Renormalization group approach to half-filled bilayer graphene

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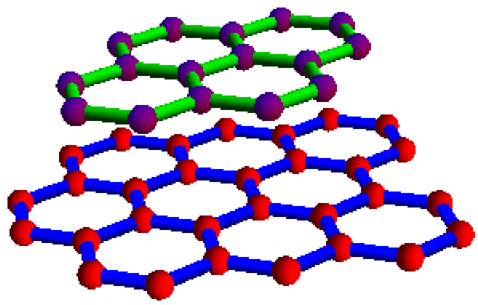
Etude du graphène bi-couche par la technique du groupe de renormalisation



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Abstract

Graphene is a recently discovered material that has remarkable properties with many expected technological applications. In this report we study two Hubbard models of bilayer graphene using a renormalization group technique developed by G. Benfatto and G. Gallavotti. We find that while one of the models behaves like the single-layer case, the other exhibits divergences in the flow of the running coupling constants. This is made apparent by setting up a scale decomposition and computing the β-function at second order, and proving its flow is unstable.

Résumé

Le graphène est un matériau découvert récemment qui a des propriétés remarquables et beaucoup de potentielles applications technologiques. On étudiera dans ce rapport deux modèles de Hubbard de graphène bi-couche en utilisant une technique de groupe de renormalisation développé par G. Benfatto et G. Gallavotti. Il s'avère que l'un de ces modèles se comporte comme le graphène mono-couche, alors que l'autre donne lieu à des divergences dans le flot des constantes de couplage. Ceci est rendu apparent en mettant en place une décomposition en échelles et en calculant la fonction-β au deuxième ordre, et en démontrant que son flot est instable.

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Introduction

Graphene is a recently discovered material that has remarkable properties with many expected technological applications. It is made of a single layer of carbon atoms in a hexagonal grid, so it is extremely thin and flexible, yet it is mechanically resistant. It also has a high conductivity, so one can imagine making flexible screens by mixing graphene with a thin sheet of plastic. A.K. Geim and K.S. Novoselov were awarded the Nobel prize for physics in 2010 for isolating a sheet of graphene from graphite and characterizing its conduction properties [KN04].

It is interesting to study why these properties arise in graphene [Wal47, Sem84, JG94, IH09]. A. Giuliani and V. Mastropietro proved the existence of the thermodynamic limit for the ground state energy and the free energy using an interacting Hubbard model [AG10], and computed correlations [AG10, AG] as well as proved that the conductivity does not depend on the details of the model [AG11]. In this report, we will investigate the question of the existence of the thermodynamic limit for the free energy of a Hubbard model of bilayer graphene which consists of two stacked layers of graphene. The final goal is to understand the difference in the conductivity properties between single and multiple layers of graphene. To do this we investigate the perturbation theory for the free energy, and study the possible sources of divergences which we re-sum using a finite number of *running coupling constants*. After the re-summation, we check whether the theory is finite by computing the β-function. This computation was done in [Vaf10, OV10] with a somewhat different method from the one explained in this report. Essentially, we only use [Vaf10, OV10] to compare the final results.

The model we will consider is defined by a Hamiltonian that is the sum of a *free* term that accounts for the movement or *hopping* of the electrons from one atom to another and an *interaction* term that accounts for the interaction between electrons. In the free term we will consider two cases: the first in which there are three types of hopping: an intra-layer one with amplitude $\gamma_0 = 3.2$ and two inter-layer ones with amplitudes $\gamma_1 = 0.4$ and $\gamma_3 = 0.3$ [MD02]; and another where the third hopping type will be neglected i.e. $\gamma_3 = 0$. The first model portrays graphene more accurately, but the second is sufficient for "large" energies (see below for a more detailed explanation). Note that the first case resembles strongly the single-layer case, at least in the small coupling regime, but the second one exhibits fundamental differences, so we will focus more attention on it. Furthermore one can imagine that the value of the constants could be changed by modifying other parameters in the experiments, e.g. the inter-layer spacing. It is therefore not unreasonable to consider a model with $\gamma_3 = 0$ and $\gamma_1 \neq 0$. Moreover the analysis of the $\gamma_3 = 0$ case may provide some insight beyond the small interaction regime of the $\gamma_3 \neq 0$ model (see [Mas, Vaf10]).

If we neglect the interaction term, the free energy can be explicitly computed since the free Hamiltonian can be diagonalized and the four spectral bands can be explicitly calculated. This can however not be done when we consider the interacting Hamiltonian, so we will treat the interaction as a perturbation and use a technique referred to as *functional renormalization group*, which was first used for fermionic systems by G. Benfatto and G. Gallavotti in [GB90]. The idea is to write the free energy formally and investigate its convergence by re-summing certain terms from which divergences could a priori appear. These a priori possible divergences come from the presence of small denominators in the propagators, located at the *Fermi points*, i.e. the momenta for which the bands are equal to zero. The zeros of the dispersion relation are linear in the single-layer and in the $\gamma_3 \neq 0$ cases; they are quadratic in the $\gamma_3 = 0$ case. There are eight such singularities for $\gamma_3 \neq 0$ and two for $\gamma_3 = 0$.

To set up the technique, it is convenient to express the free energy using Grassmann integrals which can be expressed as a sum over Feynman graphs. To isolate the possible sources of divergences, we decompose momentum space into *scales*, where the momenta at large positive scales are large, and momenta at small negative scales are close to the singularities of the small denominators. The so called *addition principle* of Grassmann integrals allows us to write the free energy by integrating scale after scale. Each integration leads to new values for the parameters of the effective interaction. In that way, in order to prove the existence of the thermodynamic limit, it is sufficient to find bounds on the effective interaction at each scale and prove that they converge as the scale goes to $-\infty$. We isolate the possible divergences by expressing the effective interaction at each scale using a sum over Gallavotti-Nicolò trees which describe how the vertices of each Feynman graph can be grouped together into clusters at each scale, therefore enabling us to avoid overlapping divergences (see below). Using this method, we discriminate the terms in the effective interaction that can give rise to divergences and which can not. The latter are treated as corrections and are called *irrelevant*, the former are either *relevant* or *marginal*. However, to check whether the effective interaction is bounded, we must explicitly compute the scale dependence of the relevant and marginal terms, and check the bound. We call the coefficients in front of each of these terms the *running coupling constants* (r.c.c.). The function that gives the r.c.c.'s at scale h as a function of the r.c.c.'s at scale $h+1$ is called the *β-function*.

The difference between single-layer and bilayer graphene arises at small scales, so the integration of the large scales will not be developed in this report.

The previously described method shows that for small scales, the possible sources of divergences are the quadratic terms in the Grassmann fields in the $\gamma_3 \neq 0$ case and quadratric and quartic terms in the $\gamma_3 = 0$ case. This is where the fundamental difference between these two cases appears. Dealing with only quadratic terms in the $\gamma_3 \neq 0$ case can be done in the same way as in the single-layer case ([AG10, Giu10]). But in the case where we also have to worry about the quartic terms, we have to compute the β-function for them. Therefore the computations are quite different from the one done for the single-layer model. Therefore we will investigate the $\gamma_3 = 0$ case in more detail and compute the β-function for the four-field terms to second order.

Before we start the computation it is important to notice that since the singularities only exist in two of the four bands, the integration, which is done a priori over a four-component spinor, can be executed instead over a two-component spinor. To prove this, we change variables to the eigenbasis of the free Hamiltonian and explicitly perform the integration at first order in the amplitude of the interaction U over the eigenvectors of the two bands that are non-singular.

Our computation of the second order β-function shows (in agreement with [Vaf10, OV10]) that the flow of the r.c.c.'s is unstable. This means that if their initial value is different from 0, then as h goes to $-\infty$ they diverge. Therefore γ_3 is essential to perform a perturbative expansion of the free energy. Setting $\gamma_3 = 0$ can provide information on the dominant quantum instabilities of the system at low temperatures.

1. The model

The lattice

Bilayer graphene is made of two layers of graphene stacked in the following way:

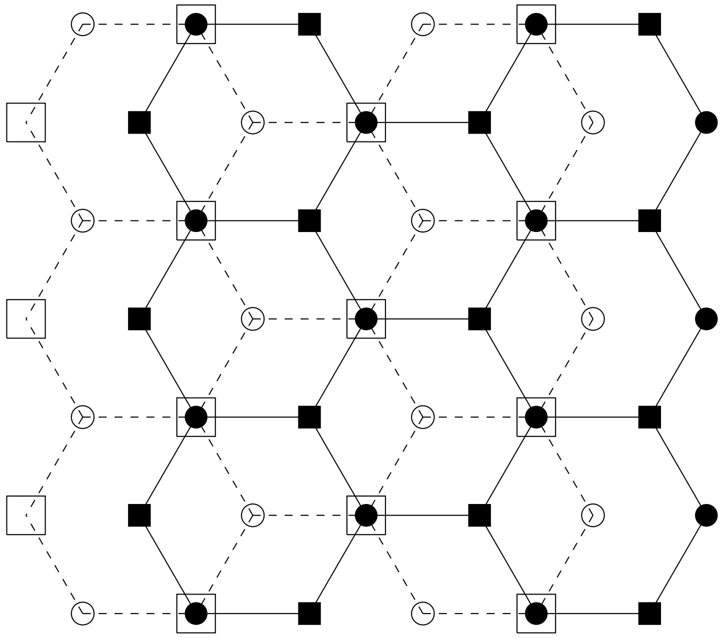


fig 1.1: ● and ■ represent atoms on the lower layer and ○ and □ represent atoms on the upper layer. Full lines join nearest neighbors within the lower layer and dashed lines join nearest neighbors within the upper layer.

We can see that the crystal can be constructed by repeating an elementary cell at every integer combination of

$$\vec{l}_1 := \left(\frac{3}{2}, \frac{\sqrt{3}}{2}, 0\right), \ \vec{l}_2 := \left(\frac{3}{2}, -\frac{\sqrt{3}}{2}, 0\right) \tag{1.1}$$

where we have chosen the length unit to be equal to the distance between two nearest neighbors in a plane. The elementary cell consists of four atoms at the following coordinates

$$(-1, 0, c); \ (0, 0, 0); \ (0, 0, c); \ (1, 0, 0)$$

given relatively to the center of the cell. c is the spacing between layers; it can be measured experimentally, and has a value of approximately 2.

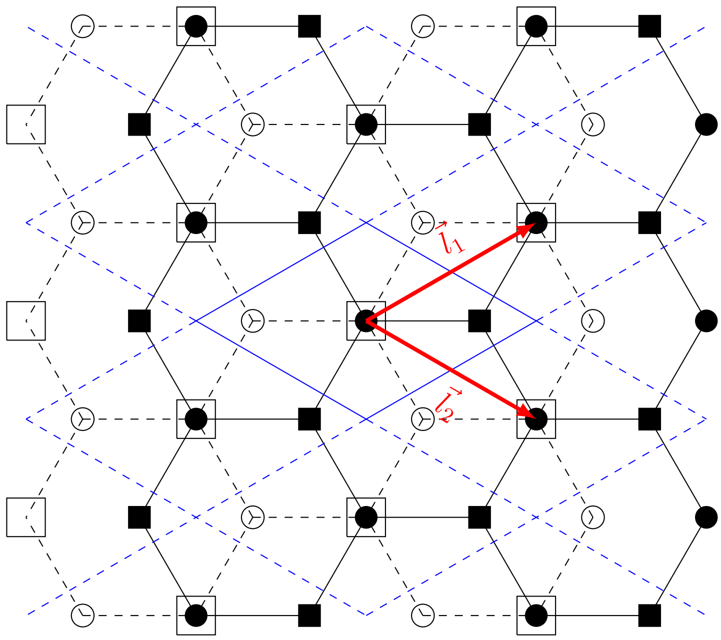


fig 1.2: decomposition of the crystal in elementary cells, represented by blue rhombi.

In each elementary cell, we call the atom ● of type a, ■ of type b, ○ of type $\tilde{a}$ and □ of type $\tilde{b}$.

We then define the lattice

$$\Lambda := \left\{ n_1\vec{l}_1 + n_2\vec{l}_2,\ (n_1, n_2) \in [|0, L-1|]^2 \right\} \tag{1.2}$$

where $[|a,b|] = \{n \in \mathbb{Z},\ a \leqslant n \leqslant b\}$. L is a strictly positive integer that determines the size of the crystal, that we will eventually send to ∞. In our model, we shall consider periodic boundary conditions, so if $\vec{x} \in \Lambda$, we identify $\vec{x}$ with $\vec{x} + m_1 L\vec{l}_1 + m_2 L\vec{l}_2$ for all $(m_1, m_2) \in \mathbb{Z}^2$.

We introduce the nearest neighbor vectors:

$$\begin{aligned} &\vec{\delta}_1^0 := (1,0,0), \quad \vec{\delta}_2^0 := \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right), \quad \vec{\delta}_3^0 := \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right) \\ &\vec{\delta}^1 := (0,0,c). \end{aligned} \tag{1.3}$$

The Hamiltonian

We want a model for the dynamics of the electrons in the lattice. To accomplish this, we postulate a Hamiltonian in second quantized form, which means that the Hamiltonian is an operator on $\mathbb{C}^{8|\Lambda|}$ constructed using *fermionic annihilation* and *creation* operators (where $|\Lambda| = L^2$ is the total number of cells).

We define for $\vec{x} \in \Lambda$, $a_{\vec{x}}$, $b_{\vec{x}+\vec{\delta}_1^0}$, $\tilde{a}_{\vec{x}+\vec{\delta}^1-\vec{\delta}_1^0}$ and $\tilde{b}_{\vec{x}+\vec{\delta}^1}$ the annihilation operators around the atoms of type a, b, $\tilde{a}$ and $\tilde{b}$ respectively. The corresponding creation operators are their adjoint operators.

We write the Hamiltonian as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \tag{1.4}$$

where $\mathcal{H}_0$ is called the *free Hamiltonian* and $\mathcal{H}_I$ is the *interaction*. We postulate the form of the free Hamiltonian using the *tight binding approximation*:

$$\begin{aligned}\mathcal{H}_0 := -\gamma_0 \sum_{\substack{\vec{x}\in\Lambda\\ j=1,2,3}} \left(a^\dagger_{\vec{x}} b_{\vec{x}+\vec{\delta}_j^0} + b^\dagger_{\vec{x}+\vec{\delta}_j^0} a_{\vec{x}} \right) - \gamma_0 \sum_{\substack{\vec{x}\in\Lambda\\ j=1,2,3}} \left(\tilde{a}^\dagger_{\vec{x}+\vec{\delta}^1-\vec{\delta}_j^0} \tilde{b}_{\vec{x}+\vec{\delta}^1} + \tilde{b}^\dagger_{\vec{x}+\vec{\delta}^1} \tilde{a}_{\vec{x}+\vec{\delta}^1-\vec{\delta}_j^0} \right) \\ -\gamma_1 \sum_{\vec{x}\in\Lambda} \left(a^\dagger_{\vec{x}} \tilde{b}_{\vec{x}+\vec{\delta}^1} + \tilde{b}^\dagger_{\vec{x}+\vec{\delta}^1} a_{\vec{x}} \right) - \gamma_3 \sum_{\substack{\vec{x}\in\Lambda\\ j=1,2,3}} \left(b^\dagger_{\vec{x}+\vec{\delta}_1^0} \tilde{a}_{\vec{x}+\vec{\delta}_1^0+\vec{\delta}^1+\vec{\delta}_j^0} + \tilde{a}^\dagger_{\vec{x}+\vec{\delta}_1^0+\vec{\delta}^1+\vec{\delta}_j^0} b_{\vec{x}+\vec{\delta}_1^0} \right)\end{aligned} \tag{1.5}$$

The interpretation of (1.5) is that $a^\dagger_{\vec{x}} b_{\vec{x}+\vec{\delta}_j^0}$ describes the *hopping* of an electron between two sites: an electron on site $\vec{x} + \vec{\delta}_j^0$ is destroyed and one is created on site $\vec{x}$, which corresponds to a movement of an electron from one site to the next. The γ's correspond to three different types of hopping that are represented in the following figure:

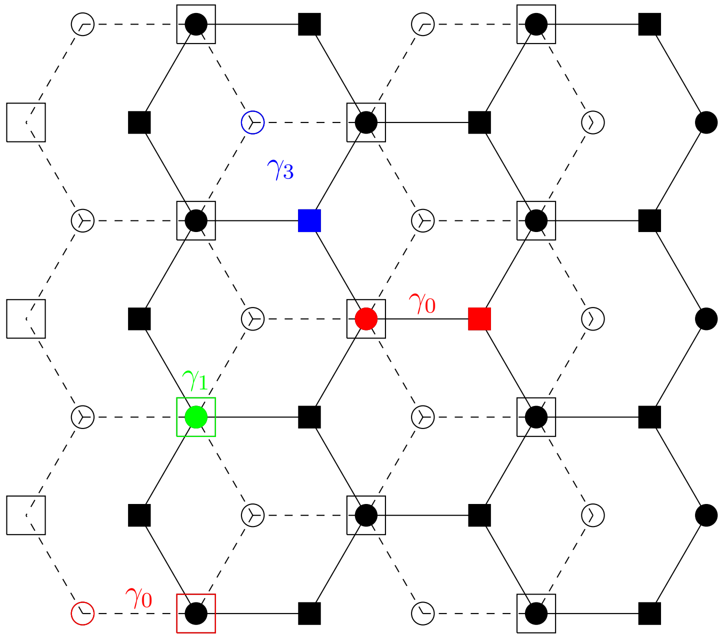


fig 1.3: the different types of hopping. The intra-layer hopping γ_0 is in red: ● ↔ ■ and ○ ↔ □ ; the vertical inter-layer hopping γ_1 is in green: ● ↔ □ ; the other inter-layer hopping γ_3 is in blue: ○ ↔ ■ .

We define the number operators: $n^\alpha_{\vec z} := \alpha^\dagger_{\vec z}\alpha_{\vec z}$ for $\alpha \in \{a, b, \tilde a, \tilde b\}$ and $\vec z \in \mathbb{R}^3$, and postulate the form of the interaction to be of an extended *Hubbard* form:

$$\mathcal{H}_I := U \sum_{(\vec x,\vec y)\in\Lambda^2} \sum_{(\alpha,\alpha')\in\{a,b,\tilde a,\tilde b\}^2} v(\vec x + \vec\delta_\alpha - \vec y - \vec\delta_{\alpha'}) \left(n^\alpha_{\vec x+\vec\delta_\alpha} - \frac{1}{2}\right)\left(n^{\alpha'}_{\vec y+\vec\delta_{\alpha'}} - \frac{1}{2}\right) \tag{1.6}$$

where $\vec\delta_a := \vec 0$, $\vec\delta_b := \vec\delta^0_1$, $\vec\delta_{\tilde a} := \vec\delta^1 - \vec\delta^0_1$, $\vec\delta_{\tilde b} := \vec\delta^1$ and v is a Fourier transformable function that only depends on $|\vec x - \vec y|$ (for example $v(\vec x) = e^{-|\vec x|^2}$). Since we took periodic boundary conditions on Λ, $|\vec x - \vec y|$ is the norm on the torus.

The interpretation of (1.6) is:

$Uv(\vec x + \vec\delta_\alpha - \vec y - \vec\delta_{\alpha'}) n^\alpha_{\vec x+\vec\delta_\alpha} n^{\alpha'}_{\vec y+\vec\delta_{\alpha'}}$ represents the repulsive interaction between an electron at $\vec x + \vec\delta_\alpha$ and another at $\vec y + \vec\delta_{\alpha'}$, which corresponds to a screened Coulomb interaction. Notice that in order for the interaction to be repulsive, Uv must be positive, however the model also makes sense for negative Uv provided $|Uv|$ is small enough.

$-U\left(\sum_{\vec{y}} v(\vec{x}+\vec{\delta}_\alpha-\vec{y}-\vec{\delta}_{\alpha'})\right)\frac{1}{2}n^\alpha_{\vec{x}+\vec{\delta}_\alpha}$ acts as a chemical potential term. The presence of the term $\dfrac{1}{2}$ corresponds to *half-filled* bands (since it ensures particle-hole symmetry).

2. The free theory

The first step is to understand the *free* system, i.e. the model for $U=0$. We compute the eigenvalues and eigenvectors of the free Hamiltonian.

