Incommensurate Twisted Bilayer Graphene: emerging quasi-periodicity and stability

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We consider a lattice model of Twisted Bilayer Graphene (TBG). The presence of incommensurate angles produces an emerging quasi-periodicity manifesting itself in large momenta Umklapp interactions that almost connect the Dirac points. We rigorously establish the stability of the semimetallic phase via a Renormalization Group analysis combined with number theoretical properties of irrationals, similar to the ones used in Kolmogorov-Arnold-Moser (KAM) theory for the stability of invariant tori. The interlayer hopping is weak and short ranged and the angles are chosen in a large measure set. The result provides a justification, in the above regime, to the effective continuum description of TBG in which large momenta interlayer interactions are neglected.

I. INTRODUCTION

The discovery that at certain angles Twisted Bilayer Graphene (TBG) develops superconductivity [1] has generated much interest in such materials both for technological and theoretical reasons [2]. It was predicted, using continuum models obtained by keeping only the dominant harmonics in the lattice model [3], [4],[5], that at such angles some strongly correlated behaviour should appear, but not of superconducting type. The mechanism behind the superconductivity remains elusive.

Taking the lattice into account breaks several symmetries of the continuum description [6], [7] leading to effects like the possible shift of Fermi points. More interestingly, for generic angles, excluding a special set [8], [9], one has an incommensurate structure; in such a case Bloch band theory does not apply and one has an emergent quasi-periodicity [10]-[14], with some feature in common with the one in fermions with quasi-periodic potentials [15]-[17]. It is known that electronic quasi-periodic systems have remarkable properties. In 1d they produce a metal-insulator transition [18],[19],[20]. The interplay with a many body interaction produces peculiar phases with anomalous gaps or many body localization [21]–[26]. Quasi-periodicity has been studied also in Weyl semimetals [27], [28],[29] or in the 2d Ising model [30], [31], [32]. It is therefore natural to expect that quasi-periodicity plays an important role in the interacting phases of TBG. Most theoretical analyis are however based on continuum effective description, do not distinguish between commensurate and incommensurate angles, and are based on the assumption that lattice effects preserve the semimetallic phase [3], [4], [5].

We consider a lattice model for TBG consisting of two graphene layers one on top of the other and rotated by an angle θ. The momenta involved in the two-particle scattering process are of the form $k_1 - k_2 + G + G' = 0$ with $G = l_1 b_1 + l_2 b_2 \equiv lb$, $G' = m_1 b'_1 + m_2 b'_2 \equiv mb'$, $l \equiv (l_1, l_2) \in \mathbb{Z}^2$, $m \equiv (m_1, m_2) \in \mathbb{Z}^2$, and b_1, b_2 are the vectors of the reciprocal lattice and $b'_i = R^T(\theta) b_i$ the reciprocal lattice of the twisted layer in which $R(\theta)$ is the rotation matrix; the terms involving non zero G, G' are also known as Umklapp interactions. Note that, apart from special angles, G' is not commensurate with G and the effect of the mismatch of the lattices is very similar to the effect of a quasi-periodic potential. This is quite clear comparing for instance with the conservation law of 1d fermions with Aubry-André potential $\cos 2\pi\omega x$, which is $k_1 - k_1 + 2l\pi + 2\pi\omega m = 0$ with ω irrational.

It is expected that the relevant processes in TBG are the ones connecting the Dirac points as closely as possible, that is the terms that minimize the quantity $|G + G' + p_{F,i} - p'_{F,j}|$ where $p_{F,i}$ $p'_{F,j}$ are the Dirac points of the two layers. The approximation at the basis of the effective models [3], [4],[5] consists in taking restricting the interaction to only the terms $G = G' = 0$ or $G = b_1, G' = -b'_1$ or $G = b_2, G' = -b'_2$ and taking the continuum limit, based on the fact that larger values of G or G' are exponentially depressed [33]. However in the incommensurate case, Umklapp terms with very large values of G, G' make $|G + G' + p_{F,i} - p'_{F,j}|$ arbitrarely small, producing almost relevant processes which can destroy the semimetallic behaviour. In the 1d Aubry-André model the processes that produce small values for $2\varepsilon p_F + 2l\pi + 2\pi\omega m$, $\varepsilon = 0, \pm 1$ are indeed the ones producing the insulating behaviour at large coupling, while at weak coupling the metallic regime persists. Similarly the persistence or not of the semimetallic regime in TBG depends on the relevance or irrelevance of the terms involving large G, G' that almost connect the Dirac points. This fact cannot be understood only on the basis of perturbative arguments; it is indeed a non perturbative phenomenon which can be established only by the convergence or divergence of the whole series expansion. Despite the similarity of quasi-periodic potentials and incommensurate TBG, there are crucial differences like the higher dimensionality of TBG and the fact that

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the frequencies are not independent parameters but are functions of a single parameter, the angle between the layers, and this produces rather different small divisors.

The aim of this paper is to investigate when the quasi-metallic phase is stable against the large momentum processes in the incommensurate case. The analysis is based on Renormalization Group methods combined with number theoretical properties of irrationals, similar to the ones used in Kolmogorov-Arnold-Moser (KAM) theory for the stability of invariant tori. Due to the difficulty of getting information on the single particle spectrum, we analyze the behavior of the Euclidean correlations, which provide information on the spectrum close to the Fermi points. Such methods are robust enough to be extended to many-body systems, as it was done for the interacting Aubry-André model [26] or in Weyl semimetals [29]. Our main result is the proof of the stability of the semimetallic phase in a large measure set of angles in the incommensurate case.

The paper is organized in the following way. In Section II the lattice model of TBG is presented. In Section III a perturbative expansion for the correlations is derived. In Section IV the emerging quasi-periodicity and the small divisor problem is described, together with the required (number theoretical) Diophantine conditions. Section V contains a statement of the main result and in Section VI the Renormalization Group derivation is presented. The Appendices detail the more technical aspects of the analysis.

II. INCOMMENSURATE TBG

We consider the lattice TBG model introduced in [3], [4]. We focus on this model for the sake of definiteness but our methods could be applied more generally. We consider two graphene layers rotated with respect to one another by an angle θ around a point $\xi = (0, 1/2)$ (that is, the point between an a and b atom, chosen so that the twisted model preserves the C_2T symmetry in Appendix D 1). The Hamiltonian of the system will be written as

$$H = H_1 + H_2 + V \tag{1}$$

where H_1 and H_2 are hopping Hamiltonians within the layers 1 and 2 respectively and V is an interlayer hopping term. The first graphene layer is defined on the lattice $\mathcal{L}_1 := \{n_1 A_1 + n_2 A_2,\ n_1, n_2 \in \mathbb{Z}\}$ with $A_1 = \frac{1}{2}(3, \sqrt{3})$, $A_2 = \frac{1}{2}(3, -\sqrt{3})$. We introduce the nearest-neighbor vectors: $\delta_1 = (1, 0)$, $\delta_2 = \frac{1}{2}(-1, \sqrt{3})$, $\delta_2 = \frac{1}{2}(-1, -\sqrt{3})$. We will write the Hamiltonian in second quantized form: for $x \in \mathcal{L}_1$, we introduce *annihilation operators* $c_{1,x,a}$ and $c_{1,x,b}$ corresponding respectively to annihilating a fermion located at x and $x + \delta_1$. The nearest neighbor hopping Hamiltonian is

$$H_1 = -t \sum_{x \in \mathcal{L}_1} \sum_{i=0}^{2} (c^\dagger_{1,x,a} c_{1,x+A_i,b} + c^\dagger_{1,x+A_i,b} c_{1,x,a}) \tag{2}$$

where $A_0 := 0$ (note that $\delta_1 - \delta_2 = A_2, \delta_2 - \delta_3 = A_3$). We will do much of the computation in Fourier space, and we here introduce the Fourier transform $\hat{c}_{1,k,\alpha}$ of $c^\pm_{1,x,\alpha}$ in such a way that, for $\alpha \in \{a, b\}$,

$$c_{1,x,\alpha} = \frac{1}{|\hat{\mathcal{L}}_1|} \int_{\hat{\mathcal{L}}_1} dk\ e^{-ik(x-\xi)} \hat{c}_{1,k,\alpha} \tag{3}$$

with $|\hat{\mathcal{L}}_1| = 8\pi^2/3\sqrt{3}$, and $\hat{\mathcal{L}}_1 := \mathbb{R}^2/(b_1\mathbb{Z} + b_2\mathbb{Z})$ in which

$$b_1 = \tfrac{2\pi}{3}(1, \sqrt{3}), \quad b_2 = \tfrac{2\pi}{3}(1, -\sqrt{3}). \tag{4}$$

In Fourier space,

$$H_1 = \frac{t}{|\hat{\mathcal{L}}_1|} \int_{\hat{\mathcal{L}}_1} dk\ \left(\Omega(k) \hat{c}^\dagger_{1,k,a} \hat{c}_{1,k,b} + \Omega^*(k) \hat{c}^\dagger_{1,k,b} \hat{c}_{1,k,a} \right) \tag{5}$$

with $\Omega(k_x, k_y) := 1 + 2e^{-i\frac{3}{2}k_x} \cos(\frac{\sqrt{3}}{2}k_y)$.

The second graphene layer is rotated by an angle θ around the point $\xi = (0, 1/2)$, that is, it is defined on the lattice

$$\mathcal{L}_2 = \xi + R(\theta)(\mathcal{L}_1 - \xi), \quad R(\theta) = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \tag{6}$$

(we use the shorthand throughout this paper that $c_\theta \equiv \cos\theta, s_\theta \equiv \sin\theta$). The annihilation operators in the second layer are denoted by $c_{2,x,a}$ and $c_{2,x,b}$. The hopping Hamiltonian of this second layer is

$$H_2 = -t \sum_{x \in \mathcal{L}_2} \sum_{i=0}^{2} (c^\dagger_{2,x,a} c_{2,x+RA_i,b} + c^\dagger_{2,x+RA_i,b} c_{2,x,a}) \tag{7}$$

where $R \equiv R(\theta)$. We define the Fourier transform in the second layer: if $b'_1 := Rb_1$, $b'_2 := Rb_2$ and

$$c_{2,x,\alpha} = \frac{1}{|\hat{\mathcal{L}}_1|} \int_{\hat{\mathcal{L}}_2} dk\ e^{-ik(x-\xi)} \hat{c}_{2,k,\alpha} \tag{8}$$

we find

$$\begin{aligned} H_2 = \frac{t}{|\hat{\mathcal{L}}_1|} \int_{\hat{\mathcal{L}}_2} dk\ \cdot \\ \cdot \left(\Omega(R^T k) \hat{c}^\dagger_{2,k,a} \hat{c}_{2,k,b} + \Omega^*(R^T k) \hat{c}^\dagger_{2,k,b} \hat{c}_{2,k,a} \right) . \end{aligned} \tag{9}$$

In the absence of interlayer coupling the two graphene layers are decoupled; the single particle spectrum for layer 1 is $\pm|\Omega(k)|$ and the Fermi points are given by the relation $\Omega(p^\pm_{F,1}) = 0$ with

$$p^\pm_{F,1} = \frac{2\pi}{3}(1, \pm\tfrac{1}{\sqrt{3}}) \tag{10}$$

for momenta close to such points one has $|\Omega(k)| \sim \frac{3}{2}t|k - p^\pm_{F,1}|$, that is the dispersion relation is almost linear (relativistic) up to quadratic corrections, forming approximate *Dirac cones.* In the same way the dispersion relation for layer 2 is $\pm|\Omega(R^T k)|$; the Fermi points are $\Omega(R^T p^{\pm}_{F,2}) = 0$ with $p^{\pm}_{F,2} = R(p^{\pm}_{F,1})$ and $|\Omega(R^T k)| \sim \frac{3}{2}t|k - p^{\pm}_{F,2}|$. We are interested in understanding how these four Dirac cones are modified in the presence of the interlayer hopping.

We couple the 2 layers by an interlayer hopping Hamiltonian, which couples atoms of type a to atoms of type b:

$$V = \lambda \sum_{x_1 \in \mathcal{L}_1} \sum_{x_2' \in \mathcal{L}_2} \sum_{\alpha \in \{a,b\}} \varsigma(x_1 + d_\alpha - x_2' - Rd_\alpha) \cdot \cdot (c^\dagger_{1,x_1,\alpha} c_{2,x_2',\alpha} + c^\dagger_{2,x_2',\alpha} c_{1,x_1,\alpha}) \tag{11}$$

$d_a = (0,0), d_b = \delta_1$, and $\varsigma(x) = \varsigma(-x)$,

$$\varsigma(x_1 - x_2) = \int_{\mathbb{R}^2} \frac{dq}{4\pi^2}\, e^{iq(x_1-x_2)} \widehat{\varsigma}(q), \quad |\widehat{\varsigma}(q)| \le e^{-\kappa|q|}. \tag{12}$$

We restrict the interlayer term to hoppings between atoms of type a to atoms of type a and type b to b for technical reasons. This is so that the "Inversion" symmetry in Appendix D 1 is satisfied. We could just as easily consider a model where the interlayer hopping occurs only between atoms of type a to atoms of type b. We could relax this restriction and allow for all possible hoppings, but we would then need to add extra counterterms (see (15)) to the model. For the sake of simplicity, we avoid this and only consider these interlayer hoppings.

