The Simplified approach to the Bose gas without translation invariance

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Abstract

The Simplified approach to the Bose gas was introduced by Lieb in 1963 to study the ground state of systems of interacting Bosons. In a series of recent papers, it has been shown that the Simplified approach exceeds earlier expectations, and gives asymptotically accurate predictions at both low and high density. In the intermediate density regime, the qualitative predictions of the Simplified approach have also been found to agree very well with Quantum Monte Carlo computations. Until now, the Simplified approach had only been formulated for translation invariant systems, thus excluding external potentials, and non-periodic boundary conditions. In this paper, we extend the formulation of the Simplified approach to a wide class of systems without translation invariance. This also allows us to study observables in translation invariant systems whose computation requires the symmetry to be broken. Such an observable is the momentum distribution, which counts the number of particles in excited states of the Laplacian. In this paper, we show how to compute the momentum distribution in the Simplified approach, and show that, for the Simple Equation, our prediction matches up with Bogolyubov's prediction at low densities, for momenta extending up to the inverse healing length.

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1. Introduction

The Bose gas is one of the simplest models in quantum statistical mechanics, and yet it has a rich and complex phenomenology. As such, it has garnered much attention from the mathematical physics community for over half a century. It consists in infinitely many identical Bosons and is used to model a wide range of physical systems, from photons in black body radiation to gasses of helium atoms. Whereas photons do not directly interact with each other, helium atoms do, and such an interaction makes studying such systems very challenging. To account for interactions between Bosons, Bogolyubov [Bo47] introduced a widely used approximation scheme that accurately predicts many observables [LHY57] *in the low density* regime. Even though Bogolyubov theory is not mathematically rigorous, it has allowed mathematical physicists to develop the necessary intuition to prove a wide variety of results about the Bose gas, such as the low density expansion of the ground state energy of the Bose gas in the thermodynamic limit-[Dy57, LY98, YY09, FS20, BCS21, FS22], as well as many other results in scaling limits other than the thermodynamic limit (see [Sc22] for a review, as well as, among many others, [LSY00, LS02, NRS16, BBe18, BBe19, DSY19, BBe20, DS20, NT21, BSS22, BSS22b, HST22, NNe22]). In this note, we will focus on the ground state in the thermodynamic limit.

In 1963, E.H. Lieb [Li63, LS64, LL64] introduced a new approximation scheme to compute properties of the ground state of Bose gasses, called the *Simplified approach*, which has recently been found to yield surprisingly accurate results [CJL20, CJL21, CHe21, Ja22]. Indeed, while Bogolyubov theory is accurate at low densities, the Simplified approach has been shown to yield asymptotically accurate results at both *low and high* densities [CJL20, CJL21] for interaction potentials that are of positive type, as well as reproduce the qualitative behavior of the Bose gas at intermediate densities [CHe21]. In addition to providing a promising tool to study the Bose gas, the derivation of the Simplified approach is different enough from Bogolyubov theory that it may give novel insights into longstanding open problems about the Bose gas.

The original derivation of the Simplified approach [Li63] is quite general, and applies to any translation invariant system (it even works for Coulomb [LS64] and hard-core [CHe21] interactions). In the present paper, we extend this derivation to systems that break translation invariance. This allows us to formulate the Simplified approach for systems with external potentials, and with a large class of boundary conditions. In addition, it allows us to compute observables in systems with translation invariance, but whose computation requires breaking the translation invariance. We will discuss an example of such an observable: the momentum distribution.

The momentum distribution $\mathcal{M}(k)$ is the probability of finding a particle in the state e^{ikx}. Bose gasses are widely expected to form a Bose-Einstein condensate, although this has still not been proven (at least for continuum interacting gasses in the thermodynamic limit). From a mathematical point of view, Bose-Einstein condensation is defined as follows: if the Bose gas consists of N particles, the average number of particles in the constant state (corresponding to $k = 0$ in e^{ikx}) is of order N. The *condensate fraction* is defined as the proportion of particles in the constant state. The momentum distribution is an extension of the condensate fraction to a more general family of states. In particular, computing $\mathcal{M}(k)$ for $k \neq 0$ amounts to counting particles that are *not* in the condensate. This quantity has been used in the recent proof [FS20, FS22] of the energy asymptotics of the Bose gas at low density. A numerical computation of the prediction of the Simplified approach for $\mathcal{M}(k)$ have been published in [Ja23].