Band structure

In order to diagonalize $\mathcal{H}_0$, we go into Fourier space (using the dual lattice $\hat{\Lambda}$) and find the four bands (see appendix A1):

$$\mathrm{spec}(\mathcal{H}_0)=\left\{\pm\Omega^+(\vec{k}),\ \pm\Omega^-(\vec{k}),\quad \vec{k}\in\hat{\Lambda}\right\} \tag{2.1}$$

where

$$\begin{aligned}
&\Omega^\epsilon(\vec{k}):=\frac{1}{\sqrt{2}}\sqrt{\zeta(\vec{k})+\epsilon\sqrt{\xi(\vec{k})}},\quad \epsilon\in\{-1,+1\}\\
&\text{with}\quad \zeta(\vec{k}):=2|\Omega_0(\vec{k})|^2+|\Omega_1(\vec{k})|^2+|\Omega_3(\vec{k})|^2=\gamma_1^2+(2\gamma_0^2+\gamma_3^2)\,|\chi(\vec{k})|^2\\
&\qquad\quad \xi(\vec{k}):=\left(|\Omega_1(\vec{k})|^2-|\Omega_3(\vec{k})|^2\right)^2+4\left|\Omega_0(\vec{k})\Omega_3(\vec{k})+\Omega_0^*(\vec{k})\Omega_1(\vec{k})\right|^2\\
&\qquad\qquad\quad =\left(\gamma_1^2-\gamma_3^2|\chi(\vec{k})|^2\right)^2+4\left|\gamma_0\gamma_3 e^{i3\vec{k}\vec{\delta}_1^0}(\chi(\vec{k}))^2+\gamma_1\gamma_0\chi^*(\vec{k})\right|^2
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
\text{and}\quad &\Omega_0(\vec{k}):=\gamma_0\sum_{j=1,2,3}e^{i\vec{k}(\vec{\delta}_j^0-\vec{\delta}_1^0)}:=\gamma_0\chi(\vec{k})=\gamma_0\left(1+2e^{-i\frac{3}{2}k_x}\cos\left(\frac{\sqrt{3}}{2}k_y\right)\right)\\
&\Omega_1(\vec{k}):=\gamma_1\\
&\Omega_3(\vec{k}):=\gamma_3\sum_{j=1,2,3}e^{i\vec{k}(2\vec{\delta}_1^0+\vec{\delta}_j^0)}=e^{3ik_x}\gamma_3\chi(\vec{k})
\end{aligned}$$

As we will see later, it is important to determine for which $\vec{k}$'s the eigenvalues of $\mathcal{H}_0$ are equal to 0. Such momenta will be called *Fermi points*.
Computations show that if $G:=\dfrac{\gamma_1\gamma_3}{\gamma_0^2}\in(0,3)$ there are eight Fermi points in the first Brillouin zone:

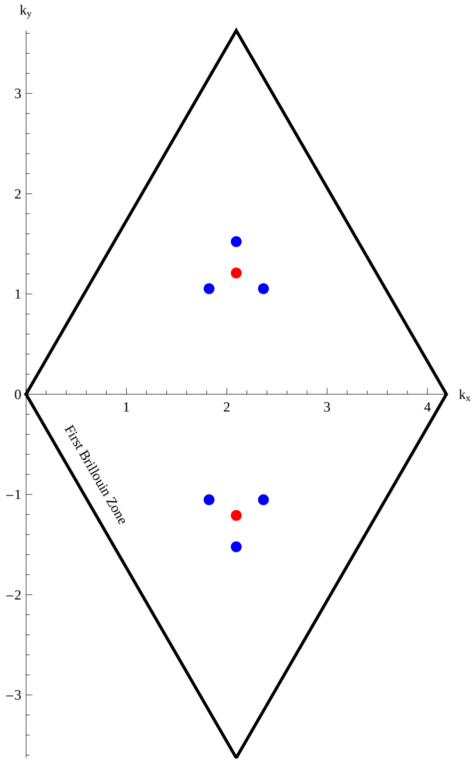


fig 2.1: the eight Fermi points for $G = 0.5$.

As G goes to 0 the blue points get closer to the red points. At $G = 0$ there are only two Fermi points which are drawn in red. The expression of the Fermi points is given in Appendix A2.

It is also important to know what the order of the singularities is. For $G \in (0, 3)$ they are linear.

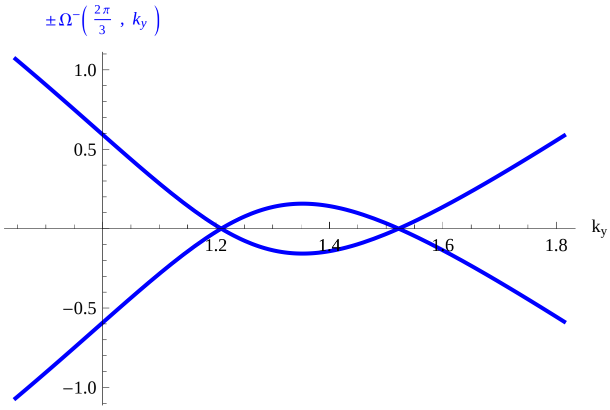


fig 2.2: plot of the vanishing band along the vertical axis for $G = 1$. The left cusp corresponds to a red Fermi point and the right one to the blue point directly above it.

However, with the values mentioned in the introduction, we have $G = 0.012$ and the Fermi points are too close to be distinguished:

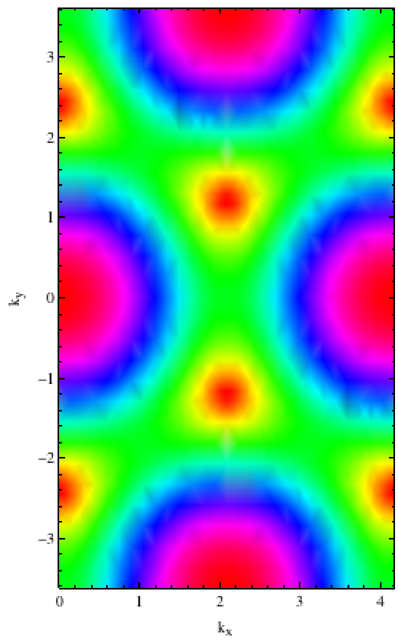


fig 2.3: plot of Ω^{-}. with the experimental values for the γ, the eight Fermi points are in two clusters.

Therefore, as long as the detectors used in experiments don't resolve very small energies, we can consider the system with $\gamma_3 = 0$. In that case, there are only two Fermi points:

$$\vec{p}_F^{\,\omega} := \left(\frac{2\pi}{3}, \omega\frac{2\pi}{3\sqrt{3}}\right), \quad \omega \in \{-1, +1\}. \tag{2.3}$$

As for the order of the singularities (see appendix A3),

$$\Omega^-(\vec{k}-\vec{p}_F^{\,\omega}) = \frac{9\gamma_0^2}{4\gamma_1}|\vec{k}-\vec{p}_F^{\,\omega}|^2 + O(\vec{k}-\vec{p}_F^{\,\omega})^3 \tag{2.4}$$

where $O(\vec{r}^{\;3}) := O(r_1^3) + O(r_1^2 r_2) + O(r_1 r_2^2) + O(r_2^3)$.
So the singularity at the Fermi points is quadratic.

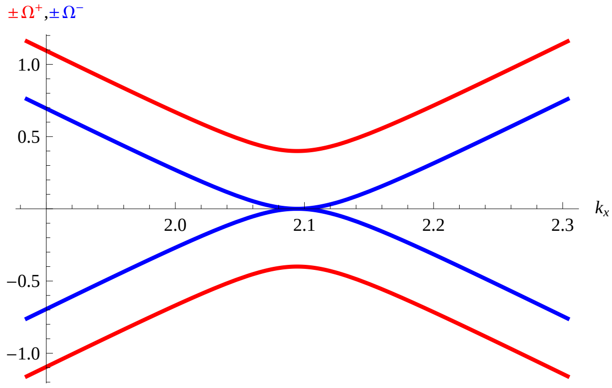


fig 2.4: band structure for $\gamma_0 = 3$, $\gamma_1 = 0.4$ and $\gamma_3 = 0$ along $k_y = \frac{2\pi}{3\sqrt{3}}$ close to $\vec{p}_F^{\,+}$.

Using the expression for the bands, one can diagonalize the free Hamiltonian, and compute the so called *non-interacting specific free energy* in the thermodynamic limit at zero temperature

$$\begin{aligned} f_0 &:= -\lim_{\beta,|\Lambda|\to\infty} \frac{1}{\beta|\Lambda|} \log\left(\mathrm{tr}\left(e^{-\beta\mathcal{H}_0}\right)\right) \\ &= -\lim_{\beta,|\Lambda|\to\infty} \frac{1}{\beta|\Lambda|} \sum_{\vec{k}\in\hat{\Lambda}} \sum_{(\omega,\epsilon)\in\{-1,+1\}^2} \log\left(1+e^{\epsilon\beta\Omega^\omega(\vec{k})}\right). \end{aligned} \tag{2.5}$$

Free propagator

The quantity of interest is the *specific free energy* for the interacting system in the thermodynamic limit at zero temperature

$$f := -\lim_{\beta,|\Lambda|\to\infty} \frac{1}{\beta|\Lambda|} \log\left(\mathrm{tr}\left(e^{-\beta\mathcal{H}}\right)\right). \tag{2.6}$$

One can prove that f can be expressed using f_0 and a *Gaussian Grassmann integral* involving an *imaginary time component*. Grassmann integrals are defined in appendix A4. Considering an imaginary time component means that all vectors have an additional component formally called *time* and we redefine the annihilation and creation operators on $\Lambda_\beta := \left(-\frac{\beta}{2}, \frac{\beta}{2}\right] \times \Lambda$: if $\mathbf{x} = (x_0, \vec{x}) \in \Lambda_\beta$, we write $\Psi^-_{\vec x} = \left(a_{\vec x}, b_{\vec x+\delta_1^0}, \tilde a_{\vec x+\delta^1-\delta_1^0}, \tilde b_{\vec x+\delta^1}\right)$ and $\Psi^+_{\vec x} = \left(a^\dagger_{\vec x}, b^\dagger_{\vec x+\delta_1^0}, \tilde a^\dagger_{\vec x+\delta^1-\delta_1^0}, \tilde b^\dagger_{\vec x+\delta^1}\right)$ and we define for $(\epsilon,\rho)\in\{-,+\}\times[|1,4|]$ and $\mathbf{x}=(x_0,\vec x)\in\Lambda_\beta$

$$\Psi^\epsilon_{\mathbf{x},\rho} := e^{x_0\mathcal{H}_0}\Psi^\epsilon_{\vec x,\rho}e^{-x_0\mathcal{H}_0}. \tag{2.7}$$

One can then prove that f can be expressed using a Grassmann integral, however since we only defined finite-dimensional Grassmann algebras, we must introduce a cutoff M, that we will send to ∞:

$$f = -\lim_{\beta,|\Lambda|\to\infty}\lim_{M\to\infty}\frac{1}{\beta|\Lambda|}\log\left(\int P_M(d\psi)e^{-\mathcal{V}(\psi)}\right)+f_0, \tag{2.8}$$

where

$$P_M(d\psi) := \frac{1}{\mathcal{N}}\prod_{\mathbf{k}\in\hat\Lambda^M_\beta}\prod_{\rho\in[|1,4|]} d\psi^-_{\mathbf{k},\rho}d\psi^+_{\mathbf{k},\rho}\prod_{\mathbf{k}\in\hat\Lambda^M_\beta}\prod_{(\rho,\rho')\in[|1,4|]^2} e^{-\frac{1}{\beta|\Lambda|}\psi^+_{\mathbf{k},\rho}\hat g^{-1}(\mathbf{k})_{\rho,\rho'}\psi^-_{\mathbf{k},\rho'}},$$

$$\hat g(\mathbf{k})_{\rho,\rho'} := \sum_{(\epsilon,\omega)\in\{-,+\}^2}\frac{\chi_0(2^{-M}|k_0|)}{-ik_0+\epsilon\Omega^\omega(\vec k)}(\eta^{\epsilon,\omega}_{\vec k})_\rho(\eta^{\epsilon,\omega}_{\vec k})^*_{\rho'}, \tag{2.9}$$

$$\text{defined for } \mathbf{k}\in\hat\Lambda^M_\beta := \left\{\mathbf{k}\in\frac{2\pi}{\beta}\left(\mathbb{Z}-\frac12\right)\times\hat\Lambda, \chi_0(2^{-M}|k_0|)>0\right\},$$

where χ_0 is a differentiable cutoff function such that

$$\chi_0(x) = \begin{matrix} 1 \text{ if } x\leqslant\frac13 \\ 0 \text{ if } x\geqslant\frac23\end{matrix}, \tag{2.10}$$

$$\eta^{\epsilon,\omega}_{\vec k} = \frac{1}{\sqrt{2((\Omega^\omega(\vec k))^2+|\Omega_0(\vec k)|^2)}}\begin{pmatrix}\Omega^\omega(\vec k)\\ \epsilon\Omega_0(\vec k)\\ \Omega_0^*(\vec k)\\ \epsilon\Omega^\omega(\vec k)\end{pmatrix}\text{ (the }\alpha\text{ are the eigenvectors of }\mathcal{H}_0), \tag{2.11}$$

$$\mathcal{V}(\psi) = U\int_{-\frac\beta2}^{\frac\beta2}dx_0dy_0\sum_{(\vec x,\vec y)\in\Lambda^2}\sum_{(\rho,\rho')\in[|1,4|]^2} v(\vec x+\vec\delta_\rho-\vec y-\vec\delta_{\rho'})\cdot \\ \cdot\psi^+_{\mathbf{x}+\vec\delta_\rho,\rho}\psi^-_{\mathbf{x}+\vec\delta_\rho,\rho}\psi^+_{\mathbf{x}+\vec\delta_{\rho'},\rho'}\psi^-_{\mathbf{x}+\vec\delta_{\rho'},\rho'}. \tag{2.12}$$

$\hat g$ is called the *free propagator*.

3. Perturbation theory and renormalization group

Renormalization strategy

Equation (2.8) gives a formal expression for the specific free energy f. We would like to see if it is finite, which is not trivial because of the factor $\dfrac{1}{-ik_0+\epsilon\Omega^-(\vec{k})}$ in (2.9) that diverges as $\mathbf{k}$ goes to $\mathbf{p}_F^{\pm}$. The technique we will use to see if f is bounded for U sufficiently small is referred to as *functional renormalization* and was first used for fermionic systems by G. Gallavotti and G. Benfatto in [GB90]. It allows us to identify the terms that could produce divergences, and to see if they can then be re-summed.

The details of the strategy are given in numerous articles, among which [Giu10, GB95, GG01, Sha91, Pol]. We will here give an outline of the technique.

First we write the integral in (2.8) as a series of *truncated expectations*

$$-\log\left(\int P_M(d\psi)e^{-\mathcal{V}(\psi)}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\mathcal{E}^T\left(\mathcal{V}(\psi);n\right) \tag{3.1}$$

where $\mathcal{E}^T\left(\mathcal{V}(\psi);n\right):=\dfrac{\partial^n}{\partial\lambda^n}\log\left(\int P_M(d\psi)e^{\lambda\mathcal{V}(\psi)}\right)\Big|_{\lambda=0}$. The point is that $\mathcal{E}^T$ can be expressed as a sum over connected Feynman graphs, so the problem is reduced to computing a series of sums of connected Feynman graphs. Notice that the series in (3.1) actually has only a finite number of terms because of the finiteness of β, $|\Lambda|$ and M, which implies that the number of Grassmann variables is finite.

Then we decompose momentum space into scales. To illustrate this, we consider a case where there is only one Fermi point at $\mathbf{0}$: we write

$$\hat{g}(\mathbf{k})=\sum_{h=-\infty}^{\infty}\left(\chi_0(2^{h+1}|\mathbf{k}|)-\chi_0(2^{h-1}|\mathbf{k}|)\right)\hat{g}(\mathbf{k})=:\sum_{h=-\infty}^{\infty}\hat{g}^{(h)}(\mathbf{k}). \tag{3.2}$$

Notice the sum over h is actually finite since $|\mathbf{k}|$ has an upper bound because of the cutoff M, and a lower bound due to the finiteness of β coming from the $+\frac{1}{2}$ in $\frac{2\pi}{\beta}\left(\mathbb{Z}+\frac{1}{2}\right)$.

We use a property of Grassmann integrals called the *addition principle* which allows us to write formally

$$\int P_M(d\psi)e^{-\mathcal{V}(\psi)}=\int P_{-\infty}(d\psi^{(-\infty)})\cdots\int P_{+\infty}(d\psi^{(+\infty)})e^{-\mathcal{V}(\psi^{(-\infty)}+\cdots+\psi^{(+\infty)})} \tag{3.3}$$

where P_h has the same expression as P_M exept that g is replaced by $g^{(h)}$. We inductively define

$$\mathcal{V}^{(h)}(\psi^{(\leqslant h)})+\beta|\Lambda|\bar{e}_{h+1}:=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\mathcal{E}^T_{h+1}\left(\mathcal{V}^{(h+1)}(\psi^{(\leqslant h+1)});n\right) \tag{3.4}$$

where $\bar{e}_h$ is the constant part of the right hand side of (3.4), $\psi^{(\leqslant h)} := \sum_{k=-\infty}^{h} \psi^{(k)}$ and $\mathcal{E}_h^T$ is defined like $\mathcal{E}^T$ but with P_h instead of P_M . It is sufficient to prove that $\frac{1}{\beta|\Lambda|}\mathcal{V}^{(h)}$ and $\bar{e}_h$ are bounded in h independently of β, $|\Lambda|$ and M to show that the thermodynamic limit for f exists. $\mathcal{V}^{(h)}$ is called the *effective interaction at scale* h. To search for such a bound we use the *tree structure* in (3.4) which allows us to write

$$\mathcal{V}^{(h)}(\psi^{(\leqslant h)}) + \beta|\Lambda|\bar{e}_{h+1} = \begin{cases} \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}_{n,h,M+1}} \mathrm{Val}(\tau) \text{ if } h \geqslant 0 \\ \sum_{n=1}^{\infty} \sum_{\tau\in\mathcal{T}_{n,h,1}} \mathrm{Val}(\tau) \text{ if } h < 0 \end{cases} \tag{3.5}$$

where $\mathcal{T}_{n,h,m}$ is the set of trees with n endpoints whose root is at "height" h and whose endpoints are all at "height" m. Each node of a given tree that is not an endpoint or the root will contribute to its value by a sum of Feynman graphs multiplied by a product of $\psi^{(\leqslant h_v-1)}$ called *external fields* (where h_v is the height (or scale) of the node). To get a good bound on the value of trees, we express this sum as the determinant of a Gram matrix, which can be easily and efficiently bounded.

The advantage of considering these trees is to avoid *overlapping divergences*: they give an unambiguous order in which to perform the renormalizations.

Using this method, one proves that for $h \geqslant 0$, $\mathcal{V}^{(h)}$ can be bounded independently of M, β and $|\Lambda|$. For $h < 0$, we must look at the order of the propagator $g^{(h)}$ to find a condition on $\mathcal{V}^{(h)}$ in order for it to be bounded. This is called *power counting*. The condition is usually expressed as a lower bound A on the number of Grassmann fields that there can be in any given product in $\mathcal{V}^{(h)}$. We then call the terms in $\mathcal{V}^{(h)}$ that satisfy this condition *irrelevant terms*, the others are called either *relevant* or *marginal terms*.

We then wish to "ignore" the irrelevant terms. Therefore we define $\mathcal{L}_0$ as the linear operator that to an irrelevant term associates 0. We write $\mathcal{R}_0 = 1 - \mathcal{L}$. $\mathcal{L}_0\mathcal{V}^{(h)}$ is to be *renormalized*, in the sense that we express $\mathcal{V}^{(h)}$ using $\mathcal{R}_0\mathcal{V}^{(h)}$ and $\mathcal{L}_0\mathcal{V}^{(h+1)}$. If we can find a bound on $\mathcal{L}_0\mathcal{V}^{(h+1)}$ such that the resulting $\mathcal{V}^{(h)}$ is bounded independently of h, then the formal expansion for f in (2.8) is finite and the theory is said to be *renormalizable*.

Things are actually a bit more complicated: it turns out it is hard to express $\mathcal{L}_0\mathcal{V}^{(h)}$ as a function of $\mathcal{L}_0\mathcal{V}^{(h+1)}$, and in fact it is not necessary. It can be shown it is sufficient to consider the linear operator $\mathcal{L}$ that to a monomial of the form $W(\mathbf{k}_1,\cdots,\mathbf{k}_n)\psi^+_{\mathbf{k}_1}\psi^-_{\mathbf{k}_2}\cdots\psi^+_{\mathbf{k}_{2n-1}}\psi^-_{\mathbf{k}_{2n}}$ associates

$$\begin{cases} 0 \text{ if } n > A \\ \psi^+_{\mathbf{k}_1}\psi^-_{\mathbf{k}_2}\cdots\psi^+_{\mathbf{k}_{2n-1}}\psi^-_{\mathbf{k}_1-\mathbf{k}_2+\cdots+\mathbf{k}_{2n-1}}\sum_{j=0}^{n}\frac{1}{j!}\mathbf{k}^j\cdot\partial^j_{\mathbf{k}}W(\mathbf{0},\cdots,\mathbf{0}) \end{cases} \tag{3.6}$$

where $\mathbf{k}_j\cdot\partial^j_{\mathbf{k}} := \sum_{j_0=0}^{n}\sum_{j_1=0}^{n-j_0} k_0^{j_0}k_1^{j_1}k_2^{j-j_0-j_1}\partial^{j_0}_{k_0}\partial^{j_1}_{k_1}\partial^{j-j_0-j_1}_{k_2}$. We define $\mathcal{R} := 1-\mathcal{L}$. It can be proven that $\mathcal{R}\mathcal{V}^{(h)}$ is bounded for all the considered models.

Notice that the local part of a polynomial has only a finite number of terms. We call the coefficients in front of each of these terms in $\mathcal{L}\mathcal{V}^{(h)}$ the *running coupling constants*.

Power counting for $\gamma_3 \neq 0$

The fundamental difference between the $\gamma_3 \neq 0$ and the $\gamma_3 = 0$ cases arises in the power counting. If $\gamma_3 \neq 0$, Ω^- is linear around the eight Fermi points. In the single-layer case, the bands are also linear around two Fermi points (see [AG10, Giu10]), so we expect the power counting to be the same in the bilayer case with $\gamma_3 \neq 0$. This means that the proof of the existence of the thermodynamic limit for small enough interactions can be performed in the same way as the single layer case, though the details of the proof are cumbersome since there are many Fermi points.

Since in the $\gamma_3 = 0$ case the bands are quadratic around the singularities, the power counting is different from the single layer case, so we must study the model with $\gamma_3 = 0$.