Note that, whereas the Fourier transform for c is defined on $\widehat{\mathcal{L}}_i$, the Fourier transform of ς is defined on all $\mathbb{R}^2$. We write V in Fourier space: we get, see App. A

$$V = \frac{\lambda}{4\pi^2|\widehat{\mathcal{L}}_1|} \sum_\alpha \Bigg(\sum_{l \in \mathbb{Z}^2} \int_{\widehat{\mathcal{L}}_1} dk\; \tau^{(1)}_{l,\alpha}(k + lb) \widehat{c}^\dagger_{1,k,\alpha} \widehat{c}_{2,k+lb,\alpha} + \sum_{m \in \mathbb{Z}^2} \int_{\widehat{\mathcal{L}}_2} dk\; \tau^{(2)}_{m,\alpha}(k + mb') \widehat{c}^\dagger_{2,k,\alpha} \widehat{c}_{1,k+mb',\alpha} \Bigg)$$

where we use the notation $lb \equiv l_1 b_1 + l_2 b_2$, $mb' \equiv m_1 b_1' + m_2 b_2'$, and

$$\tau^{(1)}_{l,\alpha}(k) := e^{i\xi lb} e^{-ik(d_\alpha - Rd_\alpha)} e^{-i\xi \sigma_{k,1} b'} \widehat{\varsigma}^*(k) \tag{13}$$

$$\tau^{(2)}_{m,\alpha}(k) := e^{i\xi mb'} e^{-ik(d_\alpha - Rd_\alpha)} e^{-i\xi \sigma_{k,2} b} \widehat{\varsigma}(k) \tag{14}$$

in which $\sigma_{k,i} \in \mathbb{Z}^2$ is the unique integer vector such that $k - \sigma_{k,1} b' \in \widehat{\mathcal{L}}_2$, $k - \sigma_{k,2} b \in \widehat{\mathcal{L}}_1$. Note that the difference of the momenta of the two fermions is given by $lb + mb'$.

The position of the Dirac points are in general modified (renormalized) by the interlayer hopping. It is conventient to fix the values of the renormalized Dirac points by properly choosing the bare ones. This can be achieved by replacing $\Omega(k)$ with $\Omega(k) + \nu_{i,\omega}$ close to each Dirac points, that is adding a counterterm has the form

$$M = \sum_{\omega \in \{+,-\}} \sum_{i=1,2} \int_{\widehat{\mathcal{L}}_i} dk\; \chi_{\omega,i}(k) (\nu_{i,\omega} \widehat{c}^\dagger_{i,k,a} \widehat{c}_{i,k,b} + \nu^*_{i,\omega} \widehat{c}^\dagger_{i,k,b} \widehat{c}_{i,k,a}) \tag{15}$$

where $\chi_{\omega,i}(k)$ is a smooth compactly supported function that is non vanishing for $||k - p^\omega_{F,i}||_i \le 1/\gamma$, for some $\gamma > 1$, in which $||.||_i$ is the norm on the torus $\widehat{\mathcal{L}}_i$.

Our main result concerns the two-point Schwinger function, which we define as follows. We first introduce a Euclidean time component: given an inverse temperature $\beta > 0$, we define for $x_0 \in [0, \beta)$,

$$c_{j,x,\alpha}(x_0) := e^{-x_0 \bar{H}} c_{j,x,\alpha} e^{x_0 \bar{H}}. \tag{16}$$

and $\bar{H} = H + M$. Combining the Euclidean time component with the spatial one, we define $\Lambda_i := [0, \infty) \times \mathcal{L}_i$. The corresponding Fourier-space operators are

$$\widehat{c}_{j,k,\alpha}(k_0) = \int_0^\beta dx_0 e^{-ix_0 k_0} \sum_{x \in \mathcal{L}_j} e^{-i(x-\xi)k} c_{j,x,\alpha}(x_0) \tag{17}$$

which is defined for $(k_0, k) \in \widehat{\Lambda}_j := \frac{2\pi}{\beta}\mathbb{Z} \times \widehat{\mathcal{L}}_j$.

Now, given $j, j' \in \{1,2\}$, $\mathbf{k} = (k_0, k) \in \Lambda_j$, the two-point Schwinger function is defined as the 2×2 matrix $S_{j,j'}(\mathbf{k})$ whose components are indexed by $\alpha, \alpha' \in \{a, b\}$:

$$(\widehat{S}_{j,j'}(\mathbf{k}))_{\alpha,\alpha'} := \frac{\mathrm{Tr}(e^{-\beta \bar{H}} T(\widehat{c}_{j,k,\alpha}(k_0), \widehat{c}^\dagger_{j',k,\alpha'}(k_0)))}{\mathrm{Tr}(e^{-\beta \bar{H}})} \tag{18}$$

where T is the time ordering operator, which is bilinear, and is defined in real-space $T(c_{j,x,\alpha}(x_0), c^\dagger_{j',y,\alpha'}(y_0)) =$

$$\begin{cases} c_{j,x,\alpha}(x_0) c^\dagger_{j',y,\alpha'}(y_0) & \text{if } x_0 < y_0 \\ -c^\dagger_{j',y,\alpha'}(y_0) c_{j,x,\alpha}(x_0) & \text{if } x_0 \geqslant y_0. \end{cases} \tag{19}$$

To compute the Schwinger functions, we will use the Grassmann integral formalism. To do so, we add an imaginary time component to all position vectors: we define $\mathbf{A}_i := (0, A_i)$. Given $\mathbf{x} = (x_0, x) \in \Lambda_j \equiv [0, \beta) \times \mathcal{L}_j$, we introduce Grassmann variables $\psi^\pm_{\mathbf{x},a,j}, \psi^\pm_{\mathbf{x},b,j}$ and their Fourier transforms

$$\psi^\pm_{j,\mathbf{x},\alpha} = \frac{1}{|\widehat{\Lambda}_j|} \int_{\widehat{\Lambda}_j} d\mathbf{k}\; e^{\pm i\mathbf{k}(\mathbf{x}-\boldsymbol{\xi})} \widehat{\psi}^\pm_{j,\mathbf{k},\alpha} \tag{20}$$

with $|\widehat{\Lambda}_j| = 8\pi^2/3\sqrt{3}$, $\boldsymbol{\xi} = (0, \xi)$ and $\widehat{\Lambda}_j = \frac{2\pi}{\beta}\mathbb{Z} \times \widehat{\mathcal{L}}_j$. The Schwinger fiunctions are given by

$$(S_{j,j'}(\mathbf{x}, \mathbf{y}))_{\alpha,\alpha'} := \frac{\int P(d\psi)\; e^{-\beta(V(\psi) + M(\psi))} \psi^-_{j,\mathbf{x},\alpha} \psi^+_{j',\mathbf{y}\,\alpha'}}{\int P(d\psi)\; e^{-\beta(V(\psi)+M(\psi))}} \tag{21}$$

where $P(d\psi) = P(d\psi_1) P(d\psi_2)$ where $P(d\psi_1)$ is the Grassmann integration with propagator

$$\widehat{g}_1(\mathbf{k}) = \begin{pmatrix} -ik_0 & \Omega(k) \\ \Omega^*(k) & -ik_0 \end{pmatrix}^{-1}. \tag{22}$$

and $g_1(\mathbf{x}, \mathbf{y}) = \frac{t}{|\widehat{\mathcal{L}}_1|} \int_{\widehat{\Lambda}_1} d\mathbf{k} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \widehat{g}_1(\mathbf{k})$ and

$$\widehat{g}_1(\mathbf{k} + p^\pm_F) \sim \begin{pmatrix} -ik_0 & -v_F(-ik_1 \pm k_2) \\ -v_F(ik_1 \pm k_2) & -ik_0 \end{pmatrix}^{-1}. \tag{23}$$

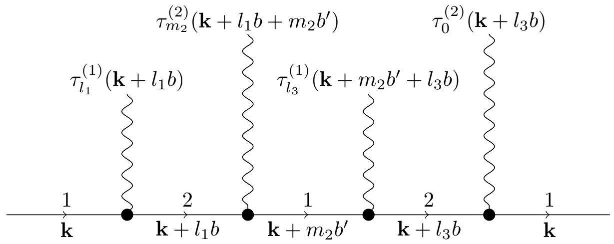

FIG. 1: Example of a Feynman diagram for $S_{1,1}(\mathbf{k})$ with 4 vertices.

$P(d\psi_2)$ has propagator

$$g_2(\mathbf{x},\mathbf{y}) = \frac{1}{|\widehat{\mathcal{L}}_1|}\int_{\widehat{\Lambda}_2} d\mathbf{k} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})}\widehat{g}_2(\mathbf{k}) \tag{24}$$

with $\widehat{g}_2(\mathbf{k}) = \begin{pmatrix} -ik_0 & \Omega(R^Tk) \\ \Omega^*(R^Tk) & -ik_0 \end{pmatrix}^{-1} \equiv \widehat{g}_1(R^T\mathbf{k})$ which is singular at $p^{\pm}_{F,2} := Rp^{\pm}_{F,1}$ and periodic in k with period b'_1, b'_2. The interaction and counterterm are rewritten formally in terms of Grassmann variables:

$$\begin{aligned} V(\psi) = \; & \frac{\lambda}{4\pi^2|\widehat{\Lambda}_1|}\sum_{\alpha} \\ & \cdot\Bigg(\sum_{l\in\mathbb{Z}^2}\int_{\widehat{\Lambda}_1} d\mathbf{k}\; \tau^{(1)}_{l,\alpha}(k+lb)\widehat{\psi}^+_{1,\mathbf{k},\alpha}\widehat{\psi}^-_{2,\mathbf{k}+l\mathbf{b},\alpha} \\ & + \sum_{m\in\mathbb{Z}^2}\int_{\widehat{\Lambda}_2} d\mathbf{k}\; \tau^{(2)}_{m,\alpha}(k+mb')\widehat{\psi}^+_{2,\mathbf{k},\alpha}\widehat{\psi}^-_{1,\mathbf{k}+m\mathbf{b}',\alpha}\Bigg) \end{aligned} \tag{25}$$

$$M(\psi) = \sum_{\omega,1}\int d\mathbf{k}\,\chi_{\omega,i}(k)(\nu_{i,\omega}\psi^+_{i,\mathbf{k},\alpha}\psi^-_{i,\mathbf{k},\beta} + \nu^*_{i,\omega}\psi^+_{i,\mathbf{k},\beta}\psi^-_{i,\mathbf{k},\alpha}) \tag{26}$$

III. PERTURBATIVE EXPANSION AND SMALL DIVISORS

To compute the Schwinger functions $S(\mathbf{k})$, we will use Feynman graph expansions. We give the rule setting $\nu = 0$ for definiteness. The graphs for this model are chain graphs of the form depicted in Figure 1

Each line s is associated a layer label $i_s \in \{1,2\}$ and two valley indices $\alpha_s, \alpha'_s \in \{a,b\}$. Each vertex has one entering line s_1 and an exiting line s_2, and we impose that these lines have different indices: $i_{s_1} = 3 - i_{s_2}$, and the same valley indices $\alpha_{s_1} = \alpha_{s_2}$. To each vertex r that has an entering line s_1 and exiting line s_2, we associate an index, which if $i_{s_1} = 1$ we denote by $l_s \in \mathbb{Z}^2$, and if $i_{s_2} = 2$ we denote by $m_s \in \mathbb{Z}^2$.

- Each internal line s with layer label 1 coming from a vertex with index m_s corresponds to the propagator $g_{1;\alpha_s,\alpha'_s}(\mathbf{k} + m_rb')$ (where $\mathbf{k}$ is the momentum at which S is being evaluated). Each internal line s with layer label 2 coming from a vertex with index l_s corresponds to the propagator $g_{2;\alpha_s,\alpha'_s}(\mathbf{k} + l_rb)$.
- Each external line s corresponds to the propagator $g_{i_s:\alpha_s,\alpha'_s}(\mathbf{k})$.
- Each vertex r that has an entering line s_1 and exiting line s_2 with $i_{s_1} = 1$ and $i_{s_2} = 2$, if s_1 comes from a vertex with index m_{r-1}, the vertex r corresponds to an interaction term $\lambda\tau^{(1)}_{l_r,\alpha_{s_2},\alpha'_{s_1}}(\mathbf{k}+m_{r-1}b'+l_rb)$.
- Each vertex r that has an entering line s_1 and exiting line s_2 with $i_{s_1} = 2$ and $i_{s_2} = 1$, if s_2 comes from a vertex with index l_{r-1}, the vertex r corresponds to an interaction term $\lambda\tau^{(2)}_{m_r,\alpha_{s_2},\alpha'_{s_1}}(\mathbf{k} + l_{r-1}b + m_rb')$.

The value of a graph is obtained by taking the product over the lines of the corresponding propagators, and multiplying them by the product over the vertices of the interaction terms. The Schwinger function is then obtained by taking a sum over all possible graphs with the correct external labels. A more explicit expression for the Schwinger function is given in Appendix C.

The persistence or not of the semimetallic behaviour depends on the convergence of the expansion. One of the more important and difficult reasons for which convergence could, in principle, not occur, is that small divisors could accumulate, as we will now explain. Note that if $\mathbf{k}_1$ and $\mathbf{k}_2$ are 2 neighboring terms their difference is given by $k_1 - k_2 + mb' + lb$; moreover the propagator are singular at the Dirac points $p^{\omega,i}_F$. Consider therefore $g_1(\mathbf{k})\tau^{(1)}_l(k+lb)g_2(\mathbf{k}+lb)$ and suppose that k is in the first Brillouin zone and is close to $p^{\omega}_{F,1}$, that is $k = p^{\omega}_{F,1} + r_1$ with $r_1 = O(\varepsilon)$ and ε is a small parameter (so that the propagator is $O(\varepsilon^{-1})$); suppose now that $k + lb + mb'$ is also in the first Brillouin zone (by periodicity we can always add mb' to achieve this) and close to $p^{\omega'}_{F,2}$, that is $k + lb + mb' = p^{\omega}_{F,2} + r_2$ with $r_2 = O(\varepsilon)$. In principle this would produce an $O(\varepsilon^{-2})$ contribution, that is an accumulation of small divisors which could produce large contributions which can destroy convergence. Note however that

$$O(\varepsilon) = |r_1|+|r_2| \geq |r_1-r_2| = |lb+mb'-p^{\omega'}_{F,2}-p^{\omega}_{F,1}| \tag{27}$$

so that this accumulation of small divisors is possible only when the quantity in the r.h.s. is small. The effect of such terms is rather delicate to be understoord. Indeed similar terms in the case of random potential produce a localization phase in 1d, while an extended phase in hgher dimension, while a quasi periodic disorder in 1d produces an extended phase.