The main results in this paper fall into two categories. First, we will derive the Simplified approach without assuming translation invariance, see Theorem 2.2. To do so, we will make the so-called "factorization assumption", on the marginals of the ground state wavefunction, see Assumption 2.1. This allows us to derive a Simplified approach for a wide variety of situations in which translation symmetry breaking is violated, such as in the presence of external potentials. Second, we compute a prediction for the momentum distribution using the Simplified approach. The Simplified approach does not allow us to compute the ground state wavefunction directly, so to compute observables, such as the momentum distribution, we use the Hellmann-Feynman technique and add an operator to the Hamiltonian. In the case of the momentum distribution, this extra operator is a projector onto e^{ikx}, which breaks the translation invariance of the system. In Theorem 2.4, we show how to compute the momentum distribution in the Simplified approach using the general result of Theorem 2.2. In addition, we check that the prediction is credible, by comparing it to the prediction of Bogolyubov theory, and find that both approaches agree at low densities and small k, see Theorem 2.5.

The rest of the paper is structured as follows. In Section 2, we specify the model and state the main results precisely. We then prove Theorem 2.2 in Section 3, Theorem 2.4 in Section 4.1, and Theorem 2.5 in Section 4.2. The proofs are largely independent and can be read in any order.

2. The model and main results

Consider N Bosons in a box of volume V denoted by $\Omega_V := [-V^{\frac{1}{3}}/2, V^{\frac{1}{3}}/2]^3$, interacting with each other via a pair potential $v \in L_1(\Omega_V^2)$ that is symmetric under exchanges of particles: $v(x,y) \equiv v(y,x)$. The Hamiltonian acts on $L_{2,\mathrm{sym}}(\Omega_V^N)$ as

$$\mathcal{H} := -\frac{1}{2}\sum_{i=1}^N \Delta_i + \sum_{1\leqslant i<j\leqslant N} v(x_i,x_j) + \sum_{i=1}^N P_i \tag{2.1}$$

where $\Delta_i \equiv \partial_{x_i}^2$ is the Laplacian with respect to the position of the i-th particle and P_i is an extra single-particle term of the following form: given a self-adjoint operator ϖ on $L_2(\Omega_V)$,

$$P_i := \mathbb{1}^{\otimes i-1} \otimes \varpi \otimes \mathbb{1}^{\otimes N-i}. \tag{2.2}$$

For instance, if we take ϖ to be a multiplication operator by a function v_0, then $\sum_i P_i$ is the contribution of the external potential v_0. Or ϖ could be a projector onto e^{ikx}, which is what we will do below to compute the momentum distribution. Because P_i acts on a single particle, it breaks translational symmetry as soon as it is not constant.

We may impose any boundary condition on the box, as long as the Laplacian is self-adjoint. We will consider the thermodynamic limit, in which $N, V \to \infty$, such that

$$\frac{N}{V} = \rho \tag{2.3}$$

is fixed. We consider the ground state ψ_0, which is the eigenfunction of $\mathcal{H}$ with the lowest eigenvalue E_0:

$$\mathcal{H}\psi_0 = E_0\psi_0. \tag{2.4}$$

(It is a standard argument to prove that ψ_0 exists, and is both real and non-negative.)