Power counting for $\gamma_3 = 0$

As we have previously noted, the large scales can be integrated out in the same way as for single-layer graphene in [AG10, Giu10], so we concentrate on the divergences coming from $\dfrac{1}{-ik_0 \pm \Omega^-(\vec{k})}$. We have

$$\Omega^-(\vec{k}-\vec{p}^{\,\omega}_F) = \frac{9\gamma_0^2}{4\gamma_1}|\vec{k}-\vec{p}^{\,\omega}_F|^2 + O(\vec{k}-\vec{p}^{\,\omega}_F)^3 \tag{3.7}$$

so in order to perform the scale decomposition, we define

$$\|\mathbf{k}\| = \left(k_0^2 + |\vec{k}|^4\right)^{1/4} \tag{3.8}$$

and

$$g^{(h)}_\omega(\mathbf{k}) = \left(\chi_0(2^{h+1}\|\mathbf{k}\|) - \chi_0(2^{h-1}\|\mathbf{k}\|)\right)\hat{g}(\mathbf{k}-\mathbf{p}_F^\omega)$$

where χ_0 is the cutoff function defined in (2.10). We will write $f_h(\mathbf{k}) := \chi_0(2^{h+1}\|\mathbf{k}\|) - \chi_0(2^{h-1}\|\mathbf{k}\|)$. The measure of the support of f_h is "of order" 2^{4h} (by "of order" we mean that $\exists(c_1,c_2)\in\mathbb{R}^{*+2}$ such that $2^{4h}c_1 \leqslant \mathrm{Vol}(\mathrm{supp}(f_h)) \leqslant 2^{4h}c_2$; we write $\mathrm{Vol}(\mathrm{supp}(f_h)) \sim 2^{4h}$). Furthermore on the support of f_h,

$$g^{(h)}_\omega(\mathbf{k}) \sim \begin{pmatrix} 1 & 2^{-h} & 2^{-h} & 1 \\ 2^{-h} & 2^{-2h} & 2^{-2h} & 2^{-h} \\ 2^{-h} & 2^{-2h} & 2^{-2h} & 2^{-h} \\ 1 & 2^{-h} & 2^{-h} & 1 \end{pmatrix} \tag{3.9}$$

Following a method described in detail in [Giu10], we can prove that the value of the trees used to compute $\mathcal{V}^{(h)}$ can be bounded by a term proportional to

$$2^{h(2-\frac{1}{2}|P_{v_0}|)} \prod_{\substack{v \text{ not} \\ \text{endpoint}}} 2^{(h_v - h_{v'})(\frac{1}{2}|P_v|-2)}$$

where $|P_v|$ is the number of external fields at node v, v_0 is the node following the root (which is unique) and v' is the first node preceeding v that has more than one child (the bound is *proportional* since there is also a term that depends on the number of endpoints that must be small enough so as not to produce divergences, however we will not detail this here). Therefore, as long as $|P_v| > 4$, we have a bound for $\mathcal{V}^{(h)}$. This is equivalent to saying that if $\forall h < 0$, all the terms in $\mathcal{V}^{(h)}$ are a product of more than 4 Grassmann fields, the theory is finite. This can also be seen by estimating $\vec{k} \sim 2^h$ and $k_0 \sim 2^{2h}$ in (2.6).

As described in the previous section, we must define a *localization operator* $\mathcal{L}$. To do this we first write a generic interaction as

$$\begin{aligned}\mathcal{V}^{(h)}(\psi) = \frac{1}{(\beta|\Lambda|)^{2n}} \sum_{n\geq 1} \sum_{\substack{\omega_1,\cdots,\omega_{2n} \\ \rho_1,\cdots,\rho_{2n}}} \sum_{\substack{\mathbf{k}_1,\cdots,\mathbf{k}_{2n-1} \\ \mathbf{k}_i \in \hat{\Lambda}_\beta^{h,\omega_i}}} \left(\prod_{j=1}^n \psi^{(\leqslant h)+}_{\mathbf{k}_{2j-1},\rho_{2j-1},\omega_{2j-1}} \psi^{(\leqslant h)-}_{\mathbf{k}_{2j},\rho_{2j},\omega_{2j}} \right) \cdot \\ \cdot \delta\left(\sum_{j=1}^{2n} (-1)^j \left(\mathbf{k}_j + \mathbf{p}_f^{\omega_j}\right) \right) \widehat{W}^{(h)}_{2n,\underline{\rho},\underline{\omega}}(\mathbf{k}_1,\cdots,\mathbf{k}_{2n-1})\end{aligned} \tag{3.10}$$

where $\hat{\Lambda}_\beta^{h,\omega} = \{\mathbf{k} - \mathbf{p}_F^\omega,\ \mathbf{k} \in \hat{\Lambda}_\beta \text{ such that } f_h(\mathbf{k}) > 0\}$, and $\psi_{\mathbf{k},\rho,\omega}$ is such that $\psi_{\mathbf{x},\rho} = \sum_{\omega=\pm} e^{i\mathbf{p}_F^\omega\cdot\mathbf{x}}\psi_{\mathbf{x},\rho,\omega}$. The δ term comes from the conservation of momentum.

The localization operator is linear and it is such that

$$\begin{cases} \mathcal{L}\left(\prod_{j=1}^{n+3} \psi^{(\leqslant h)+}_{\mathbf{k}_{2j-1},\rho_{2j-1},\omega_{2j-1}} \psi^{(\leqslant h)-}_{\mathbf{k}_{2j},\rho_{2j},\omega_{2j}} \widehat{W}^{(h)}_{2n+6,\underline{\rho},\underline{\omega}}(\mathbf{k}_1,\cdots,\mathbf{k}_{2n+5})\right) = 0 \\ \mathcal{L}\left(\psi^{(\leqslant h)+}_{\mathbf{k},\rho_1,\omega_1}\psi^{(\leqslant h)-}_{\mathbf{k},\rho_2,\omega_2}\widehat{W}^{(h)}_{2,\underline{\rho},\underline{\omega}}(\mathbf{k})\right) = \psi^{(\leqslant h)+}_{\mathbf{k},\rho_1,\omega_1}\psi^{(\leqslant h)-}_{\mathbf{k},\rho_2,\omega_2}\left(\widehat{W}^{(h)}_{2,\underline{\rho},\underline{\omega}}(\mathbf{0}) + \mathbf{k}\cdot\partial_{\mathbf{k}}\widehat{W}^{(h)}_{2,\underline{\rho},\underline{\omega}}(\mathbf{0})\right) \\ \mathcal{L}\left(\psi^{(\leqslant h)+}_{\mathbf{k}_1,\rho_1,\omega_1}\psi^{(\leqslant h)-}_{\mathbf{k}_2,\rho_2,\omega_2}\psi^{(\leqslant h)+}_{\mathbf{k}_3,\rho_3,\omega_3}\psi^{(\leqslant h)-}_{\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3,\rho_4,\omega_4}\widehat{W}^{(h)}_{4,\underline{\rho},\underline{\omega}}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3)\right) \\ \qquad = \psi^{(\leqslant h)+}_{\mathbf{k}_1,\rho_1,\omega_1}\psi^{(\leqslant h)-}_{\mathbf{k}_2,\rho_2,\omega_2}\psi^{(\leqslant h)+}_{\mathbf{k}_3,\rho_3,\omega_3}\psi^{(\leqslant h)-}_{\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3,\rho_4,\omega_4}\widehat{W}^{(h)}_{4,\underline{\rho},\underline{\omega}}(\mathbf{0}) \end{cases} \tag{3.11}$$

We also define $\mathcal{R} := 1 - \mathcal{L}$. We can see that $\mathcal{R}\mathcal{V}^{(h)}$ is bounded since the power counting for $\mathcal{R}\mathcal{V}^{(h)}$ will have for the quartic terms an extra 2^h due to the extra $\mathbf{k}\cdot$ and an extra $2^{-(h_v-h_{v'})}$ due to the extra $\cdot\partial_{\mathbf{k}}$; and similarly for the quadratic terms.

4. The β-function computation

Integrating out the massive terms

As it has been stated earlier, there are no singularities in $\dfrac{1}{-ik_0 \pm \Omega^+(\vec{k})}$. Therefore of the four bands only two require a non-trivial renormalisation. This means that we can trivially integrate two of the four fields.
We recall that we wish to compute

$$\ln\left(\int P_M(d\psi)e^{-\mathcal{V}(\psi)}\right).$$

We change variables in the Grassmann integral to

$$\begin{pmatrix} \mu_{\mathbf{k}} \\ \bar{\mu}_{\mathbf{k}} \\ \nu_{\mathbf{k}} \\ \bar{\nu}_{\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \eta^{++}_{\vec{k}} & \eta^{-+}_{\vec{k}} & \eta^{+-}_{\vec{k}} & \eta^{--}_{\vec{k}} \end{pmatrix}^{\dagger} \begin{pmatrix} a_{\mathbf{k}} \\ b_{\mathbf{k}} \\ \tilde{a}_{\mathbf{k}} \\ \tilde{b}_{\mathbf{k}} \end{pmatrix} \tag{4.1}$$

where the η are the eigenvectors of $\mathcal{H}_0$ defined in (2.11), so (4.1) is simply the change of variables to the eigenbasis of the free Hamiltonian. μ and $\bar{\mu}$ are called the *massive* terms and are associated to the eigenvalue Ω^+ whereas ν and $\bar{\nu}$ are called the *massless* terms and are associated to the eigenvalue Ω^-. We wish to integrate out the massive terms. This is fairly simple since by definition g is diagonal in the eigenbasis of the free Hamiltonian, so $P_M(d\psi)$ can be decomposed into $P_M(d\nu)P_M(d\mu)$ and we are left with computing

$$\mathcal{V}_0(\nu,\bar{\nu}) := \ln\left(\int P_M(d\mu)e^{-\mathcal{V}(\mu,\bar{\mu},\nu,\bar{\nu})}\right) \tag{4.2}$$

If we re-express (4.2) with a series of truncated expectations as in (3.1), it is clear that at lowest order in U (first order in U), the terms in $\mathcal{V}_0$ that have a product of four massless fields are those in $\mathcal{V}$ that have four massless fields. So at first order in U, $\mathcal{V}_0$ is computed from $\mathcal{V}$ by neglecting the massive fields.

It turns out that it is even simpler: in the rest of the computation we will only be interested in the local part of $\mathcal{V}_0$, which means we are only interested in the change of variables for $\vec{k}=0$, namely

$$\begin{pmatrix} \eta^{++}_{\vec{0}} & \eta^{-+}_{\vec{0}} & \eta^{+-}_{\vec{0}} & \eta^{--}_{\vec{0}} \end{pmatrix}^\dagger = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & X^*(\vec{k}) & X(\vec{k}) & 0 \\ 0 & -X^*(\vec{k}) & X(\vec{k}) & 0 \end{pmatrix} \tag{4.3}$$

where $X(\vec{k}) := \dfrac{\chi(\vec{k})}{|\chi(\vec{k})|}$ using the χ from (2.2) (see appendix A5).

This notation is slightly abusive but it is important to notice that X does not have a limit in $\vec{0}$, hence we must keep a dependence on $\vec{k}$ in the change of variables. This is not a problem since one can prove that the local part of $\mathcal{V}_0$ after considering the change of variables in (4.3) is the same as the local part of $\mathcal{V}_0$ after considering the change of variables in (4.1) (see appendix A6).

We immediateley notice that by taking the change of variables in (4.3), the massive terms are a linear combination of a and $\tilde{b}$, and the massless terms are a linear combination of b and $\tilde{a}$. Therefore, neglecting the massive terms is simply neglecting a and $\tilde{b}$.

Just one more thing before we write $\mathcal{L}\mathcal{V}_0$: we notice that in order for the δ in (3.10) to be different from 0, all four-field terms in the effective interaction must be such that $\omega_1-\omega_2+\omega_3-\omega_4=0$. This comes from the fact that at small scales, the momenta are close to the Fermi points. Therefore if we want their sum to be equal to 0, the momenta must be “in pairs”.

After removing the massive terms and applying the localization operator to

(2.12), we find that the four-field terms in $\mathcal{V}_0$ are

$$
\begin{aligned}
&\mathcal{L}\mathcal{V}_4^{(0)}(b,\tilde{a})\\
&= U\sum_\omega \left(\int d\mathbf{y}\ v(\mathbf{y})\left(1-e^{-i(\mathbf{p}_F^+-\mathbf{p}_F^-)\cdot\mathbf{y}}\right)\right)\cdot\\
&\quad\cdot\int d\mathbf{x}\left(b^\dagger_{\mathbf{x}+\vec{\delta}_1^0,\omega}b_{\mathbf{x}+\vec{\delta}_1^0,\omega}b^\dagger_{\mathbf{x}+\vec{\delta}_1^0,-\omega}b_{\mathbf{x}+\vec{\delta}_1^0,-\omega}+\tilde{a}^\dagger_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega}\tilde{a}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega}\tilde{a}^\dagger_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega}\tilde{a}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega}\right)\\
&+U\sum_\omega 2\left(\int d\mathbf{y}\ v(\mathbf{y}+\vec{\delta}^1-2\vec{\delta}_1^0)\right)\int d\mathbf{x}\ b^\dagger_{\mathbf{x}+\vec{\delta}_1^0,\omega}b_{\mathbf{x}+\vec{\delta}_1^0,\omega}\tilde{a}^\dagger_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega}\tilde{a}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega}\\
&+U\sum_\omega 2\left(\int d\mathbf{y}\ v(\mathbf{y}+\vec{\delta}^1-2\vec{\delta}_1^0)\right)\int d\mathbf{x}\ b^\dagger_{\mathbf{x}+\vec{\delta}_1^0,\omega}b_{\mathbf{x}+\vec{\delta}_1^0,\omega}\tilde{a}^\dagger_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega}\tilde{a}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega}\\
&+U\sum_\omega 2\left(\int d\mathbf{y}\ v(\mathbf{y}+\vec{\delta}^1-2\vec{\delta}_1^0)e^{-i(\mathbf{p}_F^+-\mathbf{p}_F^-)\mathbf{y}}\right)\int d\mathbf{x}\ b^\dagger_{\mathbf{x}+\vec{\delta}_1^0,\omega}b_{\mathbf{x}+\vec{\delta}_1^0,-\omega}\tilde{a}^\dagger_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega}\tilde{a}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega}
\end{aligned}
\tag{4.4}
$$

where $\displaystyle\int d\mathbf{x} := \int_{-\beta/2}^{\beta/2} dx_0 \sum_{\vec{x}\in\Lambda}$.

Furthermore a straightforward computation (see appendix A7) shows that if $|\Lambda|$ is a multiple of 9, then

$$\int d\mathbf{y}\ v(\mathbf{y}+\vec{\delta}^1-2\vec{\delta}_1^0)e^{-i(\mathbf{p}_F^+-\mathbf{p}_F^-)\mathbf{y}} = 0. \tag{4.5}$$

This can also be seen by noticing that the term after this integral violates $\frac{2\pi}{3}$-symmetry. So we have

$$
\begin{aligned}
&\mathcal{L}\mathcal{V}_4^{(0)}(b,\tilde{a})\\
&= U_0\sum_\omega\int d\mathbf{x}\left(b^\dagger_{\mathbf{x}+\vec{\delta}_1^0,\omega}b_{\mathbf{x}+\vec{\delta}_1^0,\omega}b^\dagger_{\mathbf{x}+\vec{\delta}_1^0,-\omega}b_{\mathbf{x}+\vec{\delta}_1^0,-\omega}\right.\\
&\qquad\qquad\left.+\tilde{a}^\dagger_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega}\tilde{a}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega}\tilde{a}^\dagger_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega}\tilde{a}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega}\right)\\
&+U_1\sum_\omega\int d\mathbf{x}\ b^\dagger_{\mathbf{x}+\vec{\delta}_1^0,\omega}b_{\mathbf{x}+\vec{\delta}_1^0,\omega}\tilde{a}^\dagger_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega}\tilde{a}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega}\\
&+U_1\sum_\omega\int d\mathbf{x}\ b^\dagger_{\mathbf{x}+\vec{\delta}_1^0,\omega}b_{\mathbf{x}+\vec{\delta}_1^0,\omega}\tilde{a}^\dagger_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega}\tilde{a}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega}.
\end{aligned}
\tag{4.6}
$$

with

$$\begin{cases} U_0 = \int d\mathbf{y}\ v(\mathbf{y}) \left(1 - e^{-i(\mathbf{p}_F^+ - \mathbf{p}_F^-)\cdot\mathbf{y}}\right) \\ U_1 = \int d\mathbf{y}\ v(\mathbf{y} + \vec{\delta}^1 - 2\vec{\delta}_1^0) \end{cases} \tag{4.7}$$

Furthermore, the propagator becomes

$$\tilde{g}_\omega(\mathbf{k}) = \frac{1}{k_0^2 + |\Omega_\omega(\mathbf{k})|^2} \begin{pmatrix} ik_0 & \Omega_\omega(\mathbf{k}) \\ \Omega_\omega^*(\mathbf{k}) & ik_0 \end{pmatrix} \tag{4.8}$$

with $\Omega_\omega(\mathbf{k}) = (X(\mathbf{k} - \mathbf{p}_F^\omega))^2 \Omega^-(\mathbf{k} - \mathbf{p}_F^\omega)$.

The β-function

The computation of the β-function is based on equation (3.4) or more precisely on the local part of equation (3.4):

$$\mathcal{L}\mathcal{V}^{(h)}(\psi^{(\leqslant h)}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \mathcal{L}\mathcal{E}_{h+1}^T \left(\mathcal{L}\mathcal{V}^{(h+1)}(\psi^{(\leqslant h+1)}) + \mathcal{R}\mathcal{V}^{(h+1)}(\psi^{(\leqslant h+1)}); n\right). \tag{4.9}$$

Using (4.9) we inductively prove

$$\begin{aligned}
&\mathcal{L}\mathcal{V}_4^{(h)}(b, \tilde{a}) \\
&= \lambda_1^{(h)} \sum_\omega \int d\mathbf{x} \Big(b^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}_1^0,\omega} b^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}_1^0,\omega} b^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}_1^0,-\omega} b^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}_1^0,-\omega} \\
&\qquad\qquad + \tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega} \tilde{a}^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega} \tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega} \tilde{a}^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega} \Big) \\
&+ \lambda_2^{(h)} \sum_\omega \int d\mathbf{x}\ b^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}_1^0,\omega} b^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}_1^0,\omega} \tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega} \tilde{a}^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega} \\
&+ \lambda_3^{(h)} \sum_\omega \int d\mathbf{x}\ b^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}_1^0,\omega} b^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}_1^0,\omega} \tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega} \tilde{a}^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega} \\
&+ \lambda_4^{(h)} \sum_\omega \int d\mathbf{x} \Big(b^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}_1^0,\omega} \tilde{a}^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega} b^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}_1^0,-\omega} \tilde{a}^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega} \\
&\qquad\qquad + \tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,\omega} b^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}_1^0,\omega} \tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x}+\vec{\delta}^1-\vec{\delta}_1^0,-\omega} b^{(\leqslant h)}_{\mathbf{x}+\vec{\delta}_1^0,-\omega} \Big)
\end{aligned} \tag{4.10}$$

and find the expression of the $\lambda^{(h)}$ as a function of the $\lambda^{(h+1)}$.

The computation of the β-function is carried out as follows: the truncated expectation is multilinear, in the sense that $\mathcal{E}^T(\mathcal{V}_1+\mathcal{V}_2;n)$ should be thought of as $\mathcal{E}^T(\mathcal{V}_1+\mathcal{V}_2;\cdots;\mathcal{V}_1+\mathcal{V}_2)$ which could also be written $\langle(\mathcal{V}_1+\mathcal{V}_2)^n\rangle^T$. Therefore

$$\begin{aligned}\mathcal{L}\mathcal{E}^T_{h+1}&\left(\mathcal{L}\mathcal{V}^{(h+1)}(\psi^{(\leqslant h+1)})+\mathcal{R}\mathcal{V}^{(h+1)}(\psi^{(\leqslant h+1)});n\right)\\&=\sum_{(\mathcal{A}_1,\cdots,\mathcal{A}_n)\in\{\mathcal{L},\mathcal{R}\}^n}\mathcal{L}\mathcal{E}^T_{h+1}\left(\mathcal{A}_1\mathcal{V}^{(h+1)}(\psi^{(\leqslant h+1)});\cdots;\mathcal{A}_n\mathcal{V}^{(h+1)}(\psi^{(\leqslant h+1)})\right).\end{aligned}\tag{4.11}$$

In order for the iteration to work, we neglect the terms where at least one of the $\mathcal{A}_j$ is equal to $\mathcal{R}$. In this approximation, it is clear that to compute the β-function to second order in the r.c.c.'s, we need only consider $n=1$ and $n=2$. We then use the fact that $\psi^{(\leqslant h+1)}=\psi^{(\leqslant h)}+\psi^{(h+1)}$ to rewrite $\mathcal{L}\mathcal{V}^{(h+1)}(\psi^{(\leqslant h+1)})$ as a polynomial in $\psi^{(h+1)}$ with coefficients made with products of complex numbers and $\psi^{(\leqslant h)}$ fields. The coefficients in these polynomials come out of $\mathcal{E}^T_{h+1}$ by multi-linearity, and we are left with computing $\mathcal{E}^T_{h+1}$ on monomials in $\psi^{(h+1)}$ using Feynman graphs.