IV. DIOPHANTINE CONDTIONS

As we noted above, the accumulation of small divisors can only occur when $|lb + mb' + p^{\omega}_{F,i} - p^{\omega'}_{F,j}|$ is small. For generic values of θ (for which s_θ or c_θ is irrational), this happens when l, m are large enough, and the basic issue for stability is if small divisors are balanced by the fact that the interlayer hopping is weak for large values of l, m. We therefore need to quantify the relation between the size of $lb+mb'+p^{\omega}_{F,i}-p^{\omega'}_{F,j}$ and that of l, m, which we do using number theoretical considerations. One such consideration is the so called *Diophantine* condition, which consists in restricting ourselves to values of θ for which, for any ω, ω', i, j

$$|p^{\omega}_{F,i}-p^{\omega'}_{F,j}+lb+mb'| \geq \frac{C_0}{|y|^\tau}. \tag{28}$$

where y is either $y=l$ or $y=m$ and $y\neq 0$, If this is satisfied, then when $|p^{\omega}_{F,i}-p^{\omega'}_{F,j}+lb+mb'| = O(\epsilon)$, then

$$O(\varepsilon) \geq C_0|l|^{-\tau}. \tag{29}$$

Therefore $|l| \geq \varepsilon^{-1/\tau}$, and, due to the exponential decay of $\widehat{\varsigma}$ in $\tau^{(1)}$ (see (12)), so $|\tau^{(1)}_l(k+lb)| \leq e^{-\kappa\varepsilon^{-1/\tau}}$ which compensates the small divisors ε^{-2}. This simple argument says that small divisors due to adjacent propagators do not accumulate. Of course this argument is not sufficient to prove the convergence of the series, and then the persistence of cones; it says only that two adjacent propagators cannot be simultaneously small but it says nothing about non adjacent ones, which is the generic case. This requires a Renormalization Group analysis, as we will discuss below.

A basic preliminary question is whether there are θ's such that (28) holds. Such condition is usually assumed in KAM theory or in the case of fermions with quasi periodic potentials; in such cases the frequencies are independent so that the fact that there is a large measure set of them verifying (28) follows from standard notions of number theory. In the present case however the issue is much more subtle as all the frequencies depend on a single parameters θ. Nevertheless, we prove the following lemma.

Lemma IV.1. *For every interval $[\theta_0,\theta_1]\subset[0,2\pi)$, the set $\mathcal{D} := \{\theta\in[\theta_0,\theta_1]$ satisfying (28)$\}$ has measure $1-O(C_0/(\theta_1-\theta_0)^2)$, so that choosing C_0 small enough with respect to $\theta_1-\theta_0$, $\mathcal{D}$ has large relative measure.*

We will provide a full proof of this lemma in Appendix B. For the moment, let us discuss the main idea of this proof. In order to keep things simple, we will focus on the case $i=j$ and $\omega=\omega'$ here. Let $M := lb+mb'$ and we wish to obtain a lower bound on $|M|$. To do so, we use a simple inequality: $\forall x, y\in\mathbb{R}$,

$$\sqrt{x^2+y^2} \geq \frac{\sqrt{3}}{2}x - \omega\frac{1}{2}y \tag{30}$$

and so, since

$$M = \tfrac{2\pi}{3}\big(\begin{array}{l} l_1+l_2+m_1\varphi_1+m_2\varphi_2, \\ \sqrt{3}(l_1-l_2+m_1\varphi_3-m_2\varphi_4)\big)\end{array} \tag{31}$$

with $\varphi_1 = c_\theta - s_\theta\sqrt{3}$, $\varphi_2 = c_\theta + s_\theta\sqrt{3}$, $\varphi_3 = c_\theta + s_\theta/\sqrt{3}$, $\varphi_4 = c_\theta - s_\theta/\sqrt{3}$, we have

$$|M| \geq \frac{2\pi}{\sqrt{3}}(l_\omega + m\cdot f_\omega) \tag{32}$$

with $l_+ \equiv l_1$ and $l_- \equiv l_2$, and

$$f_\omega(\theta) := \left(\tfrac{\varphi_1-\omega\varphi_3}{2},\ \tfrac{\varphi_2+\omega\varphi_4}{2}\right). \tag{33}$$

Thus, we wish to impose a condition on θ such that $g_{l_\omega,m}(\theta) := |l_\omega + m\cdot f_\omega(\theta)| \geqslant C_1|m|^{-\tau}$. The measure of the complement of the set where this is true is bounded by

$$\sum^{*}_{m,l_\omega}\int_{-C_1|m|^{-\tau}}^{C_1|m|^{-\tau}} \frac{1}{g'_{l_\omega,m}}dg_{l_\omega,m} \tag{34}$$

where $g'_{l_\omega,m}$ is the derivative of $g_{l_\omega,m}$, and $\sum^*_{k,l}$ has the constraint that $\exists\theta\in[\theta_0,\theta_1]$ such that $g_{l_\omega,m}(\theta)\in[-C_1|k|^{-\tau}, C_1|k|^{-\tau}]$. Therefore we get the bound, taking into account the sum over l_ω,

$$\sum^{*}_{m} 2C_1\frac{|m|^{1-\tau}}{\min_\theta|m\cdot f'_\omega(\theta)|} \tag{35}$$

Now, in order for this bound to be useful, we need a good bound on $m\cdot f'_\omega$. Now $m\cdot f'_\omega(\theta)$ can be small, but only if m is large enough. This suggests we should control it using another Diophantine condition, but we would run into an infinite problem: we would need mf''_ω to be bounded below, for which we would impose an extra Diophantine condition, which would itself require a Diophantine condition on the third derivatives, et cætera. We can avoid this problem as follows. If $f'_\omega(0) = |f'_\omega(0)|(\cos\beta,\sin\beta)$ is non vanishing and θ is small then $(m\cdot f'_\omega/|f'_\omega|)\sim|m|\cos(\theta_m)$ where θ_m is the angle between β and m. We can distinguish in the sum (35) the sum over m such that $|\theta_m|, |\theta_m-\pi| < \pi/4$ and the complementary set $|\theta_m-\pi/2|, |\theta_m-3\pi/2| < \pi/4$. In the first sum, $(m\cdot f'_\omega/|f'_\omega|)$ will be greater than $|m|$ up to some constant; we impose a Diophantine condition only for the second term: for $|\theta_m-\pi/2|, |\theta_m-3\pi/2| < \pi/4$ we assume that $|l_\omega + m\cdot f'_\omega| \leq C_1|m|^{-\tau}$. Again we will end-up with a condition like (35) involving $m\cdot(f'_\omega/|f'_\omega|)'$ which in principle could be arbitrarily small. However $(f'/|f'|)'$ is orthogonal to $f'_\omega/|f'_\omega|$ hence $m\cdot(f'_\omega/|f'_\omega|)'\sim|m|\cos(\theta_m+\pi/2)$ which, in this region, is greater than $|m|$.

Thus, we construct a large-measure set of θ's that satisfy $|l_\omega + m\cdot f_\omega(\theta)| \geqslant C_1|m|^{-\tau}$. A detailed proof can be found in Appendix B.

V. STABILITY OF THE SEMIMETALLIC PHASE

We are now ready to state our main result.

Theorem *For any interval $[\theta_0,\theta_1]\subset[0,2\pi)$, any $C_0>0$, there exists a subset of $[\theta_0,\theta_1]$ whose measure is at* *least $1-O(C_0/(\theta_0-\theta_1)^2)$ such that (28) holds, and an ε_0 (depending on C_0,θ_0,θ_1), such that, for any $|\lambda|\le\varepsilon_0$, for a suitable choice of $\nu_{i,\omega}$ ($\widehat{S}_{j,j}(\mathbf{k}+\mathbf{p}^{\omega}_{F,j})) =$*

$$\begin{pmatrix} -iZ_{j,\omega}k_0 & (iv_{j,\omega}k_1 - w_{j,\omega}\omega k_2) \\ (-iv^*_{j,\omega}k_1 - w^*_{j,\omega}\omega k_2) & -iZ_{j,\omega}k_0 \end{pmatrix}^{-1} (1+O(|k|^{\alpha})) \tag{36}$$

with $0\le\alpha\le 1$, $Z=1+O(\lambda)$ real and $v_{i,\omega}=3t/2+O(\lambda)$, $w_{i,\omega}=3t/2+O(\lambda)$, $\nu_{j,\omega}=O(\lambda)$.

This result ensures that, even taking into account the Umklapp processes involving the exchange of very high momenta due to the emerging quasi-periodicity, the Weyl semimetallic phase persists for small interalyer coupling and a large measure set of angles.

The interlayer coupling modifies the position of the Dirac points; we have properly chosen the bare Dirac points $p^{\pm}_{F,i}(\lambda)$ in absence of interlayer coupling given by $\Omega(p^{\pm}_{F,i}(\lambda))+\nu_{i,\pm}=0)$ so that their renormalized physical value is $p^{\pm}_{F,i}$ given by (10). This is essentially equivalent to say that the position of the Dirac points genericaly moves in a way depending on the angle, the layer and the coupling.

The velocities $w_{j,\omega}, v_{j,\omega}$ and the wave function normalization $Z_{j,\omega}$, are renormalizated in a way generically dependent on the layer and the angle. Note that a priori several other relevant terms could be present, but they are excluded by symmetry. The singularity of the Schiwnger function is given by $Z^2k_0^2+R(k)$ with $R(k)\sim(|v|^2k_1^2+|w|^2k_2^2)$; the singularity of the 2-point function is therefore the same as in absence of interlayer at weak coupling ensuring the stability of the semimetallic phase.

The result holds for irrational twisting angles verifying (28). The relative measure of this set can be made arbitrarely close to 1 by decreasing C_0. In the remaining sections we prove the above result by a Renormalization Group analysis, leading to a convergent expansion.

VI. THE RENORMALIZED EXPANSION

A. Multiscale decomposition

We introduce smooth cut-off functions: for $i=1,2$, $\omega=\pm$, $h\in\{-\infty,\cdots,0\}$, let $\chi_{h,i,\omega}(\mathbf{k})$ be a smooth function that vanishes outside the region $||\mathbf{k}-\mathbf{p}^{\omega}_{F,i}||\le\gamma^{h-1}$ and that is equal to 1 for $||k-\mathbf{p}^{\omega}_{F,i}||\ge\gamma^{h-2}$. The constant $\gamma>1$ will be chosen below to be large enough. Note that, in this way, the supports of $\chi_{0,i,+}$ and $\chi_{0,i,-}$ do not overlap. We define $\widehat{g}^{(\le 0)}_{i,\omega}(\mathbf{k})=\chi_{0,i,\omega}(\mathbf{k})\widehat{g}_i(\mathbf{k})$ and

$$\widehat{g}_i(\mathbf{k}) = g^{(1)}_i(\mathbf{k}) + \sum_{\omega=\pm}\widehat{g}^{(\le 0)}_{i,\omega}(\mathbf{k}) \tag{37}$$

with $\widehat{g}^{(1)}(\mathbf{k}) = (1-\sum_{\omega}\chi_{0,i,\omega}(k))\widehat{g}_i(\mathbf{k})$; this induces the Grassmann variable decomposition $\widehat{\psi}_{i,\mathbf{k},\alpha}=\widehat{\psi}^{(1)}_{i,\mathbf{k},\alpha}+\sum_{\omega=\pm}\widehat{\psi}^{(\le 0)}_{i,\mathbf{k},\alpha,\omega}$ with propagators given by $\widehat{g}^{(1)}_i$ and $\widehat{g}^{(\le 0)}_{i,\omega}$ respectively. Note that $\widehat{\psi}^{(1)}$ correspond to fermions with momenta far from the Fermi points, while $\widehat{\psi}^{(\le 0)}$ with momenta around $\pm\mathbf{p}_{F,i}$.