In order to take the thermodynamic limit, we will assume that v is uniformly integrable in V:

$$|v(x,y)| \leqslant \bar v(x,y), \qquad \int_{\mathbb{R}^3} dy\ \bar v(x,y) \leqslant c \tag{2.5}$$

where $\bar{v}$ and c are independent of V. In addition, we assume that, for any f that is uniformly integrable in V,

$$\int dx\ \varpi f(x) \leqslant c. \tag{2.6}$$

2.1. The Simplified approach without translation invariance

The crucial idea of Lieb's construction [Li63] is to consider the wave function ψ as a probability distribution, instead of the usual $|\psi|^2$. Since $\psi \geqslant 0$, this can be done by normalizing ψ by its L_1 norm. We then define the i-th marginal of ψ as

$$g_i(x_1, \cdots, x_i) := \frac{\int \frac{dx_{i+1}}{V} \cdots \frac{dx_N}{V}\ \psi(x_1, \cdots, x_N)}{\int \frac{dy_1}{V} \cdots \frac{dy_N}{V}\ \psi(y_1, \cdots, y_N)} \equiv V^i \frac{\int dx_{i+1} \cdots dx_N\ \psi(x_1, \cdots, x_N)}{\int dy_1 \cdots dy_N\ \psi(y_1, \cdots, y_N)}. \tag{2.7}$$

In particular, for $i \in \{2, \cdots, N\}$,

$$\int \frac{dx_i}{V}\ g_i(x_1, \cdots, x_i) = g_{i-1}(x_1, \cdots, x_{i-1}), \quad \int \frac{dx}{V}\ g_1(x) = 1. \tag{2.8}$$

Because of the symmetry of ψ under exchanges of particles, g_i is symmetric under $x_i \leftrightarrow x_j$.

Inspired by [Li63], we will make the following approximation.

Assumption 2.1

(*Factorization*)

For $i = 2, 3, 4$,

$$g_i(x_1, \cdots, x_i) = \prod_{1 \leqslant j < l \leqslant i} W_i(x_j, x_l) \tag{2.9}$$

with

$$W_i(x, y) = f_i(x) f_i(y)(1 - u_i(x, y)) \tag{2.10}$$

in which f_i and u_i are bounded independently of V and u_i is uniformly integrable in V:

$$|u_i(x, y)| \leqslant \bar{u}_i(x, y), \quad \int dy\ \bar{u}_i(x, y) \leqslant c_i \tag{2.11}$$

with c_i independent of V. We further assume that, for $i = 1, 2, 3$,

$$\lim_{V \to \infty} \int dx_i\ \Delta_{x_i} g_i(x_1, \cdots, x_i) = 0 \tag{2.12}$$

in other words, these boundary terms vanish in the thermodynamic limit.

In other words, g_i factorizes exactly as a product of pair terms W_i. The f_i in W_i allow for W_i to be modulated by a slowly varying density, which is the main novelty of this paper compared to [Li63]. The inequality (2.11) ensures that u_i decays sufficiently fast on the microscopic scale. Note that, by the symmetry under exchanges of particles, $u_i(x, y) \equiv u_i(y, x)$.

Here, we use the term "assumption" because it leads to the Simplified approach. However, it is really an *approximation* rather than an assumption: this factorization will certainly not hold true exactly. At best, one might expect that the assumption holds approximately in the limit of small and large ρ, and for distant points, as numerical evidence suggests in the translation invariant case. In the present paper, we will not attempt a proof that this approximation is accurate, and instead explore its consequences. Suffice it to say that this approximation is one of *statistical independence* that is reminiscent of phenomena arising in statistical mechanics when the density is low, that is, when the interparticle distances are large. In the current state of the art, we do not have much in the way of an explanation for why this statistical independence should hold; instead, we have extensive evidence, both numerical [CHe21] and analytical [CJL20, CJL21], that this approximation leads to very accurate predictions.

The equations of the Simplified approach are derived from Assumption 2.1, using the eigenvalue equation (2.4) along with

$$\int \frac{dx}{V}\ g_1(x) = 1 \tag{2.13}$$

$$\int \frac{dy}{V}\ g_2(x,y) = g_1(x) \tag{2.14}$$

$$\int \frac{dz}{V}\ g_3(x,y,z) = g_2(x,y) \tag{2.15}$$

$$\int \frac{dz}{V}\frac{dt}{V}\ g_4(x,y,z,t) = g_2(x,y) \tag{2.16}$$

(all of which follow from (2.8)) to compute u_i and f_i.