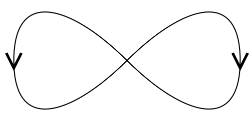
Here is an example (in momentum space):

$$\mathcal{L}\mathcal{E}^T_{h+1}\left(b^{(\leqslant h+1)\dagger}_{\mathbf{k}_1,\omega}b^{(\leqslant h+1)}_{\mathbf{k}_2,\omega}b^{(\leqslant h+1)\dagger}_{\mathbf{k}_3,-\omega}b^{(\leqslant h+1)}_{\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3,-\omega};b^{(\leqslant h+1)\dagger}_{\mathbf{k}'_1,\omega}b^{(\leqslant h+1)}_{\mathbf{k}'_2,\omega}b^{(\leqslant h+1)\dagger}_{\mathbf{k}'_3,-\omega}b^{(\leqslant h+1)}_{\mathbf{k}'_1-\mathbf{k}'_2+\mathbf{k}'_3,-\omega}\right)$$

yields a sum of terms among which

$$\begin{aligned}\mathcal{L}\mathcal{E}^T_{h+1}&\left(b^{(\leqslant h)\dagger}_{\mathbf{k}_1,\omega}b^{(h+1)}_{\mathbf{k}_2,\omega}b^{(h+1)\dagger}_{\mathbf{k}_3,-\omega}b^{(\leqslant h)}_{\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3,-\omega};b^{(h+1)\dagger}_{\mathbf{k}'_1,\omega}b^{(\leqslant h)}_{\mathbf{k}'_2,\omega}b^{(\leqslant h)\dagger}_{\mathbf{k}'_3,-\omega}b^{(h+1)}_{\mathbf{k}'_1-\mathbf{k}'_2+\mathbf{k}'_3,-\omega}\right)\\&=b^{(\leqslant h)\dagger}_{\mathbf{k}_1,\omega}b^{(\leqslant h)}_{\mathbf{k}_2,\omega}b^{(\leqslant h)\dagger}_{\mathbf{k}'_3,-\omega}b^{(\leqslant h)}_{\mathbf{k}'_1-\mathbf{k}'_2+\mathbf{k}'_3,-\omega}\mathcal{L}\mathcal{E}^T_{h+1}\left(b^{(h+1)\dagger}_{\mathbf{k}_3,\omega}b^{(h+1)}_{\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3,-\omega};b^{(h+1)\dagger}_{\mathbf{k}'_1,-\omega}b^{(h+1)}_{\mathbf{k}'_2,\omega}\right)\end{aligned}$$

which can be computed using the following Feynman graph:



whose value is

$$-\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\hat\Lambda^{h+1,\omega}_\beta}\tilde g^{(h+1)}_{-\omega}(\mathbf{k})_{1,1}\tilde g^{(h+1)}_{\omega}(\mathbf{k})_{1,1}.$$

The values of all Feynman graphs are given by

$$G^{\omega,\omega'(h+1)}_{\rho_1,\rho_2,\rho_3,\rho_4}:=\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\hat\Lambda^{h+1,\omega}_\beta}\tilde g^{(h+1)}_{\omega}(\mathbf{k})_{\rho_1,\rho_2}\tilde g^{(h+1)}_{\omega'}(\mathbf{k})_{\rho_3,\rho_4}.\tag{4.12}$$

G can be computed explicitly (see appendix A8):

$$G^{\omega,\omega'(h+1)}_{\rho_1,\rho_2,\rho_3,\rho_4}=\epsilon^{\omega,\omega'}_{\rho_1,\rho_2,\rho_3,\rho_4}m+O\left(2^h\right)\tag{4.13}$$

where $m = \frac{1}{\pi}\ln(2)\frac{\gamma_1}{9\gamma_0^2}$ and $\epsilon^{\omega,\omega'}_{\rho_1,\rho_2,\rho_3,\rho_4} \in \{-1,0,+1\}$.
We then find that up to terms of $O\left(2^h\right) + O\left(\lambda^{(h+1)3}\right)$ (see appendix A9),

$$\left\{\begin{array}{l}
\lambda_1^{(h)} = \lambda_1^{(h+1)} + 2m\lambda_1^{(h+1)}\lambda_3^{(h+1)} - 4m\lambda_1^{(h+1)}\lambda_4^{(h+1)} - m\lambda_2^{(h+1)}\lambda_3^{(h+1)} \\
\qquad\qquad\qquad -2m\lambda_4^{(h+1)}\lambda_4^{(h+1)} \\
\lambda_2^{(h)} = \lambda_2^{(h+1)} - 4m\lambda_1^{(h+1)}\lambda_3^{(h+1)} + 2m\lambda_2^{(h+1)}\lambda_3^{(h+1)} - 4m\lambda_2^{(h+1)}\lambda_4^{(h+1)} \\
\qquad\qquad\qquad +4m\lambda_4^{(h+1)}\lambda_4^{(h+1)} \\
\lambda_3^{(h)} = \lambda_3^{(h+1)} + 4m\lambda_1^{(h+1)}\lambda_1^{(h+1)} - 4m\lambda_1^{(h+1)}\lambda_2^{(h+1)} + m\lambda_2^{(h+1)}\lambda_2^{(h+1)} \\
\qquad\qquad\qquad +2m\lambda_3^{(h+1)}\lambda_3^{(h+1)} + 4m\lambda_4^{(h+1)}\lambda_4^{(h+1)} \\
\lambda_4^{(h)} = \lambda_4^{(h+1)} - 2m\lambda_1^{(h+1)}\lambda_1^{(h+1)} - 4m\lambda_1^{(h+1)}\lambda_4^{(h+1)} - \frac{1}{2}m\lambda_2^{(h+1)}\lambda_2^{(h+1)} \\
\qquad\qquad\qquad +2m\lambda_2^{(h+1)}\lambda_4^{(h+1)} + 2m\lambda_3^{(h+1)}\lambda_4^{(h+1)} - 4m\lambda_4^{(h+1)}\lambda_4^{(h+1)}.
\end{array}\right. \tag{4.14}$$

(4.14) is the *second order β-function.*

5. Discussion

In [Vaf10, OV10], O. Vafek found the same β-function as in (4.14) using a different method, although this is not immediately evident since the computations in this article were done in a different basis. In order to compare the two equations, we here give the change of variables from the λ's that have been used in this report to the g's that are used in [Vaf10, OV10]:

$$\left\{\begin{array}{l}
g_{A_1} = \frac{1}{2}\left(\lambda_2 + \lambda_3\right) + \lambda_4 \\
g_{A_2} = \lambda_1 - \frac{1}{2}\lambda_3 \\
g_{D_2} = -2\lambda_1 + \frac{1}{2}\left(\lambda_2 + \lambda_3\right) + \lambda_4 \\
g_\gamma = -\lambda_1 + \frac{1}{2}\lambda_3 + \lambda_4.
\end{array}\right. \tag{5.1}$$

A straightforward computation shows that $g_\gamma^{(h)}$ is proportional to $g_\gamma^{(h+1)}$ so if g_γ is initially equal to 0, we can forget this running coupling constant and only consider g_{A_1}, g_{A_2} and g_{D_2}. This is what is done in [OV10]. However this is only true if we consider a model where the range of the interaction is much larger than the inter-atom spacing. This means that if $v(\vec{x}) = e^{-\frac{|\vec{x}|^2}{\xi^2}}$ and the inter-atom spacing is a, then we treat the case where $\frac{a}{\xi}$ goes to 0. In this approximation, (4.6) is simplified: all the $\int v$ are equal. Therefore g_γ is initially equal to 0 so there are essentially three r.c.c.'s. However this is not true in general.

Numerical computations show that the flow of $\lambda^{(h)}$ is unstable, in the sense that $\lambda^{(h)}$ diverges as h goes to $-\infty$. This means that using the method we have used so far, we cannot prove the existence of the thermodynamic limit for the model with $\gamma_3 = 0$.

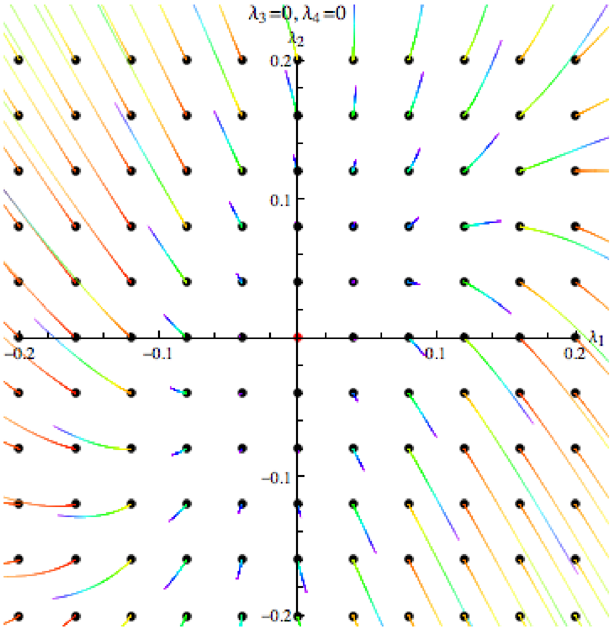


fig 5.1: flow of the β-function projected on the plane $\lambda_1 = \lambda_4 = 0$. The black discs represent the initial values from which the flow was computed. The color represents the "speed" of the flow, small velocities are in red and high velocities are in purple. We can see that the flow is stable only if the initial condition is $(0, 0, 0, 0)$ (notice that the lines that seem to stop actually diverge along one of the directions that were projected out).

Essentially this means that if we want to set up a perturbation theory, we must consider a model with $\gamma_3 \neq 0$, at least for momenta that are sufficiently close to the Fermi points. So instead of taking $\gamma_3 = 0$, we could consider a very small γ_3. Such a model would behave like the one with $\gamma_3 = 0$ for all scales larger than some cutoff scale h_c, and like the $\gamma_3 \neq 0$ model for scales less than h_c (which converges for small enough U as we have previously stated because of its similarity to the single-layer model). We may then investigate if there exists a set of initial values for the λ such that after the flow of the β-function up to h_c, the r.c.c.'s are still in the region where the $\gamma_3 \neq 0$ model converges.

In the $\gamma_3 = 0$ model, we can also determine in which direction the β-function diverges and investigate the properties of the system in that direction. This is what is done in [Vaf10].

Conclusion

In this report we have considered two different Hubbard models of bilayer graphene, one with $\gamma_3 \neq 0$ and one with $\gamma_3 = 0$. Using functional renormalization, one can prove by analogy with single-layer graphene, that the free energy has a thermodynamic limit for a small enough interaction in the $\gamma_3 \neq 0$ case. This is done by setting up a perturbation theory in the interaction. In the $\gamma_3 = 0$ case however, we have seen that there are divergences that prevent us from treating the interaction perturbatively.

The difference between these two models resides in the order of the singularities in the band structure: in the $\gamma_3 \neq 0$ case the bands are linear around the singularity whereas in the $\gamma_3 = 0$ they are quadratic. This implies that if $\gamma_3 \neq 0$ the only terms in the interaction that can possibly produce divergences are those with two fields. Such terms can be dealt with as in the single-layer case, and it turns out that there are no divergences. If $\gamma_3 = 0$ however divergencies can also appear from four-field terms. Furthermore the computation of the second order β-function for those terms shows that these divergences can actually appear.

The β-function we have computed coincides with the one in [Vaf10, OV10] that was obtained with a different method.

This result suggests that we should consider a model with a very small γ_3 instead of $\gamma_3 = 0$, in which case the momenta close to the Fermi points will be irrelevant as in the single-layer model. To treat the $\gamma_3 = 0$ case, we would have to investigate in which direction the divergence of the β-function is the fastest and study the system in that direction.

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Appendices

Appendix A1. Computation of the bands

In order to diagonalize $\mathcal{H}_0$, we go into Fourier space, so we first need to define the Fourier transform of the creation and annihilation operators:

$$\begin{aligned}
\hat{a}_{\vec{k}} &:= \sum_{\vec{x}\in\Lambda} e^{i\vec{k}\vec{x}} a_{\vec{x}} \quad &, \quad \hat{a}^\dagger_{\vec{k}} &:= \sum_{\vec{x}\in\Lambda} e^{-i\vec{k}\vec{x}} a^\dagger_{\vec{x}} \\
\hat{b}_{\vec{k}} &:= \sum_{\vec{x}\in\Lambda} e^{i\vec{k}\vec{x}} b_{\vec{x}+\vec{\delta}^0_1} \quad &, \quad \hat{b}^\dagger_{\vec{k}} &:= \sum_{\vec{x}\in\Lambda} e^{-i\vec{k}\vec{x}} b^\dagger_{\vec{x}+\vec{\delta}^0_1} \\
\hat{\tilde{a}}_{\vec{k}} &:= \sum_{\vec{x}\in\Lambda} e^{i\vec{k}\vec{x}} \tilde{a}_{\vec{x}+\vec{\delta}^1-\vec{\delta}^0_1} \quad &, \quad \hat{\tilde{a}}^\dagger_{\vec{k}} &:= \sum_{\vec{x}\in\Lambda} e^{-i\vec{k}\vec{x}} \tilde{a}^\dagger_{\vec{x}+\vec{\delta}^1-\vec{\delta}^0_1} \\
\hat{\tilde{b}}_{\vec{k}} &:= \sum_{\vec{x}\in\Lambda} e^{i\vec{k}\vec{x}} \tilde{b}_{\vec{x}+\vec{\delta}^1} \quad &, \quad \hat{\tilde{b}}^\dagger_{\vec{k}} &:= \sum_{\vec{x}\in\Lambda} e^{-i\vec{k}\vec{x}} \tilde{b}^\dagger_{\vec{x}+\vec{\delta}^1}
\end{aligned}$$

Notice that the Fourier transform of the creation and annihilation operators have the same anti-commutation formulas as the original ones and are therefore also creation and annihilation operators.
Notice as well the $+\vec{\delta}_\alpha$ in the previous definition. This is done so that the sum in $\vec{x}$ is always over Λ so that $\hat{\alpha}_{\vec{k}}$ is periodic modulo $\vec{G}_1$ and $\vec{G}_2$.
Using

$$\sum_{\vec{x}\in\Lambda} e^{i\vec{k}\vec{x}} = |\Lambda|\delta_{\vec{k},\vec{0}}, \quad \sum_{\vec{k}\in\hat{\Lambda}} e^{i\vec{k}\vec{x}} = |\Lambda|\delta_{\vec{x},0}$$

we have

$$\begin{aligned}
a_{\vec{x}} &= \frac{1}{|\Lambda|}\sum_{\vec{k}\in\hat{\Lambda}} e^{-i\vec{k}\vec{x}} \hat{a}_{\vec{k}} \ , \quad & a^\dagger_{\vec{x}} &= \frac{1}{|\Lambda|}\sum_{\vec{k}\in\hat{\Lambda}} e^{i\vec{k}\vec{x}} \hat{a}^\dagger_{\vec{k}} \\
b_{\vec{x}+\vec{\delta}^0_1} &= \frac{1}{|\Lambda|}\sum_{\vec{k}\in\hat{\Lambda}} e^{-i\vec{k}\vec{x}} \hat{b}_{\vec{k}} \ , \quad & b^\dagger_{\vec{x}+\vec{\delta}^0_1} &= \frac{1}{|\Lambda|}\sum_{\vec{k}\in\hat{\Lambda}} e^{-i\vec{k}\vec{x}} \hat{b}^\dagger_{\vec{k}} \\
\tilde{a}_{\vec{x}+\vec{\delta}^1-\vec{\delta}^0_1} &= \frac{1}{|\Lambda|}\sum_{\vec{k}\in\hat{\Lambda}} e^{-i\vec{k}\vec{x}} \hat{\tilde{a}}_{\vec{k}} \ , \quad & \tilde{a}^\dagger_{\vec{x}+\vec{\delta}^1-\vec{\delta}^0_1} &= \frac{1}{|\Lambda|}\sum_{\vec{k}\in\hat{\Lambda}} e^{i\vec{k}\vec{x}} \hat{\tilde{a}}^\dagger_{\vec{k}} \\
\tilde{b}_{\vec{x}+\vec{\delta}^1} &= \frac{1}{|\Lambda|}\sum_{\vec{k}\in\hat{\Lambda}} e^{-i\vec{k}\vec{x}} \hat{\tilde{b}}_{\vec{k}} \ , \quad & \tilde{b}^\dagger_{\vec{x}+\vec{\delta}^1} &= \frac{1}{|\Lambda|}\sum_{\vec{k}\in\hat{\Lambda}} e^{i\vec{k}\vec{x}} \hat{\tilde{b}}^\dagger_{\vec{k}}
\end{aligned}$$

So

$$\mathcal{H}_0 = -\frac{1}{|\Lambda|^2}\gamma_0 \sum_{\substack{\vec{x}\in\Lambda \\ j=1,2,3}} \sum_{(\vec{k}_1,\vec{k}_2)\in\hat{\Lambda}} \left(e^{i(\vec{k}_1-\vec{k}_2)\vec{x}-i\vec{k}_2(\vec{\delta}_j^0-\vec{\delta}_1^0)} \hat{a}_{\vec{k}_1}^{\dagger} \hat{b}_{\vec{k}_2} + e^{i(\vec{k}_2-\vec{k}_1)\vec{x}+i\vec{k}_2(\vec{\delta}_j^0-\vec{\delta}_1^0)} \hat{b}_{\vec{k}_2}^{\dagger} \hat{a}_{\vec{k}_1} \right)$$
$$-\frac{1}{|\Lambda|^2}\gamma_0 \sum_{\substack{\vec{x}\in\tilde{\Lambda} \\ j=1,2,3}} \sum_{(\vec{k}_1,\vec{k}_2)\in\hat{\Lambda}} \left(e^{i(\vec{k}_1-\vec{k}_2)\vec{x}-i\vec{k}_2(\vec{\delta}_j^0-\vec{\delta}_1^0)} \hat{\hat{a}}_{\vec{k}_1}^{\dagger} \hat{\hat{b}}_{\vec{k}_2} + e^{i(\vec{k}_2-\vec{k}_1)\vec{x}+i\vec{k}_2(\vec{\delta}_j^0-\vec{\delta}_1^0)} \hat{\hat{b}}_{\vec{k}_2}^{\dagger} \hat{\hat{a}}_{\vec{k}_1} \right)$$
$$-\frac{1}{|\Lambda|^2}\gamma_1 \sum_{\vec{x}\in\Lambda} \sum_{(\vec{k}_1,\vec{k}_2)\in\hat{\Lambda}} \left(e^{i(\vec{k}_1-\vec{k}_2)\vec{x}} \hat{a}_{\vec{k}_1}^{\dagger} \hat{\hat{b}}_{\vec{k}_2} + e^{i(\vec{k}_2-\vec{k}_1)\vec{x}} \hat{\hat{b}}_{\vec{k}_2}^{\dagger} \hat{a}_{\vec{k}_1} \right)$$
$$-\frac{1}{|\Lambda|^2}\gamma_3 \sum_{\substack{\vec{x}\in\Lambda \\ j=1,2,3}} \sum_{(\vec{k}_1,\vec{k}_2)\in\hat{\Lambda}} \left(e^{i(\vec{k}_1-\vec{k}_2)\vec{x}} e^{-i\vec{k}_2(2\vec{\delta}_1^0+\vec{\delta}_j^0)} \hat{b}_{\vec{k}_1}^{\dagger} \hat{\hat{a}}_{\vec{k}_2} + e^{i(\vec{k}_2-\vec{k}_1)\vec{x}} e^{i\vec{k}_2(2\vec{\delta}_1^0+\vec{\delta}_j^0)} \hat{\hat{a}}_{\vec{k}_2}^{\dagger} \hat{b}_{\vec{k}_1} \right)$$

$$\mathcal{H}_0 = -\frac{1}{|\Lambda|}\gamma_0 \sum_{\substack{\vec{k}\in\hat{\Lambda} \\ j=1,2,3}} \left(e^{-i\vec{k}(\vec{\delta}_j^0-\vec{\delta}_1^0)} \hat{a}_{\vec{k}}^{\dagger} \hat{b}_{\vec{k}} + e^{i\vec{k}(\vec{\delta}_j^0-\vec{\delta}_1^0)} \hat{b}_{\vec{k}}^{\dagger} \hat{a}_{\vec{k}} + e^{-i\vec{k}(\vec{\delta}_j^0-\vec{\delta}_1^0)} \hat{\hat{a}}_{\vec{k}}^{\dagger} \hat{\hat{b}}_{\vec{k}} + e^{i\vec{k}(\vec{\delta}_j^0-\vec{\delta}_1^0)} \hat{\hat{b}}_{\vec{k}}^{\dagger} \hat{\hat{a}}_{\vec{k}} \right)$$
$$-\frac{1}{|\Lambda|}\gamma_1 \sum_{\vec{k}\in\hat{\Lambda}} \left(\hat{a}_{\vec{k}}^{\dagger} \hat{\hat{b}}_{\vec{k}} + \hat{\hat{b}}_{\vec{k}}^{\dagger} \hat{a}_{\vec{k}} \right) - \frac{1}{|\Lambda|}\gamma_3 \sum_{\substack{\vec{k}\in\hat{\Lambda} \\ j=1,2,3}} e^{-i\vec{k}(2\vec{\delta}_1^0+\vec{\delta}_j^0)} \hat{b}_{\vec{k}}^{\dagger} \hat{\hat{a}}_{\vec{k}} + e^{i\vec{k}(2\vec{\delta}_1^0+\vec{\delta}_j^0)} \hat{\hat{a}}_{\vec{k}}^{\dagger} \hat{b}_{\vec{k}}$$

$$= -\frac{1}{|\Lambda|} \sum_{\vec{k}\in\hat{\Lambda}} A_{\vec{k}}^{\dagger} H_0(\vec{k}) A_{\vec{k}} \qquad \text{(A1.4)}$$

where

$$A_{\vec{k}} := \begin{pmatrix} \hat{a}_{\vec{k}} \\ \hat{b}_{\vec{k}} \\ \hat{\hat{a}}_{\vec{k}} \\ \hat{\hat{b}}_{\vec{k}} \end{pmatrix} \text{ and } H_0(\vec{k}) := \begin{pmatrix} 0 & \Omega_0^*(\vec{k}) & 0 & \Omega_1^*(\vec{k}) \\ \Omega_0(\vec{k}) & 0 & \Omega_3^*(\vec{k}) & 0 \\ 0 & \Omega_3(\vec{k}) & 0 & \Omega_0^*(\vec{k}) \\ \Omega_1(\vec{k}) & 0 & \Omega_0(\vec{k}) & 0 \end{pmatrix} \qquad \text{(A}$$

H_0 can easily be diagonalized and we find (2.1).

