impurity*

$$V = -\lambda_0 \sum_{j=1,2,3} \sum_{\alpha_1,\alpha_2,\alpha_3,\alpha_4} c^\dagger_{\alpha_1}(0)\sigma^j_{\alpha_1,\alpha_2} c_{\alpha_2}(0)\, d^\dagger_{\alpha_3}\sigma^j_{\alpha_3,\alpha_4} d_{\alpha_4}$$

  - d_α: fermionic annihilation operator
  - σ^j: Pauli matrix
  - $\lambda_0 > 0$: *ferromagnetic* case
  - $\lambda_0 < 0$: *anti-ferromagnetic* case

Kondo effect: magnetic susceptibility

- Magnetic susceptibility: response to a magnetic field h:

$$\chi(h,\beta) := \partial_h m(h,\beta).$$

($m(h,\beta)$: magnetization).

- Isolated impurity:

$$\chi^{(0)}(0,\beta) = \frac{\beta}{2} \underset{\beta\to\infty}{\longrightarrow} \infty$$

- Chain of electrons: Pauli paramagnetism:

$$\lim_{\beta\to\infty} \lim_{L\to\infty} \frac{1}{L}\chi_e(0,\beta) < \infty.$$

Kondo effect: magnetic susceptibility

- Turn on the interaction: $\lambda_0 \neq 0$. Impurity susceptibility $\chi^{(\lambda_0)}(h, \beta)$.
- Ferromagnetic interaction ($\lambda_0 > 0$):

$$\lim_{\beta \to \infty} \chi^{(\lambda_0)}(0, \beta) = \infty.$$

- Anti-ferromagnetic interaction ($\lambda_0 < 0$):

$$\lim_{\beta \to \infty} \chi^{(\lambda_0)}(0, \beta) < \infty.$$

- *Non-perturbative* effect: the qualitative behavior changes as soon as $\lambda_0 \neq 0$.

Previous results

- [J. Kondo, 1964]: third order Born approximation.

- [P. Anderson, 1970], [K. Wilson, 1975]: renormalization group approach
  - Sequence of effective Hamiltonians at varying energy scales.
  - For anti-ferromagnetic interactions, the effective Hamiltonians go to a *non-trivial fixed point*.
  - Anderson: instability of the trivial fixed point (H_0).
  - Wilson: numerical diagonalization at each step, and perturbative expansions around the trivial and non-trivial fixed points.

Current results

- Hierarchical Kondo model: idealization of the Kondo model that has the same scaling properties.

- It is *exactly solvable*: the map relating the effective theories at different scales is *explicit* (no perturbative expansions).

- With $\lambda_0 < 0$, the flow tends to a non-trivial fixed point, and $\chi^{(\lambda_0)} < \infty$ in the $\beta \to \infty$ limit (Kondo effect).

- `Remark`: [N. Andrei, 1980]: the Kondo model (suitably linearized) is exactly solvable via Bethe Ansatz.

Field theory for the Kondo model

- Partition function $Z := \mathrm{Tr}(e^{-\beta H})$.
- By introducing an extra dimension (*imaginary time*), Z can be expressed as the *Gaussian* average over a *Grassmann* algebra:

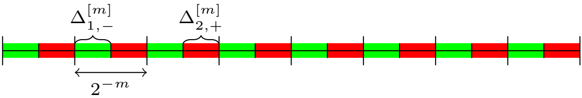
$$Z = \left\langle e^{-\int_0^\beta dt\ \mathcal{V}(t)} \right\rangle$$

where

$$\mathcal{V}(t) = -\lambda_0 \sum_{\substack{j=1,2,3 \\ \alpha_1,\alpha_2,\alpha_3,\alpha_4}} \psi^+_{\alpha_1}(0,t)\sigma^j_{\alpha_1,\alpha_2}\psi^-_{\alpha_2}(0,t)\,\varphi^+_{\alpha_3}(t)\sigma^j_{\alpha_3,\alpha_4}\varphi^-_{\alpha_4}(t)$$

with $\{\psi^\pm_\alpha(0,t), \psi^\pm_{\alpha'}(0,t')\} = 0$, $\{\varphi^\pm_\alpha(t), \varphi^\pm_{\alpha'}(t')\} = 0$.