In the translation invariant case, the factorization assumption leads to an equation for g_2 alone, as g_1 is constant. When translation invariance is broken, g_1 is no longer constant, and the Simplified approach consists in two coupled equations for g_1 and g_2. We formulate these in terms of g_1 and u_2, with

$$g_2(x,y) =: g_1(x)g_1(y)(1-u_2(x,y)). \tag{2.17}$$

Theorem 2.2

If g_i satisfies Assumption 2.1, the eigenvalue equation (2.4) and (2.13)-(2.16), then g_1 and u_2 satisfy the two coupled equations

$$\left(-\frac{\Delta}{2} + (\varpi - \langle\varpi\rangle) + 2\left(\mathcal{E}(x) - \langle\mathcal{E}(y)\rangle\right) + \frac{1}{2}\left(\bar{A}(x) - \left\langle\bar{A}\right\rangle - \bar{C}(x)\right)\right) g_1(x) + \Sigma_1(x) = 0 \tag{2.18}$$

and

$$\begin{aligned}\left(-\frac{1}{2}(\Delta_x+\Delta_y) + v(x,y) - 2\rho\bar{K}(x,y) + \rho^2\bar{L}(x,y) + \bar{R}_2(x,y)\right) g_1(x)g_1(y)(1-u_2(x,y)) + \\ +\Sigma_2(x,y) = 0\end{aligned} \tag{2.19}$$

where

$$\langle f\rangle := \int \frac{dy}{V}\ g_1(y)f(y), \quad \langle\varpi\rangle \equiv \int \frac{dy}{V}\ \varpi g_1(y) \tag{2.20}$$

$$\bar{S}(x,y) := v(x,y)(1-u_2(x,y)), \quad f_1\bar{*}f_2(x,y) := \int dz\ g_1(z)f_1(x,z)f_2(z,y) \tag{2.21}$$

$$\mathcal{E}(x) := \frac{\rho}{2}\int dy\ g_1(y)\bar{S}(x,y), \quad \bar{A}(x) := \rho^2\bar{S}\bar{*}u_2\bar{*}u_2(x,x) \tag{2.22}$$

$$\bar{C}(x) := 2\rho^2\int dz\ g_1(z)u_2\bar{*}\bar{S}(x,z) + 2\rho\int dy\ \varpi_y(g_1(y)u_2(x,y)). \tag{2.23}$$

$$\bar{K}(x,y) := \bar{S}\bar{*}u_2(x,y) \tag{2.24}$$

$$\begin{aligned}\bar{L}(x,y) := \bar{S}\bar{*}u_2\bar{*}u_2(x,y) - 2u_2\bar{*}(u_2(u_2\bar{*}\bar{S}))(x,y) + \\ +\frac{1}{2}\int dzdt\ g_1(z)g_1(t)\bar{S}(z,t)u_2(x,z)u_2(x,t)u_2(y,z)u_2(y,t)\end{aligned} \tag{2.25}$$

$$\bar R_2(x,y)=2\left(\mathcal E(x)+\mathcal E(y)-2\left<\mathcal E\right>\right)+\left(\varpi_x+\varpi_y-2\left<\varpi\right>\right)+$$
$$+\frac12\left(\bar A(x)+\bar A(y)-2\left<\bar A\right>-\bar C(x)-\bar C(y)\right)+2\rho u_2\bar\ast\left(u_2(\mathcal E-\left<\mathcal E\right>)\right)+$$
$$+\rho\int dz\ \varpi_z(g_1(z)u_2(x,z)u_2(y,z))-\rho u_2\bar\ast u_2\left<\varpi\right> \tag{2.26}$$

in which ϖ_x is the action of ϖ on the x-variable, and similarly for ϖ_y and

$$\Sigma_i \underset{V\to\infty}{\longrightarrow} 0 \tag{2.27}$$

pointwise. Furthermore, the prediction for the energy per particle is

$$e:=\left<\mathcal E\right>+\left<\varpi\right>+\Sigma_0 \tag{2.28}$$

where $\Sigma_0\to 0$ as $V\to\infty$.