Appendix A2. Expression of the Fermi points

$$
\left\{\begin{array}{l}
\vec{p}_F^{\,\pm} := \left(\frac{2\pi}{3}, \pm\frac{2\pi}{3\sqrt{3}}\right)\\
\vec{p}_{F_1}^{\,\pm} := \left(\frac{2\pi}{3}, \mp\frac{2}{\sqrt{3}}\left(\arccos\left(\frac{G-1}{2}\right)-\pi\right)\right)\\
\vec{p}_{F_2}^{\,\pm} := \left(\arccos\left(\frac{G-1}{2}\right), \pm\frac{1}{\sqrt{3}}\arccos\left(\frac{G-1}{2}\right)\right)\\
\vec{p}_{F_3}^{\,\pm} := \left(\frac{4\pi}{3}-\arccos\left(\frac{G-1}{2}\right), \pm\frac{1}{\sqrt{3}}\arccos\left(\frac{G-1}{2}\right)\right)
\end{array}\right. \tag{A2.1}
$$

Appendix A3. Order of the singularity for $\gamma_3 = 0$

First of all, notice that

$$\zeta^2(\vec{k}) = \xi(\vec{k}) \iff \chi(\vec{k}) = 0 \iff \vec{k} \in \{\vec{p}_F^{\,+}, \vec{p}_F^{\,-}\}$$

where $\vec{p}_F^{\,\pm}$ is defined in (2.3). So

$$
\begin{aligned}
\Omega^-(\vec{k}-\vec{p}_F^{\,\omega}) &= \frac{1}{\sqrt{2}}\sqrt{\gamma_1^2 + 2\gamma_0^2|\chi(\vec{k}-\vec{p}_F^{\,\omega})|^2 - \sqrt{\gamma_1^4 + 4\gamma_1^2\gamma_0^2|\chi(\vec{k}-\vec{p}_F^{\,\omega})|^2}}\\
&= \frac{\gamma_0^2}{\gamma_1}|\chi(\vec{k}-\vec{p}_F^{\,\omega})|^2 + O(|\chi(\vec{k}-\vec{p}_F^{\,\omega})|^4)
\end{aligned}
$$

Furthermore

$$|\chi(\vec{k}-\vec{p}_F^{\,\omega})|^2 = \left(\frac{3}{2}|\vec{k}-\vec{p}_F^{\,\omega}|\right)^2 + O(\vec{k}-\vec{p}_F^{\,\omega})^3$$

so

$$\Omega^-(\vec{k}-\vec{p}_F^{\,\omega}) = \frac{9\gamma_0^2}{4\gamma_1}|\vec{k}-\vec{p}_F^{\,\omega}|^2 + O(\vec{k}-\vec{p}_F^{\,\omega})^3$$

Appendix A4. Definition of Grassmann integrals

Definition: Let V be a complex vector space space, and $\mathcal{T}(V)$ its tensor space. Let $\mathcal{I}$ be the ideal generated by $u\otimes v + v\otimes u = 0$ for $(u,v)\in V^2$. The *Grassmann algebra* associated to V is

$$\mathcal{G}(V) = \frac{\mathcal{T}(V)}{\mathcal{I}}$$

We may also define the C^*-*Grassmann algebra* $\mathcal{G}^*(V)$ associated to V which is equal to $\mathcal{G}(V \oplus V^*)$

Definition: The *Grassmann integral* of a function $f : V \longrightarrow V$ with respect to $\psi \in V$, noted $\int d\psi\ f(\psi)$ is the linear mapping such that

$$\begin{cases} \int d\psi\ 1 = 0 \\ \int d\psi\ \psi = 1 \end{cases}$$

We can extend this definition to functions $f : \mathcal{G}(V) \longrightarrow \mathcal{G}(V)$: if $\{\psi_i\}$ is a basis of V,

$$\int d\psi_i\ \psi_i \prod_{j \neq i} \psi_j = \prod_{j \neq i} \psi_j$$

All other values can be computed using the anticommutation of the ψ_j and the linearity.

Lemma: The following equality will hereby be called the *Gaussian integration lemma*:

$$\int d\psi^* d\psi\ e^{-\psi^*\psi} \psi^\alpha \psi^{*\beta} = \delta_{\alpha,\beta}$$

Proof:

$$\begin{aligned} \int d\psi^* d\psi\ e^{-\psi^*\psi} \psi^{*\alpha} \psi^\beta &= \int d\psi^* d\psi\ (1 - \psi^*\psi) \psi^\alpha \psi^{*\beta} \\ &= \begin{cases} \int d\psi^* d\psi\ (1 - \psi^*\psi) = 1 & \text{if } (\alpha, \beta) = (0,0) \\ \int d\psi^* d\psi\ \psi = 0 & \text{if } (\alpha, \beta) = (1,0) \\ \int d\psi^* d\psi\ \psi^* = 0 & \text{if } (\alpha, \beta) = (0,1) \\ \int d\psi^* d\psi\ \psi\psi^* = 1 & \text{if } (\alpha, \beta) = (1,1) \end{cases} \end{aligned}$$

Lemma: the following equality will hereby be called the *Wick rule for Gaussian Grassmann integration*:

$$\int \prod_{p=1}^{n} d\psi_p^* d\psi_p \ e^{-\sum_p \psi_p^* \psi_p} \prod_p \psi_p^{\alpha_p} \psi_p^{*\beta_p} = \prod_{p=1}^{n} \delta_{\alpha_p,\beta_p}$$

For more details on exponentials of Grassmann variables, see note 4.

Proof:

$$\int \prod_{p=1}^{n} d\psi_p^* d\psi_p \ e^{-\sum_p \psi_p^* \psi_p} \prod_p \psi^{\alpha_p} \psi^{*\beta_p}$$

$$= \left(\int \prod_{p=1}^{n-1} d\psi_p^* d\psi_p \ e^{-\sum_p \psi_p^* \psi_p} \prod_{p=1}^{n-1} \psi_p^{\alpha_p} \psi_p^{*\beta_p} \right) \int d\psi^* d\psi \ e^{-\psi^* \psi} \psi^{\alpha_n} \psi^{*\beta_n}$$

since $\psi_n^* \psi_n$ commutes with any element of $\{\psi_p, \psi_p^*\}_{p \in [|1,n-1|]}$.

By using The Gaussian integration lemma, we get

$$\delta_{\alpha_n,\beta_n} \int \prod_{p=1}^{n-1} d\psi_p^* d\psi_p \ e^{-\sum_p \psi_p^* \psi_p} \prod_{p=1}^{n-1} \psi^{\alpha_p} \psi^{*\beta_p}.$$

The proof is concluded by induction.

Lemma: The variables in a Grassmann integral can be changed:

$$\int d\psi f(\psi) \quad = \quad \lambda \int d\phi \ f\left(\frac{\phi}{\lambda}\right)$$

where $\phi = \lambda\psi, \ \lambda \neq 0$

Proof: f is a polynomial in ψ, so

$$f(\psi) = a + b\psi$$

and

$$\int d\psi a + b\psi = b$$

$$\lambda \int d\phi \ \left(a + b\frac{\phi}{\lambda}\right) = b$$

Theorem: We can generalize the previous lemma to multiple Grassmann integrals:

$$\int \prod_{i=1}^{n} d\psi_i \; f(\underline{\psi}) \;=\; \det U \int \prod_{i=1}^{n} d\phi_i \; f(U^{-1}\underline{\phi})$$
$$\text{where } \underline{\phi} = U\underline{\psi}, \;\det U \neq 0$$

Proof: f is a polynomial, so

$$f(\underline{\psi}) = \sum_{i=0}^{n} \sum_{\substack{j_1,\cdots,j_i \\ j_{l+1}>j_l}} a_{j_1,\cdots,j_i} \psi_{j_1} \cdots \psi_{j_i}$$

so

$$\int \prod_{i=1}^{n} d\psi_i \; f(\underline{\psi}) = a_{1,\cdots,n}$$

and

$$\begin{aligned}
\int \prod_{i=1}^{n} d\phi_i \; f(U^{-1}\underline{\phi}) &= \int \prod_{i=1}^{n} d\phi_i \; a_{1,\cdots,n} \left(U^{-1}\underline{\phi}\right)_1 \cdots \left(U^{-1}\underline{\phi}\right)_n \\
&= a_{1,\cdots,n} \sum_{k_1,\cdots,k_n} U^{-1}_{1,k_1} \cdots U^{-1}_{n,k_n} \int \prod_{i=1}^{n} d\phi_i \; \phi_{k_1} \cdots \phi_{k_n} \\
&= a_{1,\cdots,n} \sum_{\pi} (-1)^{\pi} U^{-1}_{1,\pi(1)} \cdots U^{-1}_{n,\pi(n)} \\
&= \frac{a_{1,\cdots,n}}{\det U}
\end{aligned}$$

Lemma: Any invertible matrix M can be written as

$$M = U^{\dagger} D V$$

where U and V are unitary and D is positive definite and diagonal.

Proof: In the note entitled "`PolarRepresentationMatrices`", we prove that $M = WP$ where P is a positive definite Hermitian matrix and W is unitary. We orthodiagonalize $P = XDV$, where X and V are unitary and D is positive definite and diagonal. So

$$M = WXDV := U^{\dagger} D V$$

Theorem: The Gaussian Grassmann integration lemma may be generalized in the following way: let M be an invertible matrix,

$$\begin{cases} \displaystyle\int \prod_{p=1}^{n} d\psi_p^* d\psi_p \; e^{-\left\langle \underline{\psi}^*, M\underline{\psi} \right\rangle} = \det M \\ \displaystyle\frac{1}{\det M} \int \prod_{p=1}^{n} d\psi_p^* d\psi_p \; e^{-\left\langle \underline{\psi}^*, M\underline{\psi} \right\rangle} \psi_i \psi_j^* = \left(M^{-1}\right)_{i,j} \end{cases}$$

Proof: We use the previous lemma and write $M = U^\dagger D V$, so

$$\left\langle \underline{\psi}^*, M\underline{\psi} \right\rangle = \left\langle U^* \underline{\psi}^*, DV\underline{\psi} \right\rangle$$

we then change variables from $\underline{\psi}$, $\underline{\psi}^*$ to $\underline{\phi} = V\underline{\psi}$, $\underline{\phi}^* = U^*\underline{\psi}^*$:

$$\begin{aligned} \int \prod_{p=1}^{n} d\psi_p^* d\psi_p \; e^{-\left\langle \underline{\psi}^*, M\underline{\psi} \right\rangle} &= \int \prod_{p=1}^{n} d\phi_p^* d\phi_p \; e^{-\left\langle \underline{\phi}^*, D\underline{\phi} \right\rangle} \\ &= \prod_{p=1}^{n} \int d\phi_p^* d\phi_p \; e^{-D_{p,p}\phi_p^* \phi_p} \\ &= \prod_{p=1}^{n} D_{p,p} \int d\tilde{\phi}_p^* d\tilde{\phi}_p \; e^{-\tilde{\phi}_p^* \tilde{\phi}_p} \\ &\quad \text{with } \tilde{\phi}_p = \sqrt{D_{p,p}}\phi_p, \; \tilde{\phi}_p^* = \sqrt{D_{p,p}}\phi_p^* \\ &= \prod_{p=1}^{n} D_{p,p} \\ &= \det M \end{aligned}$$

Furthermore, $\psi_i = \sum_k V_{k,i}^* \phi_k$, $\psi_j^* = \sum_l U_{l,j} \phi_l^*$, so

$$\begin{aligned} \int \prod_{p=1}^{n} d\psi_p^* d\psi_p \; e^{-\left\langle \underline{\psi}^*, M\underline{\psi} \right\rangle} \psi_i \psi_j^* &= \sum_{k,l} V_{k,i}^* U_{l,j} \int \prod_{p=1}^{n} d\phi_p^* d\phi_p \; e^{-\left\langle \underline{\phi}^*, D\underline{\phi} \right\rangle} \phi_k \phi_l^* \\ &= \sum_{k,l} V_{k,i}^* U_{l,j} \frac{\prod_p D_{p,p}}{\sqrt{D_{k,k} D_{l,l}}} \int \prod_p d\tilde{\phi}_p^* d\tilde{\phi}_p \; e^{-\sum_p \tilde{\phi}_p^* \tilde{\phi}_p} \tilde{\phi}_k \tilde{\phi}_l^* \\ &= \sum_k V_{k,i}^* U_{k,j} \frac{\prod_p D_{p,p}}{D_{k,k}} \\ &= \det M \left(M^{-1}\right)_{i,j} \end{aligned}$$

Theorem: The Wick rule for Gaussian Grassmann integrals can be generalized in the following way: let M be an invertible matrix,

$$\frac{1}{\det M}\int\prod_{p=1}^{n} d\psi_p^* d\psi_p \ e^{-\left\langle \underline{\psi}^*, M\underline{\psi}\right\rangle}\psi_{i_1}\psi_{j_1}^*\cdots\psi_{i_k}\psi_{j_k}^* = \det G$$

where

$$G_{i,j} = \frac{1}{\det M}\int\prod_{p=1}^{n} d\psi_p^* d\psi_p \ e^{-\left\langle \underline{\psi}^*, M\underline{\psi}\right\rangle}\psi_i\psi_j^* = \left(M^{-1}\right)_{i,j}$$
$$\text{and } (i,j)\in\{(i_m,j_l),\ (m,l)\in[|1,k|]^2\}$$

Proof:

$$\int\prod_{p=1}^{n} d\psi_p^* d\psi_p \ e^{-\left\langle \underline{\psi}^*, M\underline{\psi}\right\rangle}\psi_{i_1}\psi_{j_1}^*\cdots\psi_{i_k}\psi_{j_k}^*$$
$$= \sum_{m_1,l_1,\cdots,m_k,l_k} V_{m_1,i_1}^* U_{l_1,j_1}\cdots V_{m_k,i_k}^* U_{l_k,j_k}\int\prod_{p=1}^{n} d\phi_p^* d\phi_p \ e^{-\sum_p \phi_p^*\phi_p}\phi_{m_1}\phi_{l_1}^*\cdots\phi_{m_k}\phi_{l_k}^* \cdot$$
$$\cdot\frac{\prod_p D_{p,p}}{\sqrt{D_{m_1,m_1}D_{l_1,l_1}\cdots D_{m_k,m_k}D_{l_k,l_k}}}$$

The previous Wick rule tells us that if $\{m_1,\cdots,m_k\}\neq\{l_1,\cdots,l_k\}$, we get 0. And in G, there would wither be a whole column or a whole line equal to 0, so $\det G = 0$.
If $\{m_1,\cdots,m_k\}=\{l_1,\cdots,l_k\}$, $\exists!\pi$ a permutation of the l_i such that $l_i=\pi(m_i)$ for all $i\in[|1,k|]$. And

$$\int\prod_{p=1}^{n} d\psi_p^* d\psi_p \ e^{-\left\langle \underline{\psi}^*, M\underline{\psi}\right\rangle}\psi_{i_1}\psi_{j_1}^*\cdots\psi_{i_k}\psi_{j_k}^*$$
$$= \sum_{m_1,\cdots,m_k}\sum_\pi(-1)^\pi V_{m_1,i_1}^* U_{\pi(m_1),j_1}\cdots V_{m_k,i_k}^* U_{\pi(m_k),j_k}\frac{\prod_p D_{p,p}}{D_{m_1,m_1}\cdots D_{m_k,m_k}}$$
$$= \sum_{m_1,\cdots,m_k}\sum_\pi(-1)^\pi V_{m_1,i_1}^* U_{m_1,\pi^{-1}(j_1)}\cdots V_{m_k,i_k}^* U_{m_k,\pi^{-1}(j_k)}\frac{\prod_p D_{p,p}}{D_{m_1,m_1}\cdots D_{m_k,m_k}}$$
$$= \sum_{m_1,\cdots,m_k}\sum_\pi(-1)^\pi V_{m_1,i_1}^* U_{m_1,\pi(j_1)}\cdots V_{m_k,i_k}^* U_{m_k,\pi(j_k)}\frac{\prod_p D_{p,p}}{D_{m_1,m_1}\cdots D_{m_k,m_k}}$$
$$= \det M\sum_\pi(-1)^\pi\left(M^{-1}\right)_{i_1,\pi(j_1)}\cdots\left(M^{-1}\right)_{i_k,\pi(j_k)}$$
$$= \det M\det G$$

Appendix A5. Limit of the change of variables in the $\gamma_3 = 0$ case

If we write $N^{\omega}_{\vec{k}} := \dfrac{1}{\sqrt{2((\Omega^{\omega}(\vec{k}))^2 + |\Omega_0(\vec{k})|^2)}}$

$$\begin{pmatrix} \eta^{++}_{\vec{k}} & \eta^{-+}_{\vec{k}} & \eta^{+-}_{\vec{k}} & \eta^{--}_{\vec{k}} \end{pmatrix}^{\dagger} = \begin{pmatrix} N^{+}_{\vec{k}}\Omega^{+}(\vec{k}) & N^{+}_{\vec{k}}\Omega^{*}_0(\vec{k}) & N^{+}_{\vec{k}}\Omega_0(\vec{k}) & N^{+}_{\vec{k}}\Omega^{+}(\vec{k}) \\ N^{+}_{\vec{k}}\Omega^{+}(\vec{k}) & -N^{+}_{\vec{k}}\Omega^{*}_0(\vec{k}) & N^{+}_{\vec{k}}\Omega_0(\vec{k}) & -N^{+}_{\vec{k}}\Omega^{+}(\vec{k}) \\ N^{-}_{\vec{k}}\Omega^{-}(\vec{k}) & N^{-}_{\vec{k}}\Omega^{*}_0(\vec{k}) & N^{-}_{\vec{k}}\Omega_0(\vec{k}) & N^{-}_{\vec{k}}\Omega^{-}(\vec{k}) \\ N^{-}_{\vec{k}}\Omega^{-}(\vec{k}) & -N^{-}_{\vec{k}}\Omega^{*}_0(\vec{k}) & N^{-}_{\vec{k}}\Omega_0(\vec{k}) & -N^{-}_{\vec{k}}\Omega^{-}(\vec{k}) \end{pmatrix}$$

We now look at the limit $\vec{k} \to \vec{p}^{\,\pm}_f$. First, $\vec{k} = \vec{p}^{\,\pm}_F$ if and only if $\chi(\vec{k}) = 0$.