We further decompose

$$\widehat{g}^{(\le 0)}_{i,\omega}(\mathbf{k}) = \sum_{h=-\infty}^{0}\widehat{g}^{(h)}_{i,\omega}(\mathbf{k}) \tag{38}$$

where $\widehat{g}^{(h)}_{i,\omega}(\mathbf{k}) := f_{h,i,\omega}(\mathbf{k})\widehat{g}^{(\le 0)}_{i,\omega}$ in which $f_{h,i,\omega}:=\chi_{h,i,\omega}-\chi_{h-1,i,\omega}$ is a smooth cutoff function supported in $\gamma^{h-3}\le|\mathbf{k}-\mathbf{p}^{\omega}_{F,i}|\le\gamma^{h-1}$ such that $\sum_{h=-\infty}^{0}f_{h,i,\omega}=\chi_{0,i,\omega}$. The integration is done recursively in the following way: assume that we have integrated the fields $\psi^{(1)},..,\psi^{(h-1)}$ obtaining

$$e^{W} = \int \bar{P}(d\psi^{(\le h)})e^{V^{(h)}(\psi,\phi)} \tag{39}$$

where $\bar{P}(d\psi^{(\le h)})$ is Gaussian integration with propagator $\bar{g}^{(\le h)}_{i,\omega}$ which will be defined inductively in (49), and

$$V^{(h)}(\psi,0) = \tag{40}$$
$$\sum_{i,\omega,\omega',l,\alpha,\alpha'}\int_{\widehat{\Lambda}_i} d\mathbf{k}W^{(h,\omega,\omega')}_{i,2,l,\alpha,\alpha'}(\mathbf{k})\psi^{+}_{i,\mathbf{k},\alpha,\omega}\psi^{-}_{2,\mathbf{k}+lb,\alpha',\omega'} +$$
$$\sum_{i,\omega,\omega',m,\alpha,\alpha'}\int_{\widehat{\Lambda}_i} d\mathbf{k}W^{(h,\omega,\omega')}_{i,1,m,\alpha,\alpha'}(\mathbf{k})\psi^{+}_{i,\mathbf{k},\alpha,\omega}\psi^{-}_{1,\mathbf{k}+mb',\alpha',\omega'} \tag{41}$$

According to the RG procedure, we renormalize the relevant and marginal terms; we will see below that the term with l or m non zero are actually irrelevant, due to improvements in the estimates due to the Diophantine condition. We therefore define a localization operation in the following way

$$\mathcal{L}W^{(h,\omega,\omega')}_{i,j,l}(\mathbf{k}) = \delta_{\omega,\omega'}\delta_{i,j}\delta_{l,0}[W^{(h,\omega,\omega)}_{i,i,0}(0,p^{\omega}_{F,i}) +$$
$$k_0\partial_0 W^{(h,\omega,\omega)}_{i,i,0}(0,p^{\omega}_{F,i}) + (k-p^{\omega}_{F,i})\partial W^{(h,\omega,\omega)}_{i,i,0}(0,p^{\omega}_{F,i}) \tag{42}$$

The terms for which $\mathcal{L}=0$ are called *non resonant* terms and the ones for which $\mathcal{L}\neq 0$ *resonant* terms. The terms containing derivatives are marginal ones and produce wave function or velocities renormalizations, while the terms without derivatives are the relevant terms. The action of $\mathcal{L}$ on the effective potential $V^{(h)}$ is

$$\mathcal{L}V^{(h)} = \mathcal{L}_1V^{(h)} + \mathcal{L}_2V^{(h)} \tag{43}$$

with

$$\mathcal{L}_1V^{(h)} := \sum_{i,\omega,\alpha,\alpha'}\int_{\widehat{\Lambda}_i} d\mathbf{k}\gamma^{h}\nu_{h,\omega,\alpha,\alpha',i}\psi^{+}_{\mathbf{k},\omega,i,\alpha}\psi^{-}_{\mathbf{k},\omega,i,\alpha'} \tag{44}$$

and

$$\mathcal{L}_2V^{(h)} := \sum_{i,j,\omega,\alpha,\alpha'}\int_{\widehat{\Lambda}_i} d\mathbf{k}z_{h,\omega,\alpha,\alpha',i,j}(\mathbf{k}-\mathbf{p}^{\omega}_{F,i})_j\psi^{+}_{\mathbf{k},\omega,i,\alpha}\psi^{-}_{\mathbf{k},\omega,i,\alpha'} \tag{45}$$

with

$$\nu_{h,\omega,\alpha,\alpha',i} := \gamma^{-h} W^{h,\omega,\omega}_{i,i,0,\alpha,\alpha'}(0, p^{\omega}_{F,i}) \tag{46}$$

$$z_{h,\omega,\alpha,\alpha',i,j} := -\partial_{\mathbf{k}_j} W^{h,\omega,\omega}_{i,i,0,\alpha,\alpha'}(0, p^{\omega}_{F,i}). \tag{47}$$

The form of the resonant terms is severely constrained by symmetries: as is proved in Appendix D,

$$\begin{aligned}
&\nu_{h,\omega,a,a,i} = \nu_{h,\omega,b,b,i} = 0 \quad \nu_{h,\omega,a,b,i} = \nu^*_{h,\omega,b,a,i} \\
&z_{h,\omega,b,a,i,1} = z^*_{h,\omega,a,b,i,1} \quad z_{h,\omega,b,a,i,2} = z^*_{h,\omega,a,b,i,2} \\
&z_{h,\omega,a,a,i,1} = z_{h,\omega,b,b,i,1} = z_{h,\omega,a,a,i,2} = z_{h,\omega,b,b,i,2} = 0 \\
&z_{h,\omega,b,a,i,0} = z_{h,\omega,a,b,i,0} = 0 \quad z_{h,\omega,a,a,i,0} = z_{h,\omega,b,b,i,0} \in i\mathbb{R}
\end{aligned} \tag{48}$$

The contributions from $\mathcal{L}_2 V$ are marginal, and are absorbed into the propagator at every step of the integration:

$$\begin{aligned}
&\bar{g}^{(\leq h)}_{i,\omega}(\mathbf{k}) := \chi_{h,i,\omega}(\mathbf{k}) \cdot \\
&\cdot \left((\bar{g}^{(\leq h+1)}_{i,\omega}(\mathbf{k}))^{-1} - \sum_j z_{h,\omega,\cdot,\cdot,i,j}(\mathbf{k} - \mathbf{p}^{\omega}_{F,i})_j \right)^{-1}
\end{aligned} \tag{49}$$

Thus,

$$\begin{aligned}
&\bar{g}^{(\leq h)}_{i,\omega}(\mathbf{k} + \mathbf{p}^{\omega}_{F,i}) = \chi_{h,i,\omega}(\mathbf{k})(1 + O(k)) \cdot \\
&\cdot \begin{pmatrix} -iZ_{i,\omega,h} k_0 & (iv_{i,\omega,h} k_1 - w_{i,\omega,h}\omega k_2) \\ (-iv^*_{i,\omega,h} k_1 - w^*_{i,\omega,h}\omega k_2) & -iZ_{i,\omega} k_0 \end{pmatrix}^{-1}
\end{aligned} \tag{50}$$

with

$$\begin{aligned}
Z_{i,\omega,h} &= Z_{i,\omega,h+1} - i z_{h,\omega,a,a,i,0} \\
v_{i,\omega,h} &= v_{i,\omega,h+1} + i z_{h,\omega,a,b,i,1} \\
w_{i,\omega,h} &= w_{i,\omega,h+1} + \omega z_{h,\omega,a,b,i,2}.
\end{aligned} \tag{51}$$

After absorbing $\mathcal{L}_2 V^{(h)}$ into the propagator, we are left with integrating $\mathcal{L}_1 V^{(h)}$ and

$$\mathcal{R} V^{(h)} := (1 - \mathcal{L}) V^{(h)} \tag{52}$$

so

$$e^W = \int \bar{P}(d\psi^{(\leq h)}) e^{\mathcal{L}_1 V^{(h)}(\psi) + \mathcal{R} V^{(h)}(\psi)}. \tag{53}$$

B. Feynman rules for the renormalized expansion

The renormalized expansion described above has a graphical representation that is similar to the Feynman diagram expansion from Section III. There are two main differences: first, there are two different types of vertices: "*τ-vertices*", coming from $\mathcal{R}V^{(h)}$ in (53), and "*ν-vertices*", coming from $\mathcal{L}_1 V^{(h)}$. Second, every line has a *scale label* h, corresponding to a propagator on scale h:

$$\bar{g}^{(h)}_{i,\omega}(\mathbf{k}) := f_{h,i,\omega}(\mathbf{k}) \bar{g}^{(\leq h)}_{i,\omega}(\mathbf{k}). \tag{54}$$

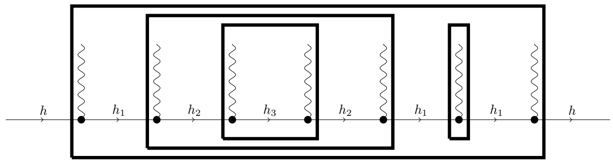

FIG. 2: An example of graph of order λ^7 with the associated clusters, denoted by thick rectangles. In this example, $h < h_1 < h_2 < h_3$.

The scale labels induce an important structure: given a diagram, we group vertices together into nested *clusters*, which are connected subgraphs in which the scales of the lines leaving the cluster are all smaller than the scales of the lines inside the cluster, see Figure 2. A cluster that is such that $\mathcal{L}$ applied to the cluster yields 0 is called *non-resonant*, otherwise it is called *resonant*. In other words, the action of $\mathcal{R} = 1 - \mathcal{L}$ is trivial on non-resonant clusters, and non-trivial on resonant ones.

Some clusters are single vertices (either ν or τ) and are called *trivial clusters*. The clusters that contain internal lines are called *non-trivial clusters*. As per the construction above, if h^{ext}_T is the largest of the scales of the external lines of a non-trivial cluster T, all its internal lines have a scale $h > h^{ext}_T$; h_T is the largest scale of the propagators internal to the cluster T. A non-trivial cluster T contains sub-clusters $\widetilde{T} \subset T$. We call a cluster $\widetilde{T} \subset T$ a *maximal cluster* if there is no other cluster $\bar{T}$ such that $\widetilde{T} \subset \bar{T} \subset T$.

For each cluster, there are two external lines that connect the cluster to other ones. In a non-resonant cluster T with external lines of type $i_1 = i_2 = 1$ and momenta k_1, k_2 with k_1 in the first Brillouin zone and $k_2 = k_1 + \widehat{m}_T b + lb$ where lb is chosen so that k_2 is in the first Brillouin zone, if $A_0, .., A_N$ are the momenta associated to the τ vertices contained in T (the ν vertices do not change momentum) one has $A_0 = k_1 + l_0 b$, $A_1 = k_1 + l_0 b + m_1 b'$, $A_2 = k_1 + m_1 b' + l_2 b$, $A_3 = k_1 + m_3 b' + l_2 b$,..., $A_N = k_1 + m_N b' + l_{N-1} b$ and $m_N = \widehat{m}_T$ with N odd. In the same way if the non-resonant cluster T has one external line of type $i_1 = 1$ and momentum k_1, and one of type $i_2 = 2$ and momentum k_2; assume that k_1 is in the first Brillouin zone and $k_2 = k_1 + \widehat{l}_T b + mb'$, where mb' is chosen such that k_2 is in the first Brillouin zone. Now with N even the momenta associated to the τ vertices in T are $A_0 = k_1 + l_0 b$, $A_1 = k_1 + l_0 b + m_1 b'$, $A_2 = k_1 + m_1 b' + l_2 b$, $A_3 = k_1 + m_3 b' + l_2 b$,...$A_{N-1} = k + m_{N-2} b' + l_{N-1} b$, $l_{N-1} = \widehat{l}_T$.

The value associated to a graph Γ is denoted by $W_\Gamma(\mathbf{k})$ and is given by the product of the propagators and ν, τ factors associated to the vertices, with the $\mathcal{R}$ operation acting on each non-resonant cluster. The effect of the $\mathcal{R}$ operation on the non-resonant clusters can be written as

$$\mathcal{R} W^h(\mathbf{k} + p^{\omega}_{F,i}) = k^2 \int_0^1 \partial^2 W(t\mathbf{k}) \tag{55}$$

C. Convergence of the series and persistence of the cones

In order to bound the contribution of a Feynman graph we note that $|\bar g^{(h)}(k)| \le C\gamma^{-h}$ (by (50)); therefore the product of propagators in the Feynman graph is bounded by

$$\prod_{T\ \mathrm{n.t.}} \gamma^{-h_T(M_T+R_T-1)} \tag{56}$$

where M_T is the number of maximal non-resonant clusters contained in T, R_T is the number of maximal resonant clusters contained in T, and $\prod_{T\ \mathrm{n.t.}}$ is a product over non-trivial clusters. Note that the trivial clusters do not contribute, as they contain no internal lines. The effect of the $\mathcal{R} := \mathbb{1}-\mathcal{L}$ (recall (42)) operation on the non-trivial resonant clusters produces an extra $\gamma^{2(h_T^{\mathrm{ext}}-h_T)}$ (the gain h_T^{ext} comes from the extra $(k-p_{F,i}^{\omega})^2$ terms (which are on the scale of the external lines) and the h_T loss from the extra second derivative (which are on the scale of the internal lines)). Therefore, we get an extra factor:

$$\prod_{T\ \mathrm{res\ n.t.}} \gamma^{2(h_T^{\mathrm{ext}}-h_T)} \tag{57}$$

where $\prod_{T\ \mathrm{res\ n.t.}}$ is the product over the non-trivial resonant clusters. In addition, each maximal resonant cluster corresponds to a weight $\gamma^h\nu_h$, so, if $|\nu_h| \leqslant C|\lambda|$, we get an extra factor:

$$\prod_{T\ \mathrm{n.t.}} \gamma^{h_T M_T^{\nu}} C|\lambda| \tag{58}$$

where M_T^{ν} is the number of maximal resonant trivial clusters in T. Thus, we get the following bound for the sum over all the labels of a renormalized graph with q vertices:

$$\sum^{*}_{\underline{h},\underline{l},\underline{m}} [L(\underline{l},\underline{m})]|\lambda|^q C^q \left(\prod_{T\,\mathrm{n.t.}} \gamma^{-h_T(M_T+R_T-1)}\right) \left(\prod_{T\ \mathrm{res\ n.t.}} \gamma^{2(h_T^{\mathrm{ext}}-h_T)}\right) \prod_{T\,\mathrm{n.t.}} \gamma^{h_T M_T^{\nu}} \tag{59}$$

where $L(\underline{l},\underline{m})$ is the norm of the product of the τ functions:

$$L(\underline{l},\underline{m}) := \prod_i |\tau(A_i)| \tag{60}$$

where A_i are the momenta, which depend on the structure of the graph, and $\sum^{*}_{\underline{h},\underline{l},\underline{m}}$ is the sum over the scales and momenta associated to the vertices, as explained in Section VI B; these sums are not independent, as they are constrained by the compact support properties of the propagators and of the Diophantine condition. Now defining $\mathbb{1}_{\Gamma\,\mathrm{res}}$ is equal to 1 if the maximal cluster T_m is resonant and 0 otherwise, we have

$$\begin{aligned}&\prod_{T\,\mathrm{n.t.}} \gamma^{-h_T R_T} \prod_{T\ \mathrm{res\ n.t.},T\neq T_m} \gamma^{h_T^{\mathrm{ext}}} \prod_{T\,\mathrm{n.t.}} \gamma^{h_T M_T^{\nu}} \le 1\\ &\prod_{T\,\mathrm{n.t.}} \gamma^{h_T} \prod_{T\ \mathrm{res\ n.t.},T\neq T_m} \gamma^{-h_T} \le \gamma^{h_{T_m}\mathbb{1}_{\Gamma\,\mathrm{res}}}\end{aligned} \tag{61}$$

and then using that if T_m is resonant and $\mathcal{L}$ is applied then $h_{T_m} = h+1$ we get

$$\prod_{T\,\mathrm{n.t.}} \gamma^{-h_T(R_T-1)}\gamma^{h_T M_T^{\nu}} \prod_{T\ \mathrm{res\ n.t.}} \gamma^{(h_T^{\mathrm{ext}}-h_T)} \le \gamma^{(h+1)\mathbb{1}_{\Gamma\,\mathrm{res}}}. \tag{62}$$

Therefore, (59) is bounded by

$$\sum^{*}_{\underline{h},\underline{l},\underline{m}} [L(\underline{l},\underline{m})]|\lambda|^q C^q \gamma^{h\mathbb{1}_{\Gamma\,\mathrm{res}}} \left(\prod_{T\,\mathrm{n.t.}} \gamma^{-h_T M_T}\right)\left(\prod_{T\ \mathrm{res\ n.t.}} \gamma^{(h_T^{\mathrm{ext}}-h_T)}\right). \tag{63}$$

As the graphs are chains, there is no problematic combinatorial factor; the main issue is the sum over $\underline{h}$; if we neglect the constraint in $\sum^{*}_{\underline{h},\underline{l},\underline{m}}$ then we will get a factor $\prod_{T\,\mathrm{n.t.}}(\sum_{h_T=-\infty}^{0}\gamma^{-h_T})$ which is divergent.