Hierarchical fields



- For each $m < 0$, we introduce fields *on scale* m indexed by an interval:

$$\psi_\alpha^\pm(\Delta_{i,\pm}^{[m]}), \quad \varphi_\alpha^\pm(\Delta_{i,\pm}^{[m]})$$

  where

$$\Delta_{i,-}^{[m]} := [2^{-m} i, 2^{-m}(i + \tfrac{1}{2}))$$

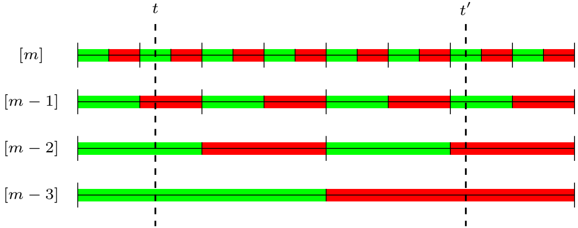
$$\Delta_{i,+}^{[m]} := [2^{-m}(i + \tfrac{1}{2}), 2^{-m}(i + 1))$$

- There are 8 fields in each $\Delta_{i,\pm}^{[m]}$.

Hierarchical fields

- Split fields over scales:

$$\psi_\alpha^\pm(t) := \sum_{m\leqslant 0} \psi_\alpha^{[m]\pm}(\Delta^{[m]}(t)), \quad \varphi_\alpha^\pm(t) := \sum_{m\leqslant 0} \varphi_\alpha^{[m]\pm}(\Delta^{[m]}(t))$$



Hierarchical propagators

- Moments:

$$\left\langle \psi_\alpha^{[m]-}(\Delta_{i,-\eta}^{[m]})\psi_\alpha^{[m]+}(\Delta_{i,\eta}^{[m]})\right\rangle = \eta 2^m$$

$$\left\langle \varphi_\alpha^{[m]-}(\Delta_{i,-\eta}^{[m]})\varphi_\alpha^{[m]+}(\Delta_{i,\eta}^{[m]})\right\rangle = \eta$$

- Full propagator:

$$\left\langle \psi_\alpha^-(t)\psi_\alpha^+(t')\right\rangle = \mathrm{sign}(t-t')2^{m_{t,t'}}, \quad \left\langle \varphi_\alpha^-(t)\varphi_\alpha^+(t')\right\rangle = \mathrm{sign}(t-t')$$

- For the (non-hierarchical) Kondo model:

$$\langle \psi_\alpha^-(0,t)\psi_\alpha^+(0,t')\rangle \approx \sum_m 2^m g_\psi^{[0]}(2^m(t-t')),$$

$$\langle \varphi_\alpha^-(t)\varphi_\alpha^+(t')\rangle \approx \sum_m g_\varphi^{[0]}(2^m(t-t'))$$

Hierarchical beta function

- Compute Z by $\mathcal{V}^{[0]}(t) := \mathcal{V}(t)$

$$e^{-\int dt\ \mathcal{V}^{[m-1]}(t)} := \left\langle e^{-\int dt\ \mathcal{V}^{[m]}(t)} \right\rangle_m$$

- Effective potential:

$$\int dt\ \mathcal{V}^{[m]}(t) = \sum_{i=1}^{2^{-m}} \mathcal{V}_{i,-}^{[m]} + \mathcal{V}_{i,+}^{[m]}$$

- Iteration

$$\left\langle e^{-\int dt\ \mathcal{V}^{[m]}(t)} \right\rangle_m = \prod_{i=1}^{2^{-m}} \left\langle e^{-(\mathcal{V}_{i,-}^{[m]} + \mathcal{V}_{i,+}^{[m]})} \right\rangle_m$$

- By anti-commutation of the fields, $e^{-\mathcal{V}_{i,\pm}^{[m]}}$ is a polynomial in the fields of order $\leqslant 8$.

Hierarchical beta function

- The computation of the beta function reduces to computing the average of a degree-16 polynomial.

- 4 running coupling constants $\ell_0, \cdots, \ell_3$:

$$e^{-\int dt\ \mathcal{V}^{[m]}(t)} = 1 + \sum_{i,\eta} \sum_{n=0}^{3} \ell_n^{[m]} O_n^{[\leqslant m]}(\Delta_{i,\eta})$$

Hierarchical beta function

- Beta function (*exact*)

$$
\begin{aligned}
C^{[m]} &= 1 + 3\ell_0^2 + 9\ell_1^2 + 9\ell_2^2 + 324\ell_3^2 \\
\ell_0^{[m-1]} &= \frac{1}{C}\Big(\ell_0 + 18\ell_0\ell_3 + 3\ell_0\ell_2 + 3\ell_0\ell_1 - 2\ell_0^2\Big) \\
\ell_1^{[m-1]} &= \frac{1}{C}\Big(\frac{1}{2}\ell_1 + 9\ell_2\ell_3 + \frac{1}{4}\ell_0^2\Big) \\
\ell_2^{[m-1]} &= \frac{1}{C}\Big(2\ell_2 + 36\ell_1\ell_3 + \ell_0^2\Big) \\
\ell_3^{[m-1]} &= \frac{1}{C}\Big(\frac{1}{2}\ell_3 + \frac{1}{4}\ell_1\ell_2 + \frac{1}{24}\ell_0^2\Big).
\end{aligned}
$$