And

$$\begin{aligned} \Omega_0 &= \gamma_0 \chi \\ \Omega^- &= \frac{\gamma_0^2}{\gamma_1}|\chi|^2 + O(|\chi|^4) \\ \Omega^+(\vec{p}^{\,\pm}_F) &= \gamma_1 \end{aligned}$$

So

$$\begin{cases} N^{+}_{\vec{k}}\Omega^{+}(\vec{k}) \longrightarrow \dfrac{1}{\sqrt{2}} \\ N^{+}_{\vec{k}}\Omega_0(\vec{k}) \longrightarrow 0 \\ N^{-}_{\vec{k}}\Omega^{-}(\vec{k}) \longrightarrow 0 \\ N^{-}_{\vec{k}}\Omega_0(\vec{k}) = \dfrac{1}{\sqrt{2}}\dfrac{\chi(\vec{k})}{|\chi(\vec{k})|} + O(|\chi(\vec{k})|) \end{cases}$$

$\big($(A5.5) does not have a limit as $\vec{k} \to \vec{p}^{\,\pm}_f\big)$

So

$$\left(\begin{array}{cccc} \eta_{\vec{k}}^{++} & \eta_{\vec{k}}^{-+} & \eta_{\vec{k}}^{+-} & \eta_{\vec{k}}^{--} \end{array}\right)^{\dagger} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & \frac{\chi^*(\vec{k})}{|\chi(\vec{k})|} & \frac{\chi(\vec{k})}{|\chi(\vec{k})|} & 0 \\ 0 & -\frac{\chi^*(\vec{k})}{|\chi(\vec{k})|} & \frac{\chi(\vec{k})}{|\chi(\vec{k})|} & 0 \end{pmatrix} + O(|\chi(\vec{k})|)$$

Appendix A6. Equivalency of the change of variables in (4.1) and (4.3)

We have

$$\hat{g}(\mathbf{k}) = \sum_{\omega,\epsilon} \frac{|N_{\mathbf{k}}^{\omega}|^2}{-ik_0 + \epsilon\Omega^{\omega}(\mathbf{k})} \begin{pmatrix} (\Omega^{\omega}(\mathbf{k}))^2 & \epsilon\Omega^{\omega}(\mathbf{k})\Omega_0^*(\mathbf{k}) & \Omega^{\omega}(\mathbf{k})\Omega_0(\mathbf{k}) & \epsilon(\Omega^{\omega}(\mathbf{k}))^2 \\ \epsilon\Omega^{\omega}(\mathbf{k})\Omega_0(\mathbf{k}) & |\Omega_0(\mathbf{k})|^2 & \epsilon(\Omega_0(\mathbf{k}))^2 & \Omega^{\omega}(\mathbf{k})\Omega_0(\mathbf{k}) \\ \Omega^{\omega}(\mathbf{k})\Omega_0^*(\mathbf{k}) & \epsilon(\Omega_0^*(\mathbf{k}))^2 & |\Omega_0(\mathbf{k})|^2 & \epsilon\Omega^{\omega}(\mathbf{k})\Omega_0^*(\mathbf{k}) \\ \epsilon(\Omega^{\omega}(\mathbf{k}))^2 & \Omega^{\omega}(\mathbf{k})\Omega_0^*(\mathbf{k}) & \epsilon\Omega^{\omega}(\mathbf{k})\Omega_0(\mathbf{k}) & (\Omega^{\omega}(\mathbf{k}))^2 \end{pmatrix}$$

so

$$U_{\mathbf{k}}^0 \hat{g}(\mathbf{k}) U_{\mathbf{k}}^{0\dagger} = \sum_{\omega,\epsilon} \frac{|N_{\mathbf{k}}^{\omega}|^2}{-ik_0 + \epsilon\Omega^{\omega}(\mathbf{k})} \cdot$$

$$\begin{pmatrix} (1+\epsilon)(\Omega^{\omega}(\mathbf{k}))^2 & 0 & \Omega^{\omega}(\mathbf{k})|\Omega_0(\mathbf{k})|(1+\epsilon) & 0 \\ 0 & (1-\epsilon)(\Omega^{\omega}(\mathbf{k}))^2 & 0 & \Omega^{\omega}(\mathbf{k})|\Omega_0(\mathbf{k})|(1-\epsilon) \\ \Omega^{\omega}(\mathbf{k})|\Omega_0(\mathbf{k})|(1+\epsilon) & 0 & |\Omega_0(\mathbf{k})|^2(1+\epsilon) & 0 \\ 0 & \Omega^{\omega}(\mathbf{k})|\Omega_0(\mathbf{k})|(1-\epsilon) & 0 & |\Omega_0(\mathbf{k})|^2(1-\epsilon) \end{pmatrix}$$

so

$$U^0_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}} = \sum_{\omega} 2|N^{\omega}_{\mathbf{k}}|^2 \cdot$$

$$\cdot \begin{pmatrix} \frac{(\Omega^{\omega}(\mathbf{k}))^2}{-ik_0+\Omega^{\omega}(\mathbf{k})} & 0 & \frac{\Omega^{\omega}(\mathbf{k})|\Omega_0(\mathbf{k})|}{-ik_0+\Omega^{\omega}(\mathbf{k})} & 0 \\ 0 & \frac{(\Omega^{\omega}(\mathbf{k}))^2}{-ik_0-\Omega^{\omega}(\mathbf{k})} & 0 & \frac{\Omega^{\omega}(\mathbf{k})|\Omega_0(\mathbf{k})|}{-ik_0-\Omega^{\omega}(\mathbf{k})} \\ \frac{\Omega^{\omega}(\mathbf{k})|\Omega_0(\mathbf{k})|}{-ik_0+\Omega^{\omega}(\mathbf{k})} & 0 & \frac{|\Omega_0(\mathbf{k})|^2}{-ik_0+\Omega^{\omega}(\mathbf{k})} & 0 \\ 0 & \frac{\Omega^{\omega}(\mathbf{k})|\Omega_0(\mathbf{k})|}{-ik_0-\Omega^{\omega}(\mathbf{k})} & 0 & \frac{|\Omega_0(\mathbf{k})|^2}{-ik_0-\Omega^{\omega}(\mathbf{k})} \end{pmatrix}$$

We wish to prove that

$$\lim_{\mathbf{k}\to\mathbf{0}} \Big((U_{\mathbf{0}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{0}})^{-1} - (U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1} \Big) = 0$$

We recall

$$U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}} = \begin{pmatrix} \frac{1}{-ik_0+\Omega^{+}(\mathbf{k})} & 0 & 0 & 0 \\ 0 & \frac{1}{-ik_0-\Omega^{+}(\mathbf{k})} & 0 & 0 \\ 0 & 0 & \frac{1}{-ik_0+\Omega^{-}(\mathbf{k})} & 0 \\ 0 & 0 & 0 & \frac{1}{-ik_0-\Omega^{-}(\mathbf{k})} \end{pmatrix}$$

so

$$(U^0_{\mathbf{k}}\hat g(\mathbf{k})U^{0\dagger}_{\mathbf{k}})(U_{\mathbf{k}}\hat g(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1} = 2\sum_{\omega}|N^{\omega}_{\mathbf{k}}|^2\cdot$$

$$\cdot\left(\begin{array}{cccc} (\Omega^\omega(\mathbf{k}))^2\frac{-ik_0+\Omega^+(\mathbf{k})}{-ik_0+\Omega^\omega(\mathbf{k})} & 0 & \Omega^\omega(\mathbf{k})|\Omega_0(\mathbf{k})|\frac{-ik_0+\Omega^-(\mathbf{k})}{-ik_0+\Omega^\omega(\mathbf{k})} & 0 \\ 0 & (\Omega^\omega(\mathbf{k}))^2\frac{-ik_0-\Omega^+(\mathbf{k})}{-ik_0-\Omega^\omega(\mathbf{k})} & 0 & \Omega^\omega(\mathbf{k})|\Omega_0(\mathbf{k})|\frac{-ik_0-\Omega^-(\mathbf{k})}{-ik_0-\Omega^\omega(\mathbf{k})} \\ \Omega^\omega(\mathbf{k})|\Omega_0(\mathbf{k})|\frac{-ik_0+\Omega^+(\mathbf{k})}{-ik_0+\Omega^\omega(\mathbf{k})} & 0 & |\Omega_0(\mathbf{k})|^2\frac{-ik_0+\Omega^-(\mathbf{k})}{-ik_0+\Omega^\omega(\mathbf{k})} & 0 \\ 0 & \Omega^\omega(\mathbf{k})|\Omega_0(\mathbf{k})|\frac{-ik_0-\Omega^+(\mathbf{k})}{-ik_0-\Omega^\omega(\mathbf{k})} & 0 & |\Omega_0(\mathbf{k})|^2\frac{-ik_0-\Omega^-(\mathbf{k})}{-ik_0-\Omega^\omega(\mathbf{k})} \end{array}\right)$$

We also recall

$$\begin{aligned} \chi(\mathbf{k}) &= 1-2e^{-i\frac{3}{2}k_x}\cos\left(\tfrac{\sqrt{3}}{2}k_y+\omega\tfrac{\pi}{3}\right),\ \ (\chi(\mathbf{0})=0) \\ \Omega_0(\mathbf{k}) &= \gamma_0\chi(\mathbf{k}) \\ \Omega^-(\mathbf{k}) &= \frac{\gamma_0^2}{\gamma_1}|\chi(\mathbf{k})|^2+O(|\chi(\mathbf{k})|^4) \\ \Omega^+(\mathbf{0}) &= \gamma_1 \\ (N^+_{\mathbf{k}})^2 &= \frac{1}{2\gamma_1^2} \\ (N^-_{\mathbf{k}})^2 &= \frac{1}{2\gamma_0^2|\chi(\mathbf{k})|^2}-\frac{1}{2\gamma_1^2}+O(|\chi(\mathbf{k})|^2) \end{aligned}$$

Therefore, if $\dfrac{-ik_0 + \epsilon\Omega^+(\mathbf{k})}{-ik_0 + \epsilon\Omega^-(\mathbf{k})}$ is bounded,

$$(U^0_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})(U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1} \underset{\mathbf{k}\to 0}{\longrightarrow} 1$$

so

$$\begin{aligned}(U^0_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})^{-1} - (U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1} &= (U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1}((U^0_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})(U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1} - 1) \\ &\underset{\mathbf{k}\to 0}{\longrightarrow} 0\end{aligned}$$

$\left(\text{since } \left|(U_{\mathbf{0}}\hat{g}(\mathbf{0})U^{\dagger}_{\mathbf{0}})^{-1}\right| < \infty\right)$

So

$$\begin{aligned}P(d\chi d\bar{\chi} d\psi d\bar{\psi}) &= \frac{1}{\mathcal{N}} \prod_{\omega,\rho} \prod_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} de^{\rho}_{\mathbf{k},\omega} de^{\rho\dagger}_{\mathbf{k},\omega} \exp\left(-\frac{1}{\beta|\Lambda|} \sum_{\omega,\rho} \sum_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} \underline{e}^{\dagger}_{\mathbf{k},\omega} (U^0_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})^{-1}\ \underline{e}_{\mathbf{k},\omega}\right) \\
&= \frac{1}{\mathcal{N}} \prod_{\omega,\rho} \prod_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} de^{\rho}_{\mathbf{k},\omega} de^{\rho\dagger}_{\mathbf{k},\omega} \exp\left(-\frac{1}{\beta|\Lambda|} \sum_{\omega,\rho} \sum_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} \underline{e}^{\dagger}_{\mathbf{k},\omega} (U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1}\ \underline{e}_{\mathbf{k},\omega}\right) \cdot \\
&\quad \cdot \exp\left(-\frac{1}{\beta|\Lambda|} \sum_{\omega,\rho} \sum_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} \underline{e}^{\dagger}_{\mathbf{k},\omega} \left((U^0_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})^{-1} - (U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1}\right)\ \underline{e}_{\mathbf{k},\omega}\right) \\
&= \frac{1}{\mathcal{N}} \prod_{\omega,\rho} \prod_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} de^{\rho}_{\mathbf{k},\omega} de^{\rho\dagger}_{\mathbf{k},\omega} \exp\left(-\frac{1}{\beta|\Lambda|} \sum_{\omega,\rho} \sum_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} \hat{s}_{\rho}(\mathbf{k}) e^{\rho\dagger}_{\mathbf{k},\omega} e^{\rho}_{\mathbf{k},\omega}\right) \cdot \\
&\quad \cdot \exp\left(-\frac{1}{\beta|\Lambda|} \sum_{\omega,\rho} \sum_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} \underline{e}^{\dagger}_{\mathbf{k},\omega} \left((U^0_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})^{-1} - (U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1}\right)\ \underline{e}_{\mathbf{k},\omega}\right)\end{aligned}$$

where $\hat{s}_{\rho}(mbfk) := \left(U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}}\right)_{\rho}$.

We wish to compute the four field terms in

$$\mathcal{L}\mathcal{V}(\psi,\bar{\psi}) :=$$

$$\mathcal{L}\log\frac{1}{\mathcal{N}}\int\prod_{\omega}\prod_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} d\chi_{\mathbf{k},\omega}d\chi^{\dagger}_{\mathbf{k},\omega}d\bar{\chi}_{\mathbf{k},\omega}d\bar{\chi}^{\dagger}_{\mathbf{k},\omega}\exp\left(-\frac{1}{\beta|\Lambda|}\sum_{\omega,\rho}\sum_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}}\hat{s}_{\rho}(\mathbf{k})e^{\rho\dagger}_{\mathbf{k},\omega}e^{\rho}_{\mathbf{k},\omega}\right)\cdot$$

$$\cdot\exp\left(-\frac{1}{\beta|\Lambda|}\sum_{\omega,\rho}\sum_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}}\underline{e}^{\dagger}_{\mathbf{k},\omega}\left((U^{0}_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})^{-1}-(U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1}\right)\ \underline{e}_{\mathbf{k},\omega}\right)e^{-\mathcal{V}_0(\chi,\bar{\chi},\psi,\bar{\psi})}$$

to the first order in U.

$$\mathcal{L}\mathcal{V}(\psi,\bar{\psi}) = \mathcal{L}\log\frac{1}{\mathcal{N}}\int\prod_{\omega}\prod_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}} d\chi_{\mathbf{k},\omega}d\chi^{\dagger}_{\mathbf{k},\omega}d\bar{\chi}_{\mathbf{k},\omega}d\bar{\chi}^{\dagger}_{\mathbf{k},\omega}\cdot$$
$$\cdot\exp\left(-\frac{1}{\beta|\Lambda|}\sum_{\omega,\rho}\sum_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}}\hat{s}_{\rho}(\mathbf{k})e^{\rho\dagger}_{\mathbf{k},\omega}e^{\rho}_{\mathbf{k},\omega}\right)e^{-\tilde{\mathcal{V}}(\chi,\bar{\chi},\psi,\bar{\psi})}$$

where

$$\tilde{\mathcal{V}}(\chi,\bar{\chi},\psi,\bar{\psi}) = \frac{1}{\beta|\Lambda|}\sum_{\omega,\rho}\sum_{\mathbf{k}\in\mathcal{B}^{0,\omega}_{\beta,L}}\underline{e}^{\dagger}_{\mathbf{k},\omega}\left((U^{0}_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})^{-1}-(U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1}\right)\ \underline{e}_{\mathbf{k},\omega}$$
$$+\mathcal{V}_0(\chi,\bar{\chi},\psi,\bar{\psi})$$

so

$$\mathcal{L}\mathcal{V}(\psi,\bar{\psi}) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\mathcal{L}\mathcal{E}^{T}_{\chi}\left(\tilde{\mathcal{V}}(\chi,\bar{\chi},\psi,\bar{\psi});n\right)$$

We look at the terms for $n\geqslant 2$: to stay at first order in U, there must be at least $n-1$ factors in $\mathcal{E}^{T}_{\chi}$ proportional to an element in

$$\left\{\left((U^{0}_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})^{-1}-(U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1}\right)_{\rho_1,\rho_2},\quad (\rho_1,\rho_2)\in[|1,4|]^2\right\}$$

and in order for $\mathcal{E}^{T}_{\chi}$ to be different from 0, $(\rho_1,\rho_2)\notin[|3,4|]^2$ (since $\mathcal{E}^{T}_{\chi}(cste\,;\chi\cdots)=0$).

Also, we cannot have all the (ρ_1, ρ_2) in $[|1,2|]^2$, or else the four ψ would be generated in $\mathcal{V}_0$, and that would yield 0 since $\mathcal{V}_0$ is of order 4. Therefore, when we localize, the terms will be proportionnal to an element in

$$\left\{ \lim_{\mathbf{k}\to\mathbf{0}} \left((U^0_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})^{-1} - (U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1} \right)_{\rho_1,\rho_2}, (\rho_1,\rho_2) \in [|1,4|]^2 \setminus \left([|1,2|]^2 \cup [|3,4|]^2\right) \right\}$$

(since there will be a term in $\chi_{\mathbf{k}}\psi_{\mathbf{k}}$ which will send that $\mathbf{k}$ to 0 upon localization). Furthermore, all the elements in the previous set are equal to 0.

A short **comment**: the fact that $(\rho_1, \rho_2) \notin [|1,2|]^2$ comes from the fact that there is at least one of the factors that has $(\rho_1, \rho_2) \notin [|1,2|]^2$. So it is important to notice that

$$\left| \lim_{\mathbf{k}\to\mathbf{0}} \left((U^0_{\mathbf{k}}\hat{g}(\mathbf{k})U^{0\dagger}_{\mathbf{k}})^{-1} - (U_{\mathbf{k}}\hat{g}(\mathbf{k})U^{\dagger}_{\mathbf{k}})^{-1} \right)_{\rho_1,\rho_2} \right| < \infty \text{ for } (\rho_1,\rho_2) \in [|1,2|]^2$$

Another short **comment**: since we proved that the set in question is $\{0\}$ even if $(\rho_1, \rho_2) \in [|1,2|]^2$, those terms don't matter.

Appendix A7. Computation of the Fourier transform of v at the Fermi points

We shall prove that if L is a multiple of 3,

$$\int d\mathbf{y}\ v(\mathbf{y} + \vec{\delta}^3_1 - \vec{\delta}^0_1) e^{-i(\mathbf{p}^+_F - \mathbf{p}^-_F)\mathbf{y}} = 0$$

First of all, $\displaystyle\int d\mathbf{y} = \int_{-\frac{\beta}{2}}^{\frac{\beta}{2}} dy_0 \sum_{\vec{y}\in\Lambda}$, so we prove

$$\sum_{\vec{y}\in\Lambda} v(\vec{y} + \vec{\delta}^3_1 - \vec{\delta}^0_1) e^{-i(\vec{p}^+_F - \vec{p}^-_F)\vec{y}} = 0$$

we recall

$$\Lambda = \left\{ n_1\vec{l}_1 + n_2\vec{l}_2,\ (n_1,n_2) \in [|0, L-1|] \right\},$$

$$\vec{l}_1 = \left(\frac{3}{2}, \frac{\sqrt{3}}{2}\right),\ \vec{l}_2 = \left(\frac{3}{2}, -\frac{\sqrt{3}}{2}\right),$$

$$\vec{p}_F^+ = \left(\frac{2\pi}{3}, \frac{2\pi}{3\sqrt{3}}\right),\ \vec{p}_F^- = \left(\frac{2\pi}{3}, -\frac{2\pi}{3\sqrt{3}}\right),$$

and

$$\vec{\delta}_3^0 - \vec{\delta}_1^0 = (-2, 0, c)$$

Consider the change of variables

$$\vec{z} = \mathcal{R}(\vec{y}) = R(\vec{y}) + (R-1)\vec{\delta}_3^0 - (R-1)\vec{\delta}_1^0$$

where

$$R = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

$\left(\text{rotation by } \frac{2\pi}{3}\right)$

We prove that $\vec{z} \in \Lambda$:

$$R(\vec{y}) + (R-1)\vec{\delta}_3^0 - (R-1)\vec{\delta}_1^0 =$$

$$\begin{cases} R(n_1\vec{l}_1 + n_2\vec{l}_2) = n_2\vec{l}_1 - (n_1+n_2)\vec{l}_2 \\ (R-1)(\vec{\delta}_3^0 - \vec{\delta}_1^0) = 2\vec{l}_2 \end{cases}$$

It is obvious from that that the change of variables is bijective and

$$\vec{y} = R^{-1}(\vec{z}) + (R^{-1}-1)\vec{\delta}_3^0 - (R^{-1}-1)\vec{\delta}_1^0$$

Furthermore, notice $\mathcal{R}^3 = 1$.

So

$$\begin{aligned} \sum_{\vec{y}\in\Lambda} v\left(\vec{y} + \vec{\delta}_3^0 - \vec{\delta}_1^0\right) e^{-i(\vec{p}_F^+ - \vec{p}_F^-)\vec{y}} &= \sum_{\vec{z}\in\Lambda} v\left(R^{-1}(\vec{z} + \vec{\delta}_3^0 - \vec{\delta}_1^0)\right) e^{-i(\vec{p}_F^+ - \vec{p}_F^-)\mathcal{R}^{-1}\vec{z}} \\ &= \sum_{\vec{z}\in\Lambda} v\left(\vec{z} + \vec{\delta}_3^0 - \vec{\delta}_1^0\right) e^{-i(\vec{p}_F^+ - \vec{p}_F^-)\mathcal{R}^{-2}\vec{z}} \end{aligned}$$

and

$$\sum_{\vec{y}\in\Lambda} v\left(\vec{y} + \vec{\delta}_3^0 - \vec{\delta}_1^0\right) e^{-i(\vec{p}_F^+ - \vec{p}_F^-)\vec{y}} = \sum_{\vec{z}\in\Lambda} v\left(\vec{z} + \vec{\delta}_3^0 - \vec{\delta}_1^0\right) e^{-i(\vec{p}_F^+ - \vec{p}_F^-)\mathcal{R}^{-2}\vec{z}}$$

Finally, we prove that $\forall \vec{z} \in \Lambda$,

$$\sum_{j=0,1,2} e^{-i(\vec{p}_F^+ - \vec{p}_F^-)\mathcal{R}^{-j}\vec{z}} = 0$$

notice that $\mathcal{R}^{-j}\vec{z} = R^{-j}(\vec{z}) + (R^{-j}-1)\vec{\delta}_3^0 - (R^{-j}-1)\vec{\delta}_1^0$ and that

$$R^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},\ R^{-2} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

so

$$(R^{-1}-1)(\vec{\delta}_3^0 - \vec{\delta}_1^0) = 2\vec{l}_1$$
$$(R^{-2}-1)(\vec{\delta}_3^0 - \vec{\delta}_1^0) = 2\vec{l}_2$$

so

$$e^{-i(R^{-1}-1)(\vec{\delta}_3^0 - \vec{\delta}_1^0)} = e^{i\frac{2\pi}{3}} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$
$$e^{-i(R^{-2}-1)(\vec{\delta}_3^0 - \vec{\delta}_1^0)} = e^{i\frac{2\pi}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

and $\forall j \in \{1,2\}$,

$$(\vec{p}_F^+ - \vec{p}_F^-)R^{-j}\vec{z} = \begin{pmatrix} 0 & \frac{4\pi}{3\sqrt{3}} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & (-1)^{j+1}\frac{\sqrt{3}}{2} \\ (-1)^j\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \vec{z}$$

$$= \begin{pmatrix} (-1)^j\frac{2\pi}{3} & -\frac{2\pi}{3\sqrt{3}} \end{pmatrix} \vec{z}$$
$$= (\vec{p}_F^+ - \vec{p}_F^-)\vec{y} - \left((-1)^j\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}}\right) \cdot \vec{z}$$

however $\forall (j,m) \in \{1,2\}^2$,

$$e^{\left((-1)^j\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}}\right)\cdot \vec{l}_m} = e^{i\left((-1)^j + (-1)^{l+1}\right)\pi} = 1$$

So

$$\sum_{j=0,1,2} e^{-i(\vec{p}_F^+ - \vec{p}_F^-)\mathcal{R}^{-j}\vec{z}} = e^{-i(\vec{p}_F^+ - \vec{p}_F^-)\vec{z}}\left(1 + e^{i\frac{2\pi}{3}} + e^{-i\frac{2\pi}{3}}\right) = 0$$

A short **comment**: δ_j^l are defined in three dimensions, so for this to be notation-wise correct, the rotation should actually be

$$R = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\vec{l}_j$ and $\vec{p}_F^{\,\omega}$ have a third component equal to 0. The rest is the same since $R^{-j} - 1$ is zero along the third component.