This theorem is proved in Section 3.

Let us compare this to the equation for u in the Simplified approach in the translation invariant case [CHe21, (5)], [Ja22, (3.15)]:

$$-\Delta u(x)=(1-u(x))\left(v(x)-2\rho K(x)+\rho^2L(x)\right) \tag{2.29}$$

$$K:=u\ast S,\quad S(y):=(1-u(y))v(y) \tag{2.30}$$

$$L:=u\ast u\ast S-2u\ast(u(u\ast S))+\frac12\int dydz\ u(y)u(z-x)u(z)u(y-x)S(z-y). \tag{2.31}$$

We will prove that these follow from Theorem 2.2:

Corollary 2.3

(*Translation invariant case*)

In the translation invariant case $v(x,y)\equiv v(x-y)$ and $\varpi=0$ with periodic boundary conditions, if (2.18)-(2.18) has a unique translation invariant solution, then (2.19) reduces to (2.29) in the thermodynamic limit.

The idea of the proof is quite straightforward. Equation (2.19) is very similar to (2.29), but for the addition of the extra term $\bar R_2$. An inspection of (2.26) shows that the terms in $\bar R_2$ are mostly of the form $f-\left<f\right>$, which vanish in the translation invariant case, and terms involving ϖ, which is set to 0 in the translation invariant case. The only remaining extra term is $\bar C(x)+\bar C(y)$, which we will show vanishes in the translation invariant case due to the identity (2.14).

Theorem 2.2 is quite general, and can be used to study a trapped Bose gas, in which there is an external potential v_0. In this case, ϖ is a multiplication operator by v_0. A natural approach is to scale v_0 with the volume: $v_0(x)=\bar v_0(V^{-1/3}x)$ in such a way that the size of the trap grows as $V\to\infty$, thus ensuring a finite local density in the thermodynamic limit. Following the ideas of Gross and Pitaevskii [Gr61, Pi61], we would then expect to find that (2.18) and (2.19) decouple, and that (2.19) reduces to the translation invariant equation (2.29), with a density that is modulated over the trap. However, the presence of $\bar R_2$ in (2.19) and $\bar C$ in (2.18) breaks this picture. Further investigation of this question is warranted.

2.2. The momentum distribution

The momentum distribution for the Bose gas is defined as

$$\mathcal{M}^{(\mathrm{Exact})}(k) := \frac{1}{N}\sum_{i=1}^{N} \langle\psi_0|\, P_i \,|\psi_0\rangle \tag{2.32}$$

where

$$\varpi f := \epsilon|e^{ikx}\rangle\langle e^{ikx}|f \equiv \epsilon e^{ikx}\int dy\ e^{-iky}f(y) \tag{2.33}$$

and P_i is defined as in (2.2):

$$P_i\psi(x_1,\cdots,x_N) = \epsilon e^{ikx_i}\int dy_y\ e^{iky_i}\psi(x_1,\cdots,x_{i-1},y_i,x_{i+1},\cdots,x_N) \tag{2.34}$$

Equivalently,

$$\mathcal{M}^{(\mathrm{Exact})}(k) = \frac{\partial}{\partial\epsilon}\frac{E_0}{N}\Big|_{\epsilon=0} \tag{2.35}$$

where E_0 is the energy in (2.4) for the Hamiltonian (2.1). Using the Simplified approach, we do not have access to the ground state wavefunction, so we cannot compute $\mathcal{M}$ using (2.32). Instead, we use the Hellmann-Feynman theorem, which consists in adding $\sum_i P_i$ to the Hamiltonian. However, doing so breaks the translational symmetry. This is why Theorem 2.2 is needed to compute the momentum distribution. (A similar computation was done in [CHe21], but, there, the derivation of the momentum distribution for the Simplified approach was taken for granted.)