However, we have still not taken advantage of the Diophantine condition. In order to do that we note that $|\tau^{(i)}(k)| \le Ce^{-\kappa|k|}$ from the exponential decay or $\widehat{\varsigma}(k)$, see (12), (13)-(14); we can write

$$e^{-\kappa/2|k|} = \prod_{h=-\infty}^{0} e^{-\kappa 2^h|k|} \tag{64}$$

Consider the product of τ factors:

$$L(\underline{l},\underline{m}) \equiv \prod_i |\tau(A_i)| \le \prod_i e^{-\kappa|A_i|/2} \prod_{h=-\infty}^{1} e^{-\kappa 2^h|A_i|} \tag{65}$$

which we estimate as

$$L(\underline{l},\underline{m}) \le \prod_i e^{-\kappa|A_i|/2} \prod_{T\ \mathrm{nonres}\ni i} e^{-\kappa 2^{h_T}|A_i|} \tag{66}$$

where in the last product i is the label of the points τ contained in T. We then exchange the products:

$$L(\underline{l},\underline{m}) \le \left(\prod_i e^{-\kappa|A_i|/2}\right) \prod_{T\ \mathrm{nonres}} \prod_{i\in T} e^{-\kappa 2^{h_T}|A_i|} \tag{67}$$

Let us consider a non resonant cluster with external lines of type 1; we have

$$|A_0|+|A_1|+\ldots \ge |A_0-A_1+A_2-A_3\ldots| = |\widehat{m}_T b'| \tag{68}$$

(see Section VI B) so that

$$\prod_{i\in T} e^{-\kappa 2^{h_T}|A_i|} \le e^{-\kappa 2^{h_T}|\widehat{m}_T b'|} \tag{69}$$

A similar analysis holds for $i_1 = i_2 = 2$ with $\widehat{l} \neq 0$. In the same way if the cluster T has $i_1 = 1$ and $i_2 = 2$ and k_1, k_2 are the momenta of the external lines using that $|A_0| + |A_1| + \dots$

$$\geq |A_0 - A_1 + A_2 - A_3 \dots| \geq |k_1 + \widehat{l}b| \geq |\widehat{l}b| - \frac{4\pi}{3} \quad (70)$$

(where we used that $|k_1| \leq \frac{4\pi}{3}$) as k_1 is in the first Brillouin zone

$$\prod_{i \in T} e^{-\kappa 2^{h_T}|A_i|} \leq e^{-\kappa 2^{h_T}(|\widehat{l}_T b| - \frac{4\pi}{3})}. \quad (71)$$

In addition, the Diophantine condition imposes a constraint between the momenta of the external lines of a cluster T and the l, m labels associated to its internal vertices. Consider a non resonant cluster with the following indices:

1. If $i_1 = i_2 = 1$ and k_1, k_2 are the momenta of the external lines; assume that k_1 is in the first Brillouin zone and $k_1 =: \bar{k}_1 + p_{F,1}^{\omega_1}$. Moreover $k_2 := k_1 + \widehat{m}_T b' + lb =: \bar{k}_2 + p_{F,1}^{\omega_2}$, with l chosen such that k_2 is in the first Brillouin zone; then, if $\widehat{m}_T \neq 0$, by (28),
$$\begin{aligned} &2\gamma^{h_T^{ext}} \geq |\bar{k}_1| + |\bar{k}_2| \geq |\bar{k}_1 - \bar{k}_2| = \\ &|p_{F,1}^{\omega_1} - p_{F,1}^{\omega_2} + \widehat{m}_T b' + lb| \geq \frac{C_0}{|\widehat{m}_T|^\tau} \end{aligned} \quad (72)$$
so that, if $\widehat{m}_T \neq 0$
$$|\widehat{m}_T| \geq (\tfrac{1}{2} C_0 \gamma^{-h_T^{\text{ext}}})^{\frac{1}{\tau}} \quad (73)$$
and so
$$|\widehat{m}_T b'| \geq c_1 \gamma^{-h_T^{\text{ext}}/\tau} \quad (74)$$
for some constant $c_1 > 0$. On the other hand if $\widehat{m}_T = 0$ but $\omega_1 \neq \omega_2$ then $l = 0$ so $\gamma^{h_T^{\text{ext}}} \geq \frac{1}{2}|p_{F,1}^{\omega_1} - p_{F,1}^{\omega_2}| = \frac{2\pi}{3\sqrt{3}}$ (recalling (10)). Thus, this eventuality does not occur provided γ is large enough.

2. In the case $i_1 = 1$ and $i_2 = 2$ and k_1, k_2 are the momenta of the external lines; assume that k_1 is in the first Brillouin zone and $k_1 =: \bar{k}_1 + p_{F,1}^{\omega_1}$. Moreover $k_2 := k_1 + \bar{l}b + mb' =: \bar{k}_2 + p_{F,2}^{\omega_2}$, with m chosen in such a way that k_2 is in the first Brillouin zone; then, if $\bar{l} \neq 0$, by (28),
$$\begin{aligned} &2\gamma^{h_T^{ext}} \geq |\bar{k}_1| + |\bar{k}_2| \geq |\bar{k}_1 - \bar{k}_2| = \\ &|p_{1,F}^{\omega_1} - p_{2,F}^{\omega_2} + \widehat{l}_T b + mb'| \geq \frac{C_0}{|\widehat{l}_T|^\tau} \end{aligned} \quad (75)$$
so that
$$|\widehat{l}_T| \geq (\tfrac{1}{2} C_0 \gamma^{-h_T^{\text{ext}}})^{\frac{1}{\tau}} \quad (76)$$
and so
$$|\widehat{l}_T b| \geq c_1 \gamma^{-h_T^{\text{ext}}/\tau} \quad (77)$$
If $\widehat{l}_T = 0$ then $2\gamma^{h_T^{\text{ext}}} \geq O(\theta)$ for $\omega_1 = \omega_2$ and $2\gamma^{h_T^{\text{ext}}} \geq O(1)$ for $\omega_1 = -\omega_2$. Thus, provided $\gamma \gg \theta^{-1}$, these eventualities do not present themselves provided γ is large enough.

3. A similar analysis holds for $i_1 = 2, i_1 = 1$, and $i_1 = i_2 = 2$.

Thus,

$$\prod_{i \in T} e^{-\kappa 2^{h_T}|A_i|} \leq e^{-\kappa 2^{h_T}(c_1 \gamma^{-h_T^{\text{ext}}/\tau} - \frac{4\pi}{3})} \quad (78)$$

which, provided γ is large enough, yields

$$\prod_{i \in T} e^{-\kappa 2^{h_T}|A_i|} \leq e^{-c_2 \gamma^{-h_T^{\text{ext}}/\tau}} \quad (79)$$

for some constant c_2. Therefore,

$$L(\underline{l}, \underline{m}) \leq e^{-c_2 \gamma^{-h/\tau} \mathbb{1}_{\Gamma \text{ nonres}}} \prod_i e^{-\kappa|A_i|/2} \prod_{T \text{ n.t.}} e^{-c_2 M_T \gamma^{-h_T/\tau}} \quad (80)$$

where $\mathbb{1}_{\Gamma \text{ nonres}}$ is equal to 1 if the maximal cluster is non resonant and 0 otherwise. Note that, provided γ is chosen to be large enough, $e^{-c_2 \gamma^{-h/\tau} \mathbb{1}_{\Gamma \text{ nonres}}} \leq \gamma^{3h \mathbb{1}_{\Gamma \text{ nonres}}}$, so

$$L(\underline{l}, \underline{m}) \leq \gamma^{3h \mathbb{1}_{\Gamma \text{ nonres}}} \prod_i e^{-\kappa|A_i|/2} \prod_{T \text{ n.t.}} e^{-c_2 M_T \gamma^{-h_T/\tau}}. \quad (81)$$

Furthermore, using the bound $e^{-\alpha x} \leq (\frac{\beta}{\alpha})^\beta e^{-\beta} x^{-\beta}$ with $\beta = 3\tau M_T$, we find

$$e^{-c_2 M_T \gamma^{-h_T/\tau}} \leq (\frac{c_2 e^1}{3\tau})^{-3\tau M_T} \gamma^{3 M_T h_T}. \quad (82)$$

In addition, $\sum_{T \text{ n.t.}} M_T \leq q$, since the clusters are nested in each other and for two clusters to be different they must differ by at least one vertex. Now, let us introduce M_T^τ as the number of maximal non-resonant trivial clusters (i.e. maximal τ-vertices) contained in T, and use the trivial bound $3M_T \leq 2M_T + M_T^\tau$ along with (82) to obtain

$$\prod_{T \text{ n.t.}} e^{-c_2 M_T \gamma^{-h_T/\tau}} \leq C_3^q \cdot \prod_{T \text{ n.t.}} \gamma^{h_T(2M_T + M_T^\tau)} \quad (83)$$

Thus, plugging this into (63), we find

$$\begin{aligned} &\gamma^h \sum_{\underline{h}, \underline{l}, \underline{m}}^{*} [L]^{\frac{1}{2}} |\lambda|^q (CC_3)^q \gamma^{h \mathbb{1}_{\Gamma \text{ res}}} \gamma^{3h \mathbb{1}_{\Gamma \text{ nonres}}} \\ &[\prod_{T \text{ res}} \gamma^{(h_T^{\text{ext}} - h_T)}] \prod_{T \text{ n.t.}} \gamma^{h_T(2M_T + M_T^\tau)}. \end{aligned} \quad (84)$$

In addition,

$$\prod_{T \text{ n.t.}} \gamma^{2h_T M_T} = \gamma^{-2h \mathbb{1}_{\Gamma \text{ nonres}}} \prod_{T \text{ nonres}} \gamma^{2h_T^{\text{ext}}} \quad (85)$$

$$\gamma^h \sum^*_{\substack{\underline{h},\\ \underline{l},\underline{m}}} [L]^{\frac{1}{2}} |\lambda|^q (CC_3)^q [\prod_{T\,\mathrm{n.t.}} \gamma^{(h_T^{\mathrm{ext}}-h_T)}] \prod_{T\,\mathrm{n.t.}} \gamma^{h_T M_T^\tau} \quad (86)$$

The crucial point is that the sum over the scales h can be performed summing over all the differences, taking into account that the scale h is fixed. Finally the sum over the l, m is done using the factor $[L(\underline{l},\underline{m})]^{\frac{1}{2}}$. (The gain term $\gamma^{h_T M_T^\tau}$ is dropped, as it does not lead to any significant gain.) In conclusion the bound on a graph with q vertices is $C_4^q \gamma^h |\lambda|^q$ assuming that $|\nu_h|, |Z_h-1|, |v_h-1|, |w_h-1| \leq C|\lambda|$.