Appendix A8. Computation of the Feynmann graphs

We wish to compute

$$\begin{cases} G_1^{\omega,\omega'(h)} := \dfrac{1}{\beta|\Lambda|} \displaystyle\sum_{\mathbf{k}_3 \in \mathcal{B}_{\beta,L}^{h+1}} \tilde{g}_\omega^{(h)}(\mathbf{k}_3)_{1,1} \tilde{g}_{\omega'}^{(h)}(\mathbf{k}_3)_{1,1} \\ G_2^{\omega,\omega'(h)} := \dfrac{1}{\beta|\Lambda|} \displaystyle\sum_{\mathbf{k}_3 \in \mathcal{B}_{\beta,L}^{h+1}} \tilde{g}_\omega^{(h)}(\mathbf{k}_3)_{1,2} \tilde{g}_{\omega'}^{(h)}(\mathbf{k}_3)_{1,2} \\ G_3^{\omega,\omega'(h)} := \dfrac{1}{\beta|\Lambda|} \displaystyle\sum_{\mathbf{k}_3 \in \mathcal{B}_{\beta,L}^{h+1}} \tilde{g}_\omega^{(h)}(\mathbf{k}_3)_{1,2} \tilde{g}_{\omega'}^{(h)}(\mathbf{k}_3)_{2,1} \end{cases}$$

with

$$\tilde{g}_\omega^{(h)}(\mathbf{k}) = \frac{f_h(\mathbf{k})}{k_0^2 + |\Omega_\omega(\mathbf{k})|^2} \begin{pmatrix} ik_0 & \Omega_\omega(\mathbf{k}) \\ \Omega_\omega^*(\mathbf{k}) & ik_0 \end{pmatrix} + O(r.g)$$

where $O(r.g) = \displaystyle\sum_{j=1,2,3} O(\lambda_j^{(h)})$

Throughout the computation we will use the following identities:

$$\Omega_\omega^-(\mathbf{k}) = \frac{9\gamma_0^2}{4\gamma_1} |\vec{k}|^2 + O(\vec{k}^3)$$

where $O(\vec{k}^3) = O(k_1^3) + O(k_1^2 k_2) + O(k_1 k_2^2) + O(k_2^3)$, and $|\Omega_\omega(\mathbf{k})| = \Omega_\omega^-(\mathbf{k})$,

$$\int\limits_{\|\mathbf{k}\|\leqslant R} d\mathbf{k}\; k_0^l k_1^m k_2^n = C_{l,m,n} R^{n+m+2l+4}$$

The proof of (A8.2) consists in changing variables to $\left(\frac{k_0}{R^2}, \frac{k_1}{R}, \frac{k_2}{R}\right)$.

$\mathbf{G_1}$:

$$G_1^{\omega,\omega'(h+1)} = \frac{1}{\beta|\Lambda|} \sum_{k_0 \in \frac{2\pi}{\beta}\left(\mathbb{Z}+\frac{1}{2}\right)}^{f_{h+1}(\mathbf{k})>0} \sum_{\vec{k}\in\hat{\Lambda}} - \frac{\left(f_{h+1}(\mathbf{k})k_0\right)^2}{\left(k_0^2 + |\Omega_\omega(\mathbf{k})|^2\right)\left(k_0^2 + |\Omega_{\omega'}(\mathbf{k})|^2\right)} + O(r.g)$$

To simplify, we take the thermodynamic limit ($L \to \infty$ and $\beta \to \infty$). We then use (A8.1) and find that if $\epsilon_\omega(\vec{k}) = (\Omega_\omega^-(\mathbf{k}))^2 - \left(\frac{3\gamma_0}{2\sqrt{\gamma_1}}|\vec{k}|\right)^4$ (notice $\epsilon_\omega(\vec{k}) = O(\vec{k}^5)$), then

$$G_1^{h+1} = -\frac{1}{8\pi^3} \int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k}\; \frac{\left(f_{h+1}(\mathbf{k})k_0\right)^2}{\left(k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4 + \epsilon_\omega(\vec{k})\right)\left(k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4 + \epsilon_{\omega'}(\vec{k})\right)} + O(r.g)$$

$$= -\frac{1}{8\pi^3} \int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \left(\frac{f_{h+1}(\mathbf{k})k_0}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)^2 \cdot$$

$$\cdot \left(1 + O\left(\frac{\epsilon_\omega(\vec{k})}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right) + O\left(\frac{\epsilon_{\omega'}(\vec{k})}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)\right) + O(r.g)$$

Furthermore

$$f_{h+1}(\mathbf{k}) > 0 \Longleftrightarrow \|\mathbf{k}\| \in \left(\frac{1}{3}2^h, \frac{4}{3}2^h\right)$$

so

$$\frac{1}{81}2^{4h} \leqslant k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4 \leqslant \frac{64}{81}2^{4h}$$

and using (A8.2),

$$\begin{aligned} \int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k}\quad \left(f_{h+1}(\mathbf{k})k_0\right)^2 k_1^n k_2^{5m-n} &\leqslant \int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k}\quad k_0^2 k_1^n k_2^{5m-n} \\ &\leqslant C_{2,n,5m-n}\left(\left(\frac{4}{3}2^h\right)^{5m+8} - \left(\frac{1}{3}2^h\right)^{5m+8}\right) \\ &= O\left(2^{(5m+8)h}\right) \end{aligned}$$

So $\forall \omega'' \in \{+1,-1\}$,

$$\int_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \left(\frac{f_{h+1}(\mathbf{k}) k_0}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4} \right)^2 O\left(\frac{\epsilon_{\omega''}(\vec{k})}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4} \right) = \sum_{m=0}^{\infty} O\left(2^{mh}\right) = O\left(2^h\right)$$

and

$$G_1^{\omega,\omega'(h+1)} = -\frac{1}{8\pi^3} \int_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \left(\frac{f_{h+1}(\mathbf{k}) k_0}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4} \right)^2 + O\left(2^h\right) + O(r.g)$$

By playing with the definition of χ_0, we can forget about the f_{h+1} factor in the integrals by putting them in the $O\left(2^h\right)$. So

$$G_1^{\omega,\omega'(h+1)} = -\frac{1}{8\pi^3} \int_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \left(\frac{k_0}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4} \right)^2 + O\left(2^h\right) + O(r.g)$$

We introduce the following change of variables:

$$\begin{pmatrix} k_0 \\ \rho \\ \phi \end{pmatrix} = \begin{pmatrix} k_0 \\ \left(k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4\right)^{1/4} \\ \phi \end{pmatrix} := F(k_0, k_1, k_2)$$

where ϕ is the element of $\dfrac{\mathbb{R}}{2\pi\mathbb{Z}}$ such that

$$\begin{cases} k_1 = \dfrac{2\sqrt{\gamma_1}}{3\gamma_0}\left(\rho^4 - k_0^2\right)^{1/4}\cos(\phi) \\[2ex] k_2 = \dfrac{2\sqrt{\gamma_1}}{3\gamma_0}\left(\rho^4 - k_0^2\right)^{1/4}\sin(\phi) \end{cases}$$

The Jacobian of F^{-1} is

$$JF^{-1} = \begin{pmatrix} 1 & -\frac{\sqrt{\gamma_1}}{3\gamma_0}\frac{\cos(\phi)}{(\rho^4-k_0^2)^{3/4}}k_0 & -\frac{\sqrt{\gamma_1}}{3\gamma_0}\frac{\sin(\phi)}{(\rho^4-k_0^2)^{3/4}}k_0 \\ 0 & \frac{2\sqrt{\gamma_1}}{3\gamma_0}\frac{\cos(\phi)}{(\rho^4-k_0^2)^{3/4}}\rho^3 & \frac{2\sqrt{\gamma_1}}{3\gamma_0}\frac{\sin(\phi)}{(\rho^4-k_0^2)^{3/4}}\rho^3 \\ 0 & -\frac{2\sqrt{\gamma_1}}{3\gamma_0}\left(\rho^4-k_0^2\right)^{1/4}\sin(\phi) & \frac{2\sqrt{\gamma_1}}{3\gamma_0}\left(\rho^4-k_0^2\right)^{1/4}\cos(\phi) \end{pmatrix}$$

whose determinant is

$$\begin{aligned} \det(JF^{-1}) &= \frac{4\gamma_1}{9\gamma_0^2}\frac{\rho^3}{\sqrt{\rho^4-k_0^2}}\left(\cos^2(\phi)+\sin^2(\phi)\right) \\ &= \frac{4\gamma_1}{9\gamma_0^2}\frac{\rho^3}{\sqrt{\rho^4-k_0^2}} \end{aligned}$$

In these new variables, we have

$$\begin{aligned} G_1^{\omega,\omega'(h+1)} &= -\frac{1}{4\pi^2}\int_{\frac{1}{3}2^h}^{\frac{4}{3}2^h} d\rho \int_{-\rho^2}^{\rho^2} dk_0\, \frac{4\gamma_1}{9\gamma_0^2}\frac{\rho^3}{\sqrt{\rho^4-k_0^2}}\frac{k_0^2}{\rho^8} + O\left(2^h\right) + O(r.g) \\ &= -\frac{1}{4\pi^2}\frac{4\gamma_1}{9\gamma_0^2}\int_{\frac{1}{3}2^h}^{\frac{4}{3}2^h} d\rho \int_{-\rho^2}^{\rho^2} dk_0\, \frac{k_0^2}{\rho^5\sqrt{\rho^4-k_0^2}} + O\left(2^h\right) + O(r.g) \\ &\underset{(u_o=k_0/\rho^2)}{=} -\frac{1}{4\pi^2}\frac{4\gamma_1}{9\gamma_0^2}\int_{\frac{1}{3}2^h}^{\frac{4}{3}2^h} d\rho \int_{-1}^{1} du_0\, \frac{1}{\rho}\frac{u_0^2}{\sqrt{1-u_0^2}} + O\left(2^h\right) + O(r.g) \\ &= -\frac{1}{2\pi^2}\ln(2)\frac{4\gamma_1}{9\gamma_0^2}\int_{-1}^{1} du_0\, \frac{u_0^2}{\sqrt{1-u_0^2}} + O\left(2^h\right) + O(r.g) \end{aligned}$$

and

$$\int_{-1}^{1} dx\, \frac{x^2}{\sqrt{1-x^2}} \underset{x=cos(\theta)}{=} -\int_{\pi}^{0} d\theta \sin(\theta) \frac{\cos^2(\theta)}{|\sin(\theta)|}$$

$$= \int_{0}^{\pi} d\theta\, \frac{1+\cos(2\theta)}{2}$$

$$= \frac{\pi}{2}$$

So

$$G_1^{\omega,\omega'(h+1)} = -\frac{1}{4\pi^2} \ln(2)\pi \frac{4\gamma_1}{9\gamma_0^2} + O\left(2^h\right) + O(r.g)$$

$\mathbf{G_2}$:

$$G_2^{\omega,\omega'(h+1)} = \frac{1}{\beta|\Lambda|} \sum_{k_0 \in \frac{2\pi}{\beta}\left(\mathbb{Z}+\frac{1}{2}\right)}^{f_{h+1}(\mathbf{k})>0} \sum_{\vec{k}\in\hat{\Lambda}} \left(\frac{f_{h+1}(\mathbf{k})\Omega_\omega(\mathbf{k})}{k_0^2+|\Omega_\omega(\mathbf{k})|^2}\right)\left(\frac{f_{h+1}(\mathbf{k})\Omega_{\omega'}(\mathbf{k})}{k_0^2+|\Omega_{\omega'}(\mathbf{k})|^2}\right) + O(r.g)$$

$$= \frac{1}{\beta|\Lambda|} \sum_{k_0 \in \frac{2\pi}{\beta}\left(\mathbb{Z}+\frac{1}{2}\right)}^{f_{h+1}(\mathbf{k})>0} \sum_{\vec{k}\in\hat{\Lambda}} \frac{\left(f_{h+1}(\mathbf{k})\right)^2 \Omega_\omega^-(\mathbf{k})\Omega_{\omega'}^-(\mathbf{k})}{\left(k_0^2+|\Omega_\omega^-(\mathbf{k})|^2\right)\left(k_0^2+|\Omega_{\omega'}^-(\mathbf{k})|^2\right)} \cdot$$

$$\cdot \left(X_\omega(\mathbf{k})\right)^2 \left(X_{\omega'}(\mathbf{k})\right)^2 + O(r.g)$$

So if we define $\epsilon_\omega(\vec{k}) = (\Omega_\omega^-(\mathbf{k}))^2 - \left(\frac{3\gamma_0}{2\sqrt{\gamma_1}}|\vec{k}|\right)^4$ (notice $\epsilon_\omega(\vec{k}) = O(\vec{k}^5)$), and $\varpi_{\omega,\omega'}(\vec{k}) = \Omega_\omega^-(\mathbf{k})\Omega_{\omega'}^-(\mathbf{k}) - \left(\frac{3\gamma_0}{2\sqrt{\gamma_1}}|\vec{k}|\right)^4$ (notice that $\varpi_{\omega,\omega'}(\vec{k}) = O(\vec{k}^5)$) then

$$G_2^{\omega,\omega'(h+1)} = \frac{1}{8\pi^3} \int_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \left(\frac{f_{h+1}(\mathbf{k})\frac{9\gamma_0^2}{4\gamma_1}|\vec{k}|^2}{k_0^2+\frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)^2 \left(X_\omega(\mathbf{k})\right)^2 \left(X_{\omega'}(\mathbf{k})\right)^2 \cdot$$

$$\cdot \left(1 + O\left(\frac{\varpi_{\omega,\omega'}(\vec{k})}{k_0^2+\frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right) + O\left(\frac{\epsilon_\omega(\vec{k})}{k_0^2+\frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)\right.$$

$$\left. + O\left(\frac{\epsilon_{\omega'}(\vec{k})}{k_0^2+\frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)\right) + O(r.g)$$

Similarly to G_1, we have $\forall m \in \mathbb{N},\ n \in [|0,5m|],\ \alpha \in [|0,2|]$

$$\int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \quad \left(f_{h+1}(\mathbf{k})\right)^2 k_1^{n+2\alpha} k_2^{5m-n+2(2-\alpha)} = O\left(2^{(5m+8)h}\right)$$

so

$$G_2^{\omega,\omega'(h+1)} = \frac{1}{8\pi^3} \int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \left(\frac{f_{h+1}(\mathbf{k})\frac{9\gamma_0^2}{4\gamma_1}|\vec{k}|^2}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)^2 \left(X_\omega(\mathbf{k})\right)^2 \left(X_{\omega'}(\mathbf{k})\right)^2 + O\left(2^h\right) + O(r.g)$$

However, if

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$$

then

$$\chi_\omega(R\mathbf{k}) = -e^{i\frac{3}{2}k_1 + i\frac{\sqrt{3}}{2}k_2 + i\omega\frac{\pi}{3}} \chi_\omega(\mathbf{k})$$

so

$$\Omega_\omega^-(R\mathbf{k}) = \Omega_\omega^-(\mathbf{k})$$

$$\left(X_\omega(R\mathbf{k})\right)^4 = e^{i6k_1 + i2\sqrt{3}k_2 + i\omega\frac{4\pi}{3}} \left(X_\omega(\mathbf{k})\right)^4 = e^{i\omega\frac{4\pi}{3}} \left(X_\omega(\mathbf{k})\right)^4 + O\left(2^h\right)$$

$$\begin{aligned}\left(X_\omega(R\mathbf{k})\right)^2 \left(X_{-\omega}(R\mathbf{k})\right)^2 &= e^{i6k_1 + i2\sqrt{3}k_2} \left(X_\omega(\mathbf{k})\right)^2 \left(X_{-\omega}(\mathbf{k})\right)^2 \\ &= \left(X_\omega(\mathbf{k})\right)^2 \left(X_{-\omega}(\mathbf{k})\right)^2 + O\left(2^h\right)\end{aligned}$$

Therefore, if $\omega = \omega'$,

$$\begin{aligned} G_2^{\omega,\omega(h+1)} &= \frac{1}{24\pi^3} \int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \left(\frac{f_{h+1}(\mathbf{k})\frac{9\gamma_0^2}{4\gamma_1}|\vec{k}|^2}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)^2 \left(X_\omega(\mathbf{k})\right)^4 \left(\sum_{j=0}^{2} e^{i\omega\frac{4\pi j}{3}}\right) + O\left(2^h\right) + O(r.g) \\ &= 0 + O\left(2^h\right) + O(r.g) \end{aligned} \tag{A8.4}$$

If $\omega = -\omega'$, we compute the limit of $\left(X_\omega(\mathbf{k})\right)^2 \left(X_{-\omega}(\mathbf{k})\right)^2$ as $\mathbf{k}$ goes to 0. We take the limit on the curve

$$\begin{cases} k_1 = t\cos(\theta) \\ k_2 = t\sin(\theta) \end{cases}$$

parametrised by t, for an arbitrary $\theta \in [0, 2\pi[$.

$$\begin{aligned}
\chi_\omega(\mathbf{k}) &= 1 - 2e^{-i\frac{3}{2}k_1} \cos\left(\tfrac{\sqrt{3}}{2}k_2 + \tfrac{\omega\pi}{3}\right) \\
&= 1 - 2e^{-i\frac{3}{2}\cos(\theta)t} \cos\left(\tfrac{\sqrt{3}}{2}\sin(\theta)t + \tfrac{\omega\pi}{3}\right) \\
&= 1 - 2e^{-i\frac{3}{2}\cos(\theta)t} \left(\frac{1}{2}\cos\left(\tfrac{\sqrt{3}}{2}\sin(\theta)t\right) + \omega\frac{\sqrt{3}}{2}\sin\left(\tfrac{\sqrt{3}}{2}\sin(\theta)t\right)\right) \\
&= 1 - 2\left(1 - i\frac{3}{2}\cos(\theta)t\right)\left(\frac{1}{2} + \omega\frac{3}{4}\sin(\theta)t\right) + O(t^2) \\
&= \frac{3}{2}\left(i\cos(\theta) - \omega\sin(\theta)\right)t + O(t^2)
\end{aligned}$$

so

$$\begin{aligned}
\left(\chi_\omega(\mathbf{k})\right)^2 \left(\chi_{-\omega}(\mathbf{k})\right)^2 &= \frac{81}{16}\left(i\cos(\theta) - \omega\sin(\theta)\right)^2 \left(i\cos(\theta) + \omega\sin(\theta)\right)^2 t^4 + O(t^5) \\
&= \frac{81}{16}t^4 + O(t^5)
\end{aligned}$$

furthermore

$$|\chi_\omega(\mathbf{k})|^2 |\chi_{-\omega}(\mathbf{k})|^2 = \frac{81}{16}t^4 + O(t^5)$$

so $\forall\theta \in [0, 2\pi[$,

$$\left(X_\omega(k_0, t\cos(\theta), t\sin(\theta))\right)^2 \left(X_{-\omega}(k_0, t\cos(\theta), t\sin(\theta))\right)^2 \underset{t\to 0}{\longrightarrow} 1$$

Therefore

$$\left(X_\omega(\mathbf{k})\right)^2 \left(X_{-\omega}(\mathbf{k})\right)^2 = 1 + O(\vec{k}) = 1 + O\left(2^h\right)$$

So

$$G_2^{\omega,-\omega(h+1)} = \frac{1}{8\pi^3} \int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \left(\frac{f_{h+1}(\mathbf{k})\frac{9\gamma_0^2}{4\gamma_1}|\vec{k}|^2}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)^2 + O\left(2^h\right) + O(r.g)$$

As for G_1 we change variables and find

$$G_2^{\omega,-\omega(h+1)} = \frac{1}{4\pi^2}\int\limits_{\frac{1}{3}2^h}^{\frac{4}{3}2^h} d\rho \int\limits_{-\rho^2}^{\rho^2} dk_0 \, \frac{4\gamma_1}{9\gamma_0^2}\frac{\rho^3}{\sqrt{\rho^4-k_0^2}}\frac{\rho^4-k_0^2}{\rho^8} + O\left(2^h\right) + O(r.g)$$

$$= \frac{1}{4\pi^2}\frac{4\gamma_1}{9\gamma_0^2}\int\limits_{\frac{1}{3}2^h}^{\frac{4}{3}2^h} d\rho \int\limits_{-\rho^2}^{\rho^2} dk_0 \, \frac{\sqrt{\rho^4-k_0^2}}{\rho^5} + O\left(2^h\right) + O(r.g)$$

$$\underset{(u_o=k_0/\rho^2)}{=} \frac{1}{4\pi^2}\frac{4\gamma_1}{9\gamma_0^2}\int\limits_{\frac{1}{3}2^h}^{\frac{4}{3}2^h} d\rho \int\limits_{-1}^{1} du_0 \, \frac{1}{\rho}\sqrt{1-u_0^2} + O\left(2^h\right) + O(r.g)$$

$$= \frac{1}{2\pi^2}\ln(2)\frac{4\gamma_1}{9\gamma_0^2}\int\limits_{-1}^{1} du_0 \, \sqrt{1-u_0^2} + O\left(2^h\right) + O(r.g)$$

and

$$\int\limits_{-1}^{1} dx \, \sqrt{1-x^2} \underset{x=cos(\theta)}{=} -\int\limits_{\pi}^{0} d\theta \sin(\theta)|\sin(\theta)|$$

$$= \int\limits_{0}^{\pi} d\theta \, \frac{1-\cos(2\theta)}{2}$$

$$= \frac{\pi}{2}$$

Therefore

$$G_2^{\omega,-\omega(h+1)} = \frac{1}{4\pi^2}\ln(2)\pi\frac{4\gamma_1}{9\gamma_0^2} + O\left(2^h\right) + O(r.g)$$