Hierarchical beta function

- Beta function (*exact*)

$$C^{[m]} = 1 + 3\ell_0^2 + 9\ell_1^2 + 9\ell_2^2 + 324\ell_3^2$$

$$\ell_0^{[m-1]} = \frac{1}{C}\left(\ell_0 + 18\ell_0\ell_3 + 3\ell_0\ell_2 + 3\ell_0\ell_1 - 2\ell_0^2\right)$$

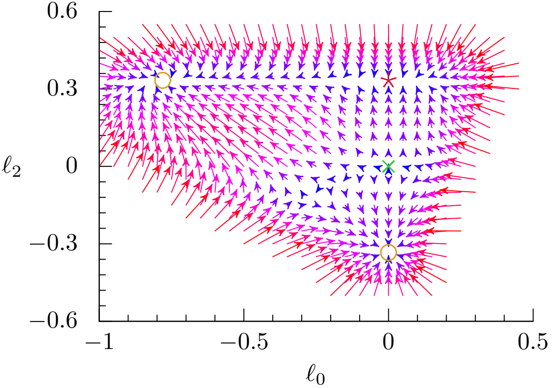
$$\ell_1^{[m-1]} = \frac{1}{C}\left(\frac{1}{2}\ell_1 + 9\ell_2\ell_3 + \frac{1}{4}\ell_0^2\right)$$

$$\ell_2^{[m-1]} = \frac{1}{C}\left(2\ell_2 + 36\ell_1\ell_3 + \ell_0^2\right)$$

$$\ell_3^{[m-1]} = \frac{1}{C}\left(\frac{1}{2}\ell_3 + \frac{1}{4}\ell_1\ell_2 + \frac{1}{24}\ell_0^2\right).$$

relevant, marginal, irrelevant

Flow



Fixed points: $\boldsymbol{\ell}^{(0)}$, $\boldsymbol{\ell}^{(+)}$, $\boldsymbol{\ell}^{*}$, $\boldsymbol{\ell}^{(-)}$.

Fixed points

- $\boldsymbol{\ell}^{(0)}$: unstable.
- $\boldsymbol{\ell}^{(+)}$: ferromagnetic ($\lambda_0 > 0$).
- $\boldsymbol{\ell}^{*}$: anti-ferromagnetic ($\lambda_0 < 0$).

Susceptibility

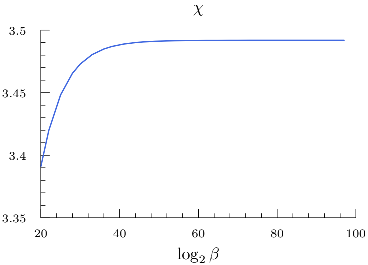
- Add magnetic field h on the impurity.
- New term in the potential:

$$-h \sum_{j \in \{1,2,3\}} \boldsymbol{\omega}_j \int dt \sum_{\alpha,\alpha'} \varphi^+_\alpha(t) \sigma^j_{\alpha,\alpha'} \varphi^-_{\alpha'}(t).$$

- 9 running coupling constants.
- The susceptibility can be computed by deriving $C^{[m]}$ with respect to h.

Kondo effect

- Fix $h = 0$.
- At $\boldsymbol{\ell}^{(+)}$, the susceptibility diverges as β.
- At $\boldsymbol{\ell}^{*}$, the susceptibility remains finite in the $\beta \to \infty$ limit.



Open questions

- Magnetic field on the chain as well. This requires defining the hierarchical model to reflect the x-dependence of $\psi(x, t)$.
- Rigorous renormalization group analysis for the Kondo model (non-hierarchical).
- The exact solvability of the hierarchical Kondo model is merely a consequence of the fermionic nature of the system. Other fermionic hierarchical models can be studied to investigate other non-perturbative phenomena, e.g. high-T_c superconductivity.

Epilogue: `meankondo`

- The computation in the h-dependent case requires computing 100 Feynman diagrams.
- By adding the field on the entire chain (open problem), this number increases to 1089.
- Software to perform the computation: `meankondo`.
- `meankondo` can be configured to study any fermionic hierarchical model.

http://ian.jauslin.org/software/meankondo/