D. Beta function and Schwinger functions

We are left with checking our assumption on Z_h, ν_h, w_h, v_h. We know that $v_{i,\omega,h} = v_{i,\omega,h+1} - iz_{h,\omega,a,b,i,1}$ with $z_{h,\omega,a,b,i,1}$ expressed by the sum of renormalized Feynman graphs Γ such that the maximal scale of the clusters is $h+1$, an extra derivative is applied (which costs a factor γ^{-h}) and the momenta of the external lines is fixed equal to p_F^ω. Moreover by the compact support of the propagator there is at least a τ vertex, as the $k=0$ value of a graph wih only ν vertice is zero; therefore the analogue of (86) becomes

$$\sum^*_{\substack{\underline{h},\\ \underline{l},\underline{m}}} [L]^{\frac{1}{2}} |\lambda|^q (CC_3)^q [\prod_{T\,\mathrm{n.t.}} \gamma^{(h_T^{\mathrm{ext}}-h_T)}] \gamma^{2h_{T^*}} \quad (87)$$

where T^* is the non trivial cluster containing a τ vertex whose scale is the largest possible (we now use the gain $\gamma^{h_T M_T^\tau}$ dropped in the bound (86)). In addition, summing the differences $h_T^{\mathrm{ext}} - h_T$ along a sequence of clusters that goes from h to h_{T^*} and discarding the others, we bound

$$\sum_T (h_T^{\mathrm{ext}} - h_T) \leqslant h - h_{T^*} \quad (88)$$

and so (87) is bounded by

$$\gamma^{\frac{h}{2}} \sum^*_{\substack{\underline{h},\\ \underline{l},\underline{m}}} [L]^{\frac{1}{2}} |\lambda|^q (CC_3)^q \prod_{T\,\mathrm{n.t.}} \gamma^{\frac{1}{2}(h_T^{\mathrm{ext}}-h_T)}. \quad (89)$$

Estimating the sum as above, we find that $|z_{h,\omega,a,b,i,1}| \leq C_5 \lambda \gamma^{\frac{h}{2}}$, and $v_{i,\omega,h} = v_{i,\omega,0} - i\sum_{h'} z_{h',\omega,a,b,i,1}$ hence $v_{i,\omega,h} = v_{i,\omega,0} + O(\lambda)$; moreover the limiting value is reached exponentially fast $v_{i,\omega,h} = v_{i,\omega,-\infty} + O(\lambda\gamma^{h/2})$. A similar argument holds for Z_h, w_h. Not

It remain to discuss the flow of ν_h; we can write, see (46),

$$W^{h,\omega,\omega}_{i,i,0,\alpha,\alpha'}(0, p^\omega_{F,i}) = \gamma^{h+1}\nu_{h+1} + \widetilde{W}^{h,\omega,\omega}_{i,i,0,\alpha,\alpha'}(0, p^\omega_{F,i}) \quad (90)$$

where $\widetilde{W}$ is given by the sum of the terms with a number of vertices greater or equal to 2; therefore

$$\nu_{h,\omega,\alpha,\alpha',i} = \gamma \nu_{h+1,\omega,\alpha,\alpha',i} + \beta^h_\nu \quad (91)$$

with $\beta^h_\nu = \gamma^{-h} \widetilde{W}^{h,\omega,\omega}_{i,i,0,\alpha,\alpha'}(0, p^\omega_{F,i})$ is given by the sum with $q \geq 2$ of terms bounded by (89). We have to prove that it is possible to choose the counterterms $\nu_{\omega,\alpha,\alpha',i}$ so that $\nu_{h,\omega,\alpha,\alpha',i}$ is bounded by $C\lambda$ for any scale h. Indeed from (91) we get, $h \leq -1$

$$\nu_{h,\omega,\alpha,\alpha',i} = \gamma^{-h}(\nu_{\omega,\alpha,\alpha',i} + \sum_{i=h}^{-1} \gamma^i \beta^i_\nu) \quad (92)$$

and choosing $\nu_{\omega,\alpha,\alpha',i}$ so that $\nu_{-\infty,\omega,\alpha,\alpha',i} = 0$ we get

$$\nu_{h,\omega,\alpha,\alpha',i} = -\gamma^{-h} \sum_{i=-\infty}^{h} \gamma^i \beta^i_\nu \quad (93)$$

and by using a fixed point argument we can show that there is a sequence such that $|\nu_{h,\omega,\alpha,\alpha',i}| \leq C\lambda\gamma^{\frac{h}{2}}$.

The application of the above bounds to the 2-point function, in order to derive (36), is straightforward. The 2-point function can be written as

$$\widehat{S}_{j,j}(\mathbf{k} + \mathbf{p}^\omega_{F,j})) = \sum_{h=-\infty}^{0} [\widehat{g}^{(h)}((\mathbf{k} + \mathbf{p}^\omega_{F,j})) + r^h(\mathbf{k}) \quad (94)$$

where $r^h(\mathbf{k})$ includes the contribution of term withs at least a vertex. We can replace in $g^{(h)}((\mathbf{k} + \mathbf{p}^\omega_{F,j}))$ the $v_{i,\omega,h}, w_{i,\omega,h}, Z_{i,\omega,h}$ with $v_{i,\omega,-\infty}, w_{i,\omega,-\infty}, Z_{i,\omega,-\infty}$ obtaining the dominant term in (36); the subdominant term is obtained both from the term containing the difference betwen $v_{i,\omega,h} - v_{i,\omega,-\infty}$, $z_{i,\omega,h} - w_{i,\omega,-\infty}$, $Z_{i,\omega,h} - Z_{i,\omega,-\infty}$, which have an extra factor $O(\lambda\gamma^{h/2})$, or the terms with at least a vertex which have at least a ν or a non resonant trivial vertex, with an extra $O(\gamma^{h/2})$ from the bounds after (87).

VII. CONCLUSION

Theoretical analyses of TBG are based on the assumption of the stability of the Weyl semimetallic phase, leading to the formulation of continuum effective models. However in lattice TBG models wth generic angles there is an emerging quasi-periodicity manifesting themself in large momenta Umklapp interactions that almost connect the Dirac points, similar to the ones appearing in electronic systems with quasi-periodic potential. Such terms are neglected in the continuum semimetallic approximations.

In this paper we have rigorously established the stability of the semimetallic phase in a lattice model, taking into full account the large momenta Umklapp interactions. The analysis is based on number theoretical properties of irrationals combined with a Renormalization

Group analysis, and requires that the interlayer hopping is weak and short ranged and the the angles are chosen in a large measure set. The effect of the interaction is to produce a finite renormalization of the Dirac points and velocities. Non perturbative effects are excluded as the series are shown to be convergent. Compared to the Aubry-André or similar models, the number theoretical analysis is much more involved due to the peculiar structure of the small divisors.

The stability of the Weyl phases provides a justification of the use of continnum models under the above assumptions. In addition, the present analysis paves the way to a more accurate evaluation of the velocities as functions of the angles, talking into account lattice or higher orders effects, and the effect of many body interactions, whose interplay with the emerging quasi-periodicity could lead to interesting phases.

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Appendix A: Fourier transform of the interlayer hopping

We write V in Fourier space: we get

$$V = \frac{\lambda}{|\widehat{\mathcal{L}}_1|^2} \sum_{x_1\in\widehat{\mathcal{L}}_1} \sum_{x_2'\in\Lambda_2} \sum_{\alpha\in\{a,b\}} \int_{\mathbb{R}^2} \frac{dq}{4\pi^2} \int_{\widehat{\mathcal{L}}_1} dk_1 \int_{\widehat{\mathcal{L}}_2} dk_2' \\ e^{i(k_1x_1-k_2'x_2'+q(x_1-x_2'))} e^{iq(d_\alpha-Rd_\alpha)} e^{i\xi(k_2'-k_1)} \widehat{\varsigma}(q) \widehat{c}^\dagger_{1,k_1,\alpha} \widehat{c}_{2,k_2',\alpha} + \\ e^{-i(k_1x_1-k_2'x_2'-q(x_1-x_2'))} e^{iq(d_\alpha-Rd_\alpha)} e^{-i\xi(k_2'-k_1)} \widehat{\varsigma}(q) \widehat{c}^\dagger_{2,k_2',\alpha} \widehat{c}_{1,k_1,\alpha} \tag{A1}$$

and using the Poisson summation formula

$$\sum_{x_1\in\mathcal{L}_1} e^{i(k_1+q)x_1} = |\widehat{\mathcal{L}}_1| \sum_{l\in\mathbb{Z}^2} \delta(k_1+q+lb) \tag{A2}$$

where we use the shorthand $lb \equiv l_1b_1 + l_2b_2$, and

$$\sum_{x_2'\in\mathcal{L}_2} e^{-i(k_2+q)x_2'} = |\widehat{\mathcal{L}}_1| \sum_{m\in\mathbb{Z}^2} \delta(k_2+q+mb') \tag{A3}$$

Noting that $\widehat{c}_{2,k_1+lb-mb',\alpha} \equiv \widehat{c}_{2,k_1+lb,\alpha}$, $\widehat{c}_{1,k_2'+mb'-lb,\alpha} \equiv \widehat{c}_{1,k_2'+mb',\alpha}$ we finally obtain (13).

Rewriting (A1) in terms of Grassmann variables, with the added imaginary time component, reads

$$V = \frac{\beta\lambda}{|\widehat{\Lambda}_1|^2} \sum_{x_1\in\mathcal{L}_1} \sum_{x_2'\in\mathcal{L}_2} \sum_{\alpha\in\{a,b\}} \int_{\mathbb{R}^2} \frac{dq}{4\pi^2} \int_{\widehat{\Lambda}_1} d\mathbf{k}_1 \int_{\widehat{\Lambda}_2} d\mathbf{k}_2' \; \delta(k_{1,0}-k_{2,0}')\cdot \\ \cdot\Big(e^{i(k_1x_1-k_2x_2'+q(x_1-x_2'))} e^{iq(d_\alpha-Rd_\alpha)} e^{i\xi(k_2'-k_1)} \widehat{\varsigma}(q) \widehat{\psi}^+_{1,\mathbf{k}_1,\alpha} \widehat{\psi}^-_{2,\mathbf{k}_2',\alpha} + \\ + e^{-i(k_1x_1-k_2x_2'-q(x_1-x_2'))} e^{iq(d_\alpha-Rd_\alpha)} e^{-i\xi(k_2'-k_1)} \widehat{\varsigma}(q) \widehat{\psi}^+_{2,\mathbf{k}_2',\alpha} \widehat{\psi}^-_{1,\mathbf{k}_1,\alpha} \Big) \tag{A4}$$

where we use the notation $\mathbf{k}_1 = (k_{1,0}, k_1)$ and $\mathbf{k}_2' = (k_{2,0}, k_2)$. Again, using the Poisson formula, we find (25).

Appendix B: Proof of Lemma IV.1

To prove lemma IV.1, we will first prove a general result on a Diophantine condition for a generic function from $[0, 2\pi)$ to $\mathbb{R}^2$. We will then apply this result to $|p^\omega_{F,i}, p^{\omega'}_{F,j} + lb + mb'|$, viewed as a function of θ, for the various values of i, j, ω, ω'.

1. Diophantine condition from $\mathbb{R}$ to $\mathbb{R}^2$

Let us consider an interval $[\theta_0,\theta_1]\subset[0,2\pi]$, and define, given constants $C_1>0,\tau>4$ that are fixed once and for all, two twice-differentiable functions $x:[0,2\pi)\to\mathbb{R}$, $f:[0,2\pi)\to\mathbb{R}^2$, and a subset $\Omega(x,f)\subset[\theta_0,\theta_1]$,

$$\mathcal{D}(x,f):=\{\theta\in\Omega(x,f):\ \forall k\in\mathbb{Z}^2\setminus\{0\},\ \forall l\in\mathbb{Z},\ |x(\theta)+l+k\cdot f(\theta)|\geqslant C_1|k|^{-\tau}\} \tag{B1}$$

We will show that, provided Ω is chosen appropriately, under certain conditions on f and x, $\mathcal{D}$ has a large measure. The novelty of this result is that f takes values in $\mathbb{R}^2$, but is a function of a single variable; if f were a function from $\mathbb{R}^n$ to $\mathbb{R}^n$, then the fact that $\mathcal{D}$ has a large measure would follow from standard arguments []. Our result is stated for $\mathbb{R}^2$, but it could easily be adapted to any other dimension, provided f takes a single real-valued argument.

In order to make our argument work, we will assume that $f'(\theta)$ (the derivative of f) remains inside a cone, that is, we assume that $\exists\xi\in\mathbb{R}^2$ with $|\xi|=1$ and $\alpha\in[0,\frac{\pi}{4})$ such that, $\forall\theta\in[\theta_0,\theta_1]$,

$$f'(\theta)\in\mathcal{C}_\xi(\alpha):=\{y\in\mathbb{R}^2,\ |y\cdot\xi|>|y|\cos(\alpha)\}. \tag{B2}$$

We take the set $\Omega(x,f)$ in (B1) to be

$$\Omega(x,f):=\{\theta\in[\theta_0,\theta_1]:\ \forall k\in\zeta,\ |x'(\theta)+k\cdot f'(\theta)|\geq C_3|f'(\theta)||k|^{-\epsilon}\} \tag{B3}$$

where $C_3>0$ is a constant, $\epsilon\in(1,\tau-3)$, and

$$\zeta:=\mathbb{Z}^2\setminus(\{0\}\cup\mathcal{C}_\xi(\tfrac{\pi}{4})) \tag{B4}$$

(the reason why we choose Ω in this way will become apparent in the proof of Lemma B.1 below).

Lemma B.1. *If the following estimates hold:*

$$\min_{\theta\in[\theta_0,\theta_1]}|f'(\theta)|>0,\ \min_{\theta\in[\theta_0,\theta_1]}|\tfrac{\partial}{\partial\theta}(\tfrac{f'(\theta)}{|f'(\theta)|})|>0 \tag{B5}$$

$\forall\theta\in[\theta_0,\theta_1]$,

$$\min_{0<|k|<R_1}|x'(\theta)+k\cdot f'(\theta)|-C_3|f'(\theta)||k|^{-\epsilon}>0 \tag{B6}$$

with

$$R_1:=\frac{C_3+\frac{|x'(\theta)|}{|f'(\theta)|}}{\cos(\alpha+\frac{\pi}{4})} \tag{B7}$$

and, for some $\eta>0$,

$$\min_{0<|k|<R_2}|\tfrac{\partial}{\partial\theta}(\tfrac{x'(\theta)}{|f'(\theta)|})+k\cdot\tfrac{\partial}{\partial\theta}(\tfrac{f'(\theta)}{|f'(\theta)|})|-\eta|k||\tfrac{\partial}{\partial\theta}(\tfrac{f'(\theta)}{|f'(\theta)|})|\cos(\alpha+\tfrac{\pi}{4})>0 \tag{B8}$$

with

$$R_2:=\eta+\frac{|\frac{\partial}{\partial\theta}(\frac{x'(\theta)}{|f'(\theta)|})|}{|\frac{\partial}{\partial\theta}(\frac{f'(\theta)}{|f'(\theta)|})|\cos(\alpha+\frac{\pi}{4})} \tag{B9}$$

then the measure of the complement of $\mathcal{D}$ is bounded by

$$|[\theta_0,\theta_1]\setminus\mathcal{D}(x,f)|\leq O(C_3)+O(\tfrac{C_1}{C_3\beta}) \tag{B10}$$

where the constants in $O(\cdot)$ depend only on θ_0, θ_1, x, f, α, ϵ, τ, η. In particular, if we choose $C_3\ll\theta_1-\theta_0$ and $C_1\ll(\theta_1-\theta_0)^2$, then $\mathcal{D}(x,f)$ fills most of $[\theta_0,\theta_1]$.