$\mathbf{G_3}$:

$$G_3^{\omega,\omega'(h+1)} = \frac{1}{\beta|\Lambda|}\sum_{k_0\in\frac{2\pi}{\beta}\left(\mathbb{Z}+\frac{1}{2}\right)}^{f_{h+1}(\mathbf{k})>0}\sum_{\vec{k}\in\hat{\Lambda}}\left(\frac{f_{h+1}(\mathbf{k})\Omega_\omega(\mathbf{k})}{k_0^2+|\Omega_\omega(\mathbf{k})|^2}\right)\left(\frac{f_{h+1}(\mathbf{k})\Omega^*_{\omega'}(\mathbf{k})}{k_0^2+|\Omega_{\omega'}(\mathbf{k})|^2}\right)+O(r.g)$$

$$= \frac{1}{\beta|\Lambda|}\sum_{k_0\in\frac{2\pi}{\beta}\left(\mathbb{Z}+\frac{1}{2}\right)}^{f_{h+1}(\mathbf{k})>0}\sum_{\vec{k}\in\hat{\Lambda}}\frac{\left(f_{h+1}(\mathbf{k})\right)^2\Omega^-_\omega(\mathbf{k})\Omega^-_{\omega'}(\mathbf{k})}{\left(k_0^2+|\Omega^-_\omega(\mathbf{k})|^2\right)\left(k_0^2+|\Omega^-_{\omega'}(\mathbf{k})|^2\right)}\cdot$$

$$\cdot\left(X_\omega(\mathbf{k})\right)^2\left(X^*_{\omega'}(\mathbf{k})\right)^2+O(r.g)$$

$$= \frac{1}{8\pi^3}\int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k}\left(\frac{f_{h+1}(\mathbf{k})\frac{9\gamma_0^2}{4\gamma_1}|\vec{k}|^2}{k_0^2+\frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)^2\left(X_\omega(\mathbf{k})\right)^2\left(X^*_{\omega'}(\mathbf{k})\right)^2+O\left(2^h\right)+O(r.g)$$

If $\omega=-\omega'$, since

$$\Omega^-_\omega(R\mathbf{k}) = \Omega^-_\omega(\mathbf{k})$$

$$|X_\omega(R\mathbf{k})|^4 = |X_\omega(\mathbf{k})|^4$$

$$\left(X_\omega(R\mathbf{k})\right)^2\left(X^*_{-\omega}(R\mathbf{k})\right)^2 = e^{i\omega\frac{4\pi}{3}}\left(X_\omega(\mathbf{k})\right)^2\left(X^*_{-\omega}(\mathbf{k})\right)^2$$

we find

$$G_3^{\omega,-\omega(h+1)} = \frac{1}{24\pi^3}\int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k}\left(\frac{f_{h+1}(\mathbf{k})\frac{9\gamma_0^2}{4\gamma_1}|\vec{k}|^2}{k_0^2+\frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4}\right)^2\left(X_\omega(\mathbf{k})\right)^2\left(X^*_{-\omega}(\mathbf{k})\right)^2\left(\sum_{j=0}^{2}e^{i\omega\frac{4\pi j}{3}}\right)$$

$$+O\left(2^h\right)+O(r.g)$$

$$= 0+O\left(2^h\right)+O(r.g) \qquad \textbf{(A8.6)}$$

If $\omega=\omega'$, we find

$$\left(X_\omega(\mathbf{k})\right)^2\left(X^*_\omega(\mathbf{k})\right)^2 = 1+O(\vec{k}) = 1+O\left(2^h\right)$$

So

$$G_3^{\omega,\omega(h+1)} = \frac{1}{8\pi^3} \int\limits_{f_{h+1}(\mathbf{k})>0} d\mathbf{k} \left(\frac{f_{h+1}(\mathbf{k})\frac{9\gamma_0^2}{4\gamma_1}|\vec{k}|^2}{k_0^2 + \frac{81\gamma_0^4}{16\gamma_1^2}|\vec{k}|^4} \right)^2 + O\left(2^h\right) + O(r.g)$$

Therefore

$$G_3^{\omega,\omega(h+1)} = \frac{1}{4\pi^2}\ln(2)\pi\frac{4\gamma_1}{9\gamma_0^2} + O\left(2^h\right) + O(r.g)$$

So, combining (A8.3), (A8.4), (A8.5), (A8.6) and (A8.7), we have

$$\begin{cases} G_1^{\omega,\omega'(h+1)} = -m + O\left(2^h\right) + O(r.g) \\ G_2^{\omega,\omega'(h+1)} = \delta_{\omega,-\omega'}m + O\left(2^h\right) + O(r.g) \\ G_3^{\omega,\omega'(h+1)} = \delta_{\omega,\omega'}m + O\left(2^h\right) + O(r.g) \end{cases}$$

where

$$m = \frac{1}{\pi}\ln(2)\frac{\gamma_1}{9\gamma_0^2}$$

We also wish to compute

$$V := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1,\omega_1}} \int d\mathbf{x}\ \phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega_1',\rho_1'}\phi^{(\leqslant h)}_{\mathbf{x},\omega_2,\rho_2}\phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega_3',\rho_3'}\phi^{(\leqslant h)}_{\mathbf{x},\omega_4,\rho_4}\cdot$$
$$\cdot\left(\tilde{g}_{\omega_1}(\mathbf{k}_1)_{\rho_1,\rho_2'}\tilde{g}_{\omega_3}(-\mathbf{k}_1)_{\rho_3,\rho_4'}\delta_{\omega_1,\omega_2'}\delta_{\omega_3,\omega_4'} - \tilde{g}_{\omega_1}(\mathbf{k}_1)_{\rho_1,\rho_4'}\tilde{g}_{\omega_3}(-\mathbf{k}_1)_{\rho_3,\rho_2'}\delta_{\omega_1,\omega_4'}\delta_{\omega_3,\omega_2'}\right)$$

which appears when summing over both possible Feynman graphs. So we define

$$\tilde{G}^{\omega,-\omega'(h)}_{\rho_1,\rho_2,\rho_3,\rho_4} := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}^h_{\beta,L}} \tilde{g}^{(h)}_{\omega}(\mathbf{k}_1)_{\rho_1,\rho_2}\tilde{g}^{(h)}_{\omega'}(-\mathbf{k}_1)_{\rho_3,\rho_4} - \tilde{g}^{(h)}_{\omega}(\mathbf{k}_1)_{\rho_1,\rho_4}\tilde{g}^{(h)}_{\omega'}(-\mathbf{k}_1)_{\rho_3,\rho_2}$$

therefore if $\omega_3 = \omega_1 := \omega$, the previous equation can be written

$$V = \delta_{\omega,\omega_2'}\delta_{\omega,\omega_4'}\tilde{G}^{\omega,-\omega(h+1)}_{\rho_1,\rho_2',\rho_3,\rho_4'}\int d\mathbf{x}\ \phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega_1',\rho_1'}\phi^{(\leqslant h)}_{\mathbf{x},\omega_2,\rho_2}\phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega_3',\rho_3'}\phi^{(\leqslant h)}_{\mathbf{x},\omega_4,\rho_4}$$

First we see that for $\tilde{G}$ to be different from 0, we need there to be an even number of 1s and 2s among $\rho_1,\ \rho_2,\ \rho_3,\ \rho_4$. In other words,

$$\sum_{j=1}^4(2\rho_j - 3)\overset{!}{\in}\{-2,0,2\}$$

Furthermore,
$$\tilde{g}^{(h)}_{\omega'}(-\mathbf{k}_1)_{\rho_3,\rho_4} = -(-1)^{\rho_3+\rho_4}\tilde{g}^{(h)}_{-\omega'}(\mathbf{k}_1)_{\rho_4,\rho_3}$$
so
$$\tilde{G}^{\omega,-\omega(h)}_{\rho_1,\rho_2,\rho_3,\rho_4} = -(-1)^{\rho_3}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}^h_{\beta,L}}(-1)^{\rho_4}\tilde{g}^{(h)}_{\omega}(\mathbf{k}_1)_{\rho_1,\rho_2}\tilde{g}^{(h)}_{-\omega}(\mathbf{k}_1)_{\rho_4,\rho_3} -(-1)^{\rho_2}\tilde{g}^{(h)}_{\omega}(\mathbf{k}_1)_{\rho_1,\rho_4}\tilde{g}^{(h)}_{-\omega}(\mathbf{k}_1)_{\rho_2,\rho_3}$$
We will write $*\rho := -\rho+3$. Using (A8.8) we find, up to terms in $O\left(2^h\right)+O(r.g)$
$$\tilde{G}^{\omega,-\omega(h)}_{\rho,\rho,\rho,\rho} = 0$$
$$\tilde{G}^{\omega,-\omega(h)}_{*\rho,\rho,*\rho,\rho} = 0$$
$$\tilde{G}^{\omega,-\omega(h)}_{\rho,\rho,*\rho,*\rho} = 0$$
$$\tilde{G}^{\omega,-\omega(h)}_{*\rho,\rho,\rho,*\rho} = 0$$
So if $\omega_1=\omega_3$, $V=0$.

If $\omega_1 = -\omega_3 := \omega$,
$$V = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}^{h+1}_{\beta,L}}\int d\mathbf{x}\ \phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_1,\rho'_1}\phi^{(\leqslant h)}_{\mathbf{x},\omega_2,\rho_2}\phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_3,\rho'_3}\phi^{(\leqslant h)}_{\mathbf{x},\omega_4,\rho_4}\cdot$$
$$\cdot\left(\tilde{g}_\omega(\mathbf{k}_1)_{\rho_1,\rho'_2}\tilde{g}_{-\omega}(-\mathbf{k}_1)_{\rho_3,\rho'_4}\delta_{\omega,\omega'_2}\delta_{-\omega,\omega'_4} - \tilde{g}_\omega(\mathbf{k}_1)_{\rho_1,\rho'_4}\tilde{g}_{-\omega}(-\mathbf{k}_1)_{\rho_3,\rho'_2}\delta_{\omega,\omega'_4}\delta_{-\omega,\omega'_2}\right)$$
If $\omega'_2 = \omega = -\omega'_4$,
$$V = \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_\omega(\mathbf{k}_1)_{\rho_1,\rho'_2}\tilde{g}_{-\omega}(-\mathbf{k}_1)_{\rho_3,\rho'_4}\int d\mathbf{x}\ \phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_1,\rho'_1}\phi^{(\leqslant h)}_{\mathbf{x},\omega_2,\rho_2}\phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_3,\rho'_3}\phi^{(\leqslant h)}_{\mathbf{x},\omega_4,\rho_4}$$
$$= -(-1)^{\rho_3+\rho'_4}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_\omega(\mathbf{k}_1)_{\rho_1,\rho'_2}\tilde{g}_{\omega}(\mathbf{k}_1)_{\rho'_4,\rho_3}\int d\mathbf{x}\ \phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_1,\rho'_1}\phi^{(\leqslant h)}_{\mathbf{x},\omega_2,\rho_2}\phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_3,\rho'_3}\phi^{(\leqslant h)}_{\mathbf{x},\omega_4,\rho_4}$$
Using (A8.8), for this term to be different from 0, we need
$$\left\{\begin{array}{l}\rho'_2=\rho'_4\\ \rho_1=\rho_3\end{array}\right.\ \text{or}\ \left\{\begin{array}{l}\rho'_2=\rho_1\\ \rho'_4=\rho_3\end{array}\right.$$
If $\omega'_2 = -\omega = -\omega'_4$,
$$V = -\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_\omega(\mathbf{k}_1)_{\rho_1,\rho'_4}\tilde{g}_{-\omega}(-\mathbf{k}_1)_{\rho_3,\rho'_2}\int d\mathbf{x}\ \phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_1,\rho'_1}\phi^{(\leqslant h)}_{\mathbf{x},\omega_2,\rho_2}\phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_3,\rho'_3}\phi^{(\leqslant h)}_{\mathbf{x},\omega_4,\rho_4}$$
$$= (-1)^{\rho_3+\rho'_2}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_\omega(\mathbf{k}_1)_{\rho_1,\rho'_4}\tilde{g}_{\omega}(\mathbf{k}_1)_{\rho'_2,\rho_3}\int d\mathbf{x}\ \phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_1,\rho'_1}\phi^{(\leqslant h)}_{\mathbf{x},\omega_2,\rho_2}\phi^{(\leqslant h)\dagger}_{\mathbf{x},\omega'_3,\rho'_3}\phi^{(\leqslant h)}_{\mathbf{x},\omega_4,\rho_4}$$

Using (A8.8), for this term to be different from 0, we need

$$\begin{cases} \rho'_2 = \rho'_4 \\ \rho_1 = \rho_3 \end{cases} \text{or} \begin{cases} \rho'_4 = \rho_1 \\ \rho'_2 = \rho_3 \end{cases}$$

Appendix A9. Details of the β-function computation

From

$$\mathcal{E}^T_{h+1}\left(\phi^{(h+1)\dagger}_{\mathbf{k}_1,\omega_1,\rho_1}\phi^{(h+1)}_{\mathbf{k}_2,\omega_2,\rho_2};\phi^{(h+1)\dagger}_{\mathbf{k}'_1,\omega'_1,\rho'_1}\phi^{(h+1)}_{\mathbf{k}'_2,\omega'_2,\rho'_2}\right)$$

we get the following terms (for the terms that cancel, I write $(\omega{-}i)$ when they cancel because $\omega_3 = \omega'_4$ or $\omega_4 = \omega'_3$ can't be satisfied):

<u>*bbbb*</u>:

$$-0\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_\omega(\mathbf{k}_3)_{1,1}\tilde{g}_\omega(\mathbf{k}_3)_{1,1}\int d\mathbf{x}\ b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}b^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},-\omega} \qquad (\omega{-}i)$$

$$-4\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_\omega(\mathbf{k}_3)_{1,1}\int d\mathbf{x}\ b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},-\omega}b^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}$$

$$-0\lambda_1^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_\omega(\mathbf{k}_3)_{1,2}\tilde{g}_\omega(\mathbf{k}_3)_{1,2}\int d\mathbf{x}\ b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}b^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},-\omega} \qquad (\omega{-}i)$$

$$-4\lambda_1^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,2}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,1}\int d\mathbf{x}\ b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}b^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},-\omega}$$

$$-0\lambda_2^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\ b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}b^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},-\omega} \qquad (\omega{-}i)$$

$$-2\lambda_2^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\ b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}b^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},-\omega}$$

$$-0\lambda_3^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}^{h+1}_{\beta,L}}\tilde{g}_\omega(\mathbf{k}_3)_{2,2}\tilde{g}_\omega(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\ b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}b^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},-\omega} \qquad (\omega{-}i)$$

$\underline{\tilde{a}\tilde{a}\tilde{a}\tilde{a}}$:

$$-0\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)} \quad {\scriptstyle(\omega-i)}$$

$$-4\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}$$

$$-0\lambda_1^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)} \quad {\scriptstyle(\omega-i)}$$

$$-4\lambda_1^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,1}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,2}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-0\lambda_2^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)} \quad {\scriptstyle(\omega-i)}$$

$$-2\lambda_2^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-0\lambda_3^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)} \quad {\scriptstyle(\omega-i)}$$

$\underline{bb\tilde{a}\tilde{a}}$:

$$-8\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,2}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,1}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}$$

$$-8\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}$$

$$-4\lambda_1^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}$$

$$-4\lambda_1^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}$$

$$-4\lambda_1^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-4\lambda_1^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_2^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,1}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,2}\int d\mathbf{x}\; b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},\omega}$$

$$-2\lambda_2^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\int d\mathbf{x}\; b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},-\omega}\tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}$$

$$-2\lambda_2^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}(\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,1}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,2}+\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,2})$$

$$\int d\mathbf{x}\; b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},-\omega}$$

$$-4\lambda_2^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\int d\mathbf{x}\; b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},-\omega}\tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}$$

$$-4\lambda_2^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,2}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,2}\int d\mathbf{x}\; \tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},-\omega}$$

$$-2\lambda_3^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\int d\mathbf{x}\; b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},\omega}$$

$$-2\lambda_3^{(h+1)}\lambda_3^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\int d\mathbf{x}\; b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}$$

$$-4\lambda_3^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\int d\mathbf{x}\; b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},-\omega}$$

$$-4\lambda_3^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,2}\int d\mathbf{x}\; \tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}b^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},-\omega}$$

$$-8\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\; b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}$$

$$-8\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_\omega\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\; b^{(\leqslant h)\dagger}_{\mathbf{x},\omega}\tilde{a}^{(\leqslant h)}_{\mathbf{x},-\omega}\tilde{a}^{(\leqslant h)\dagger}_{\mathbf{x},-\omega}b^{(\leqslant h)}_{\mathbf{x},\omega}$$

$\underline{b\tilde{a}b\tilde{a}}$:

$$
\begin{aligned}
&-\lambda_2^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\int d\mathbf{x}\; b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\\
&-4\lambda_2^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\; b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\\
&-4\lambda_3^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\; b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}\\
&-4\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,2}\int d\mathbf{x}\; b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}
\end{aligned}
$$

$\underline{\tilde{a}b\tilde{a}b}$:

$$
\begin{aligned}
&-\lambda_2^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,2}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,2}\int d\mathbf{x}\; \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\\
&-4\lambda_2^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\; \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\\
&-4\lambda_3^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,2}\int d\mathbf{x}\; \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}\\
&-4\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\omega}\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{-\omega}(\mathbf{k}_3)_{2,1}\tilde{g}_{\omega}(\mathbf{k}_3)_{2,1}\int d\mathbf{x}\; \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}
\end{aligned}
$$

$\underline{bb b\tilde{a}}$: such a term would be proportional to

$$\sum_{\mathbf{k}_3\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_{\omega}(\mathbf{k}_3)_{1,1}\tilde{g}_{\omega'}(\mathbf{k}_3)_{1,2}=0$$

The same argument can be used to cancel out all terms with three b and one $\tilde{a}$ or three $\tilde{a}$ and one b.

From

$$\mathcal{E}_{h+1}^{T}\left(\phi_{\mathbf{k}_1,\omega_1,\rho_1}^{(h+1)\dagger}\phi_{\mathbf{k}_2,\omega_2,\rho_2}^{(h+1)\dagger};\phi_{\mathbf{k}_1',\omega_1',\rho_1'}^{(h+1)}\phi_{\mathbf{k}_2',\omega_2',\rho_2'}^{(h+1)}\right)$$

we get the following terms

$$-2\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\int d\mathbf{x}\ b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\int d\mathbf{x}\ b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\int d\mathbf{x}\ b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\int d\mathbf{x}\ b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\int d\mathbf{x}\ b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_1^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\int d\mathbf{x}\ b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,1}\tilde{g}_\omega(\mathbf{k}_1)_{1,2}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$+2\lambda_1^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\int d\mathbf{x}\ \tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$-2\lambda_2^{(h+1)}\lambda_2^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde{g}_\omega(\mathbf{k}_1)_{1,1}\tilde{g}_\omega(\mathbf{k}_1)_{2,2}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde{a}_{\mathbf{x},-\omega}^{(\leqslant h)}$$

$$
\begin{aligned}
&+2\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde g_\omega(\mathbf{k}_1)_{1,2}\tilde g_\omega(\mathbf{k}_1)_{2,1}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde a_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde a_{\mathbf{x},-\omega}^{(\leqslant h)}\\
&-2\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde g_\omega(\mathbf{k}_1)_{1,2}\tilde g_\omega(\mathbf{k}_1)_{2,1}\int d\mathbf{x}\ b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde a_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde a_{\mathbf{x},-\omega}^{(\leqslant h)}\\
&-2\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde g_\omega(\mathbf{k}_1)_{1,1}\tilde g_\omega(\mathbf{k}_1)_{1,1}\int d\mathbf{x}\ \tilde a_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde a_{\mathbf{x},\omega}^{(\leqslant h)}\tilde a_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde a_{\mathbf{x},-\omega}^{(\leqslant h)}\\
&+2\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde g_\omega(\mathbf{k}_1)_{1,1}\tilde g_\omega(\mathbf{k}_1)_{1,1}\int d\mathbf{x}\ \tilde a_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}\tilde a_{\mathbf{x},\omega}^{(\leqslant h)}\tilde a_{\mathbf{x},\omega}^{(\leqslant h)\dagger}\tilde a_{\mathbf{x},-\omega}^{(\leqslant h)}\\
&-2\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde g_\omega(\mathbf{k}_1)_{2,2}\tilde g_\omega(\mathbf{k}_1)_{2,2}\int d\mathbf{x}\ b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}\\
&+2\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde g_\omega(\mathbf{k}_1)_{2,2}\tilde g_\omega(\mathbf{k}_1)_{2,2}\int d\mathbf{x}\ b_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}b_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}\\
&+2\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde g_\omega(\mathbf{k}_1)_{2,1}\tilde g_\omega(\mathbf{k}_1)_{1,2}\int d\mathbf{x}\ \tilde a_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde a_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}\\
&-2\lambda_4^{(h+1)}\lambda_4^{(h+1)}\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}_1\in\mathcal{B}_{\beta,L}^{h+1}}\tilde g_\omega(\mathbf{k}_1)_{2,1}\tilde g_\omega(\mathbf{k}_1)_{1,2}\int d\mathbf{x}\ \tilde a_{\mathbf{x},-\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},\omega}^{(\leqslant h)}\tilde a_{\mathbf{x},\omega}^{(\leqslant h)\dagger}b_{\mathbf{x},-\omega}^{(\leqslant h)}
\end{aligned}
$$

From all these terms we find (4.14)