Remark B.1. *The conditions (B6) and (B8) concern a finite number of values of k. In the applications of this lemma below, we can make both of these conditions trivial by ensuring that $R_1,R_2<1$, which reduces this finite number of values for k to 0.*

Proof Let $|\mathcal{D}^c_\Omega(x,f)|$ denote the Lebesgue measure of the complement $\Omega(x,f)\setminus\mathcal{D}(x,f)$. Let

$$g_{l,k}(\theta) = |x(\theta) + l + k\cdot f(\theta)| \tag{B11}$$

in terms of which

$$|\mathcal{D}^c_\Omega(x,f)| = \sum_{k,l}^{*} \int_{-C_1|k|^{-\tau}}^{C_1|k|^{-\tau}} \frac{1}{g'_{l,k}} dg_{l,k} \tag{B12}$$

where $\sum^*_{k,l}$ has the constraint that $\exists\theta\in[\theta_0,\theta_1]$ such that $g_{k,l}(\theta)\in[-C_1|k|^{-\tau}, C_1|k|^{-\tau}]$. Therefore

$$|\mathcal{D}^c_\Omega(x,f)| \le \sum_{l,k}^{*} 2C_1 \frac{|k|^{-\tau}}{\min_{\theta\in\Omega(x,f)} |x'(\theta) + k\cdot f'(\theta)|}. \tag{B13}$$

In addition, the number of values of l such that $g_{k,l}(\theta)\in[-C_1|k|^{-\tau}, C_1|k|^{-\tau}]$ is bounded by $C_2|k|$ for some constant C_2 (which depends only on θ_0,θ_1,x,f), and so

$$|\mathcal{D}^c_\Omega(x,f)| \le 2 \sum_{k\in\mathbb{Z}^2\setminus\{0\}} C_2C_1 \frac{|k|^{1-\tau}}{\min_{\theta\in\Omega(x,f)} |x'(\theta) + k\cdot f'(\theta)|}. \tag{B14}$$

In order for this bound to be useful, we must obtain a good lower bound on $|x' + k\cdot f'|$.

To do so, Ω must be chosen appropriately: we wish for $k\cdot f'$ to stay as far away from $-x'$ as possible. Now, it cannot avoid it entirely, as $k\cdot f'$ will cover all possible values as k varies in $\mathbb{Z}^2\setminus\{0\}$. By choosing Ω as in (B3), we ensure that $k\cdot f'$ may only approach $-x'$ for large values of k. In doing so, we can estimate $\mathcal{D}^c_\Omega$: we split the sum over $\mathbb{Z}^2\setminus\{0\}$ into a sum over ζ and a sum over its complement $\zeta^c \equiv \mathbb{Z}^2\cap\mathcal{C}_\xi(\frac{\pi}{4})$, and compute a bound for each case.

If $k\in\zeta$, then, by (B3), for $\theta\in\Omega(x,f)$,

$$|x'(\theta) + k\cdot f'(\theta)| \ge C_3|f'(\theta)||k|^{-\epsilon}. \tag{B15}$$

If, on the other hand, $k\in\zeta^c\equiv\mathbb{Z}^2\cap\mathcal{C}_\xi(\frac{\pi}{4})$,

$$|k\cdot f'(\theta)| \ge |k||f'(\theta)|\cos(\alpha+\tfrac{\pi}{4}) \tag{B16}$$

so

$$|x'(\theta) + k\cdot f'(\theta)| \ge |k||f'(\theta)|\cos(\alpha+\tfrac{\pi}{4}) - |x'(\theta)|. \tag{B17}$$

We distinguish two cases once more: either

$$|k| \ge \frac{C_3 + \frac{|x'(\theta)|}{|f'(\theta)|}}{\cos(\alpha+\frac{\pi}{4})} \equiv R_1 \tag{B18}$$

(see (B7)) in which case (B15) holds true for these k's as well, or $|k| < R_1$, in which case (B15) holds by virtue of (B6). All in all, whatever the value of k, (B15) holds for all $k\in\mathbb{Z}^2\setminus\{0\}$.

Therefore, (B14) becomes

$$|\mathcal{D}^c_\Omega(x,f)| \le \frac{2C_1C_2}{C_3\min_{\theta\in[\theta_0,\theta_1]}|f'(\theta)|} \sum_{k\in\mathbb{Z}^2\setminus\{0\}} |k|^{1-\tau+\epsilon}. \tag{B19}$$

By (B5), this sum is bounded since $\epsilon < \tau - 3$.

We are left with estimating the measure of $\Omega^c(x,f) := [\theta_0,\theta_1]\setminus\Omega(x,f)$. Proceeding in a similar way as for $|\mathcal{D}^c_\Omega(x,f)|$, we find that

$$|\Omega^c(x,f)| \le 2C_3 \sum_{k\in\zeta} \frac{|k|^{-\epsilon}}{\min_{\theta\in[\theta_0,\theta_1]} |\frac{\partial}{\partial\theta}(\frac{x'(\theta)}{|f'(\theta)|}) + k\cdot\frac{\partial}{\partial\theta}(\frac{f'(\theta)}{|f'(\theta)|})|}. \tag{B20}$$

Here we see why it was necessary to introduce the set ζ in (B3): if ζ were taken to be $\mathbb{Z}^2\setminus\{0\}$, then we would run into exactly the same problem as before: the denominator in this bound would not be bounded away from 0. However, the problem that gave rise to the necessity of introducing Ω in the first place actually only occurred for the k's that are close to being orthogonal to $f'(\theta)$. So we can restrict ζ to only include the k's that are close to being orthogonal to $f'(\theta)$, which is why we define ζ as in (B4). Because $\frac{\partial}{\partial\theta}(\frac{f'}{|f'|})$ is orthogonal to f',

$$\tfrac{\partial}{\partial\theta}(\tfrac{f'}{|f'|}) \in \mathcal{C}_{\xi^\perp}(\alpha) \tag{B21}$$

and so, for $k \in \zeta$, because the maximal angle between k and $\frac{\partial}{\partial\theta}(\frac{f'}{|f'|})$ is $\alpha + \frac{\pi}{4}$,

$$|k \cdot \tfrac{\partial}{\partial\theta}(\tfrac{f'(\theta)}{|f'(\theta)|})| > |k||\tfrac{\partial}{\partial\theta}(\tfrac{f'(\theta)}{|f'(\theta)|})| \cos(\alpha + \tfrac{\pi}{4}). \tag{B22}$$

Thus,

$$|\tfrac{\partial}{\partial\theta}(\tfrac{x'(\theta)}{|f'(\theta)|}) + k \cdot \tfrac{\partial}{\partial\theta}(\tfrac{f'(\theta)}{|f'(\theta)|})| > |k||\tfrac{\partial}{\partial\theta}(\tfrac{f'(\theta)}{|f'(\theta)|})| \cos(\alpha + \tfrac{\pi}{4}) - |\tfrac{\partial}{\partial\theta}(\tfrac{x'(\theta)}{|f'(\theta)|})|. \tag{B23}$$

Therefore, if

$$|k| \geq \eta + \frac{|\frac{\partial}{\partial\theta}(\frac{x'(\theta)}{|f'(\theta)|})|}{|\frac{\partial}{\partial\theta}(\frac{f'(\theta)}{|f'(\theta)|})| \cos(\alpha + \frac{\pi}{4})} \equiv R_2 \tag{B24}$$

then

$$|\tfrac{\partial}{\partial\theta}(\tfrac{x'(\theta)}{|f'(\theta)|}) + k \cdot \tfrac{\partial}{\partial\theta}(\tfrac{f'(\theta)}{|f'(\theta)|})| > \eta|k||\tfrac{\partial}{\partial\theta}(\tfrac{f'(\theta)}{|f'(\theta)|})| \cos(\alpha + \tfrac{\pi}{4}). \tag{B25}$$

If, on the other hand, $|k| < R_2$, then (B25) holds by virtue of (B8). Thus, (B25) holds for all $k \in \zeta$. Therefore,

$$|\Omega^c(x, f)| \leq \frac{2C_3}{\eta \min\limits_{\theta\in[\theta_0,\theta_1]} |\frac{\partial}{\partial\theta}(\frac{f'(\theta)}{|f'(\theta)|})| \cos(\alpha + \frac{\pi}{4})} \sum_{k\in\mathbb{Z}^2\setminus\{0\}} |k|^{-1-\epsilon}. \tag{B26}$$

By (B5), this sum is bounded since $\epsilon > 1$. We conclude the proof by combining (B19) with (B26). ■

2. Applying the Diophantine condition to $|p^{\omega}_{F,i} - p^{\omega'}_{F,j} + lb + mb'|$

We now apply lemma B.1 repeatedly to prove lemma IV.1.

Let us first consider the case $i = j$, $\omega = \omega'$, and $y = m$, that is, we wish to find a condition on θ such that

$$|p^{\omega}_{F,i} - p^{\omega'}_{F,j} + lb + mb'| \equiv |lb + mb'| \equiv |M| \geq \frac{C_0}{|m|^\tau} \tag{B27}$$

(recall (28) and (31)). We recall (32):

$$|M| \geq \frac{2\pi}{\sqrt{3}}(l_\omega + m \cdot f_\omega) \tag{B28}$$

with $l_+ \equiv l_1$ and $l_- \equiv l_2$, and

$$f_\omega(\theta) := (\tfrac{\varphi_1 - \omega\varphi_3}{2},\ \tfrac{\varphi_2 + \omega\varphi_4}{2}). \tag{B29}$$

Now, recalling the definition (B1), we have that if $\theta \in \mathcal{D}(0, f_\omega)$, then the inequality (28) with $i = i'$, $\omega = \omega'$, and $y = m$ holds with $C_0 := \frac{2\pi}{\sqrt{3}}C_1$. We therefore just need to use Lemma B.1 to ensure that the measure of this set is large. Taking θ_0, θ_1 sufficiently small, it suffices to verify the conditions at $\theta = 0$, and conclude by continuity. In particular, when θ_0, θ_1 are small, $f'(\theta)$ will take values in a small cone $\mathcal{C}_{f'(0)}(\alpha)$ with $\alpha = O(\theta_1)$. Next, by a straightforward computation, we find

$$|f'_\omega(0)| = \sqrt{\frac{5}{3}}, \quad \left|\frac{\partial}{\partial\theta}\frac{f'_\omega(0)}{|f'_\omega(0)|}\right| = \frac{2\sqrt{3}}{5} \tag{B30}$$

Which are both non-zero, so (B5) is satisfied for small enough θ. Since $x = 0$, the other assumptions trivially hold: we choose $C_3 < 1/\sqrt{2}$ and $\eta < 1$ such that $R_1, R_2 < 1$, in which case the minima in (B6) and (B8) are taken over empty sets, so (B6) and (B8) hold trivially. Thus, by Lemma B.1, choosing $C_1 = O(\theta_1^2)$, the set $\mathcal{D}(0, f_\omega)$ has a large measure.

We now repeat the argument for the other values of i, j, y, and ω, ω'. First, note that $p_F^+ - p_F^- = \frac{1}{3}(b_1 - b_2)$ so the condition (28) holds for $\omega \neq \omega'$ whenever it holds for $\omega = \omega'$. Next, note that

$$|p^\omega_{F,i} - p^\omega_{F,j} + lb + mb'| = |R^T(p^\omega_{F,i} - p^\omega_{F,j} + lb + mb')| \tag{B31}$$

which corresponds to exchanging m and l, and flipping the sign of θ. The arguments made for θ may be adapted in a straightforward way to the case $-\theta$ so our derivation for $y = m$ also applies to $y = l$.

We are thus left with the case $i \neq j$, $\omega = \omega'$, and $y = m$. Without loss of generality, we choose $i = 1$, $j = 2$, and we wish to bound

$$|p^\omega_{F,1} - p^\omega_{F,2} + lb + mb'| \geq \frac{C_0}{|m|^\tau} \tag{B32}$$

Proceeding as we did above, we bound

$$|p^\omega_{F,1} - p^\omega_{F,2} + lb + mb'| \geq \frac{2\pi}{\sqrt{3}}\left(x_\omega(\theta) + l_1 + m \cdot f_\omega(\theta)\right) \tag{B33}$$

with

$$x_\omega(\theta) := \frac{1}{3}(1 - c_\theta) + \omega\frac{1}{\sqrt{3}}s_\theta. \tag{B34}$$

Therefore, if $\theta \in \mathcal{D}(x_\omega, f_\omega)$, then (B32) holds with $C_0 = \frac{2\pi}{\sqrt{3}}C_1$. To show that this set has a large measure, we check the assumptions of Lemma B.1, as we did above. Again, we check the assumptions at $\theta = 0$, and argue by continuity. We compute

$$|x'_\omega(0)| = \frac{1}{\sqrt{3}}, \quad \left|\frac{\partial}{\partial\theta}\frac{x'_\omega(0)}{|f'_\omega(0)|}\right| = \frac{8}{5\sqrt{15}} \tag{B35}$$

We thus find that if $C_3 < 1/\sqrt{2} - 1/\sqrt{5}$ and $\eta < 1 - 4\sqrt{2}/\sqrt{45}$, then $R_1, R_2 < 1$, so the minima in (B6) and (B8) are taken over empty sets, so (B6) and (B8) hold trivially.

All in all, we have found that if we restrict the values of θ to an intersection of Diophantine sets:

$$\theta \in \bigcap_{\omega=\pm}\bigcap_{\sigma=\pm}\mathcal{D}(0, f_\omega(\sigma\theta)) \cap \bigcap_{\omega=\pm}\bigcap_{\sigma=\pm}\mathcal{D}(x_\omega(\sigma\theta), f_\omega(\sigma\theta)) \tag{B36}$$

then (28) is satisfied for any value of i, i', ω, ω', and y with a constant $C_0 = O(\theta_1^2)$. Because each set has an arbitrarily large measure (relative to $[\theta_0, \theta_1]$), their intersection also does.

Appendix C: Naive perturbation theory

The Schwinger function is computed using perturbation theory: formally,

$$\begin{aligned}(S_{1,1}(\mathbf{k}))_{\alpha',\alpha} = \sum_{N=0}^\infty \sum_{\alpha_0,\cdots,\alpha_{2N+1}} (g_1(\mathbf{k}))_{\alpha,\alpha_0} \Bigg(\prod_{n=0}^N \Bigg(\sum_{l_{2n}} \tau^{(1)}_{l_{2n},\alpha_{2n}}(k + m_{2n-1}b' + l_{2n}b)(g_2(\mathbf{k} + l_{2n}b))_{\alpha_{2n},\alpha_{2n+1}} \cdot \\ \cdot \sum_{m_{2n+1}} \tau^{(2)}_{m_{2n+1},\alpha_{2n+1}}(k + l_{2n}b + m_{2n+1}b')((g_1(\mathbf{k} + m_{2n+1}b'))_{\alpha_{2n+1},\alpha_{2n+2}})^{\mathbb{1}_{n<N}}\Bigg)\Bigg) (g_1(\mathbf{k}))_{\alpha_{2N+1},\alpha'}\end{aligned} \tag{C1}$$

where $m_{-1} \equiv l_{-1} \equiv 0$ and $\mathbb{1}_{n<N} \in \{0, 1\}$ is equal to 1 if and only if $n < N$,

$$\begin{aligned}(S_{2,2}(\mathbf{k}))_{\alpha',\alpha} = \sum_{N=0}^\infty \sum_{\alpha_0,\cdots,\alpha_{2N+1}} (g_2(\mathbf{k}))_{\alpha,\alpha_0} \Bigg(\prod_{n=0}^N \Bigg(\sum_{m_{2n}} \tau^{(2)}_{m_{2n},\alpha_{2n}}(k + l_{2n-1}b + m_{2n}b')(g_1(\mathbf{k} + m_{2n}b'))_{\alpha_{2n},\alpha_{2n+1}} \cdot \\ \cdot \sum_{l_{2n+1}} \tau^{(1)}_{l_{2n+1},\alpha_{2n+1}}(k + m_{2n}b' + l_{2n+1}b)((g_2(\mathbf{k} + l_{2n+1}b))_{\alpha_{2n+1},\alpha_{2n+2}})^{\mathbb{1}_{n<N}}\Bigg)\Bigg) (g_2(\mathbf{k}))_{\alpha_{2N+1},\alpha'}\end{aligned} \tag{C2}$$

$$(S_{2,1}(\mathbf{k}))_{\alpha',\alpha} = \sum_{N=0}^{\infty} \sum_{\alpha_0,\cdots,\alpha_{2N}} (g_1(\mathbf{k}))_{\alpha,\alpha_0} \left(\prod_{n=0}^{N} \left(\sum_{l_{2n}} \tau^{(1)}_{l_{2n},\alpha_{2n}}(k+m_{2n-1}b'+l_{2n}b)((g_2(\mathbf{k}+l_{2n}b))_{\alpha_{2n},\alpha_{2n+1}})^{\mathbb{1}_{n<N}} \cdot \right.\right.$$
$$\left.\left. \cdot \left(\sum_{m_{2n+1}} \tau^{(2)}_{m_{2n+1},\alpha_{2n+1}}(k+l_{2n}b+m_{2n+1}b')(g_1(\mathbf{k}+m_{2n+1}b'))_{\alpha_{2n+1,2n+2}} \right)^{\mathbb{1}_{n<N}} \right)\right) (g_2(\mathbf{k}))_{\alpha_{2N,\alpha'}} \tag{C3}$$

$$(S_{1,2}(\mathbf{k}))_{\alpha',\alpha} = \sum_{N=0}^{\infty} \sum_{\alpha_0,\cdots,\alpha_{2N}} (g_2(\mathbf{k}))_{\alpha,\alpha_0} \left(\prod_{n=0}^{N} \left(\sum_{m_{2n}} \tau^{(2)}_{m_{2n},\alpha_{2n}}(k+l_{2n-1}b+m_{2n}b')((g_1(\mathbf{k}+m_{2n}b'))_{\alpha_{2n},\alpha_{2n+1}})^{\mathbb{1}_{n<N}} \cdot \right.\right.$$
$$\left.\left. \cdot \left(\sum_{l_{2n+1}} \tau^{(1)}_{l_{2n+1}}(k+m_{2n}b'+l_{2n+1}b)(g_2(\mathbf{k}+l_{2n+1}b))_{\alpha_{2n+1},\alpha_{2n+2}} \right)^{\mathbb{1}_{n<N}} \right)\right) \tag{C4}$$

Appendix D: Symmetry constraints on the resonant terms

1. Symmetries of the system

Let us state the symmetries of the model, which will play an important role in our discussion. Note that the C_2T symmetry in (c) only holds if $\xi = (0, 1/2)$.

a) *Complex conjugation*: every complex constant is conjugated and

$$\widehat{\psi}^{\pm}_{1,k_0,k,\alpha} \mapsto e^{\mp i\xi(b_1+b_2)} \widehat{\psi}^{\pm}_{1,-k_0,-k,\alpha}, \quad \widehat{\psi}^{\pm}_{2,k_0,k,\alpha} \mapsto e^{\mp i\xi(b'_1+b'_2)} \widehat{\psi}^{\pm}_{2,-k_0,-k,\alpha} \tag{D1}$$

Indeed, it is straighforward to check that H_1 and H_2 (see (5) and (9)) are left invariant by (D1) (following from the fact that $\Omega(-k) = \Omega(k)^*$). To check the invariance of the interlayer hopping term V (see (A4)), there is one subtelty: because $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ have different periodicity, we cannot simply change k_1 to $-k_1$ and k'_2 to $-k'_2$, which would not leave $\widehat{\mathcal{L}}_i$ invariant. Instead, we map k_1 to $-k_1 + b_1 + b_2$ and k'_2 to $-k'_2 + b'_1 + b'_2$. It is then straightforward to check (using (A4)) that V remains invariant under (D1), using $e^{i(1,1)bx_1} = e^{i(1,1)b'x'_2} = 1$ and periodicity.

b) *Particle-hole*

$$\widehat{\psi}^{\pm}_{1,k_0,k,\alpha} \mapsto ie^{\pm i\xi(b_1+b_2)} \widehat{\psi}^{\mp}_{1,k_0,-k,\alpha}, \quad \widehat{\psi}^{\pm}_{2,k_0,k,\alpha} \mapsto ie^{\pm i\xi(b'_1+b'_2)} \widehat{\psi}^{\mp}_{2,k_0,-k,\alpha} \tag{D2}$$

The argument is substantially the same as for the conjugation symmetry (a).

c) C_2T *symmetry*

$$\widehat{\psi}^{\pm}_{j,k_0,k,\alpha} \mapsto e^{\pm i\chi_j(k)} \widehat{\psi}^{\pm}_{j,k_0,-k,\bar{\alpha}} \tag{D3}$$

where if $\alpha = a$ then $\bar{\alpha} = b$ and if $\alpha = b$ then $\bar{\alpha} = a$, and

$$\chi_1(k) := \frac{d_b}{2}(b_1+b_2) - kd_b - \sigma_{k,2}bd_b, \quad \chi_2(k) := \frac{d_b}{2}(b'_1+b'_2) - kRd_b - \sigma_{k,1}b'd_b. \tag{D4}$$

H_1 and H_2 are invariant under (D3) for the same reason as the conjugation and particle-hole symmetries. For the interlayer hopping, it is easiest to use the expression of V in (13), which involves two integrals: one over $\widehat{\mathcal{L}}_1$ and one over $\widehat{\mathcal{L}}_2$. Let us discuss the invariance of the integral over $\widehat{\mathcal{L}}_1$, as the invariance of the other follows from a similar argument. We change variables in the integral: $k \mapsto -k+b_1+b_2$, as well as in the sum over l: $(l_1, l_2) \mapsto -(l_1+1, l_2+1)$, and in the sum over α: $\alpha \mapsto \bar{\alpha}$. This changes $\tau^{(1)}_{l,\alpha}(k+lb)$ to $\tau^{(1)}_{-l-(1,1),\bar{\alpha}}(-k-lb)$. Now, by (13),

$$\frac{\tau^{(1)}_{-l-(1,1),\bar{\alpha}}(-k-lb)}{\tau^{(1)}_{l,\alpha}(k+lb)} = e^{-i\xi(2l+(1,1))b} e^{i(k+lb)(d_{\bar{\alpha}}+d_\alpha-Rd_\alpha-Rd_{\bar{\alpha}})} e^{-i\xi(\sigma_{-k-lb,1}-\sigma_{k+lb,1})b'} \frac{\widehat{\varsigma}(k+lb)}{\widehat{\varsigma}^*(k+lb)}. \tag{D5}$$

In addition, if $k + lb - \sigma_{k+lb,1} b' \in \widehat{\mathcal{L}}_2$, then $-k - lb + (\sigma_{k+lb,1} + (1,1)) b' \in \widehat{\mathcal{L}}_2$, so

$$\sigma_{-k-lb,1} = -\sigma_{k+lb,1} - (1,1) \tag{D6}$$

and, since $d_a = 0$ and $d_b = (1,0)$,

$$d_{\bar{\alpha}} + d_\alpha - R d_\alpha - R d_{\bar{\alpha}} = d_b - R d_b \tag{D7}$$

and, since $\varsigma(x) = \varsigma(-x)$, $\widehat{\varsigma} \in \mathbb{R}$. Therefore,

$$\frac{\tau^{(1)}_{-l-(1,1),\bar{\alpha}}(-k - lb)}{\tau^{(1)}_{l,\alpha}(k + lb)} = e^{-i\xi(2l+(1,1))b} e^{i(k+lb)(d_b - R d_b)} e^{i\xi(2\sigma_{k+lb,1}+(1,1))b'}. \tag{D8}$$

Since $\xi = d_b/2$,

$$\frac{\tau^{(1)}_{-l-(1,1),\bar{\alpha}}(-k - lb)}{\tau^{(1)}_{l,\alpha}(k + lb)} = e^{ikd_b} e^{-i(k+lb)Rd_b} e^{id_b\sigma_{k+lb,1}b'} e^{i\frac{d_b}{2}(b'_1+b'_2-b_1-b_2)}. \tag{D9}$$

It is then straightforward to check that this extra phase gets canceled out exactly by $e^{\pm i\chi_j}$ in (D3) (to see this, note that if $k \in \widehat{\mathcal{L}}_1$, then $\sigma_{k,2} = 0$).

d) *Inversion*

$$\widehat{\psi}^{\pm}_{j,k_0,k,\alpha} \to i(-1)^\alpha (-1)^j \widehat{\psi}^{\pm}_{j,-k_0,k,\alpha} \tag{D10}$$

It is straightforward to check that H_1, H_2, and the interlayer hopping (using (A4)) are invariant under (D10).

2. Constraints on the resonant terms

The discrete symmetry properties seen above implies severely constraint the form of the resonant terms. In the following, we use the notation "$=^a$" to mean "by using symmetry (a) from Section D 1 (that is, Complex conjugation), it is equal to", and similarly for "$=^b$", "$=^c$", "$=^d$".

1. Using that $W_{aa}(k_0,k) =^d -W_{aa}(-k_0,k)$ we get $W_{aa}(0,p_F^\omega) = \partial_1 W_{aa}(0,p_F^\omega) = \partial_2 W_{aa}(0,p_F^\omega) = 0$. Similarly, $W_{bb}(0,p_F^\omega) = \partial_1 W_{bb}(0,p_F^\omega) = \partial_2 W_{bb}(0,p_F^\omega) = 0$.

2. From $W_{ab}(k_0,k) =^b W_{ba}(k_0,-k) =^a W^*_{ba}(-k_0,k)$ we get $W_{ab}(0,p_F^\omega) = W^*_{ba}(0,p_F^\omega)$, $\partial_1 W_{ab}(0,p_F^\omega) = \partial_1 W^*_{ba}(0,p_F^\omega)$, $\partial_2 W_{ab}(0,p_F^\omega) = \partial_2 W^*_{ba}(0,p_F^\omega)$. Moreover $W_{ab}(k_0,k) =^d W_{ab}(-k_0,k)$ hence $\partial_0 W_{ab}(0,p_F^\omega) = 0$.

3. $\partial_0 W_{aa}(k_0,k) =^a -\partial_0 W^*_{aa}(-k_0,-k) =^b -\partial_0 W^*_{aa}(-k_0,k)$ hence is pure imaginary at $k_0 = 0$; moreover $\partial_0 W_{aa}(k_0,k) =^c \partial_0 W_{bb}(k_0,-k) =^a -\partial_0 W^*_{bb}(-k_0,k)$ so that $\partial_0 W_{aa}(0,p_F^\omega) = \partial W_{bb}(0,p_F^\omega) = iz$ with z real

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