By Theorem 2.2, and, in particular, (2.28), we obtain a natural definition of the prediction of the Simplified approach for the momentum distribution:

$$\mathcal{M}(k) := \frac{\partial}{\partial\epsilon}\left(\langle\mathcal{E}\rangle + \langle\varpi\rangle\right)|_{\epsilon=0}\,. \tag{2.36}$$

Theorem 2.4

(*Momentum distribution*)

Under the assumptions of Theorem 2.2, using periodic boundary conditions, if v is translation invariant and $\varpi = 0$, then, if $k \neq 0$, in the thermodynamic limit,

$$\mathcal{M}(k) = \frac{\partial}{\partial\epsilon}\frac{\rho}{2}\int dx\ (1-u(x))v(x)\Big|_{\epsilon=0} \tag{2.37}$$

where

$$-\Delta u(x) = (1-u(x))v(x) - 2\rho K(x) + \rho^2 L(x) + \epsilon F(x) \tag{2.38}$$

where K and L are those of the translation invariant Simplified approach (2.30)-(2.31) and

$$F(x) := -2\hat{u}(-k)\cos(kx). \tag{2.39}$$

We thus compute the momentum distribution. To check that our prediction is plausible, we compare it to the Bogolyubov prediction, which can easily be derived from [LSe05, Appendix A]:

$$\mathcal{M}^{(\mathrm{Bogolyubov})}(k) = -\frac{1}{2\rho}\left(1 - \frac{k^2+2\rho\hat{v}(k)}{\sqrt{k^4+4k^2\rho\hat{v}(k)}}\right) \tag{2.40}$$

(this can be obtained by differentiating [LSe05, (A.26)] with respect to $\epsilon(k)$, which returns the number of particles in the state e^{ikx}, which we divide by ρ to obtain the momentum distribution).

Actually, following the ideas of [LHY57], we replace $\hat{v}$ by a so-called "pseudopotential", which consists in replacing v by a Dirac delta function, while preserving the scattering length:

$$\hat{v}(k) = 4\pi a \tag{2.41}$$

where the scattering length a is defined in [LSe05, Appendix C]. Thus,

$$\mathcal{M}^{(\mathrm{Bogolyubov})}(k) = -\frac{1}{2\rho}\left(1 - \frac{k^2 + 8\pi\rho a}{\sqrt{k^4 + 16\pi k^2 \rho a}}\right). \tag{2.42}$$

We prove that, for the Simple Equation, as $\rho \to 0$, the prediction for the momentum distribution coincides with Bogolyubov's, for $|k| \lesssim \sqrt{\rho a}$. The length scale $1/\sqrt{\rho a}$ is called the *healing length*, and is the distance at which pairs of particles correlate [FS20]. It is reasonable to expect the Bogolyubov approximation to break down beyond this length scale.

The momentum distribution for the Simple equation, following the prescription detailed in-[CJL20, CJL21, CHe21, Ja22], is defined as

$$\mathcal{M}^{(\mathrm{simpleq})}(k) = \frac{\partial}{\partial \epsilon}\frac{\rho}{2}\int dx\ (1 - u(x))v(x)\bigg|_{\epsilon=0} \tag{2.43}$$

where [CJL20, (1.1)-(1.2)]

$$-\Delta u(x) = (1 - u(x))v(x) - 4eu + 2\rho e u * u + \epsilon F(x), \quad e := \frac{\rho}{2}\int dx\ (1 - u(x))v(x) \tag{2.44}$$

where F was defined in (2.39).

Theorem 2.5

Assume that v is translation and rotation invariant ($v(x, y) \equiv v(|x - y|)$), and consider periodic boundary conditions. We rescale k:

$$\kappa := \frac{k}{2\sqrt{e}} \tag{2.45}$$

we have, for all $\kappa \in \mathbb{R}^3$,

$$\lim_{e\to 0} \rho\mathcal{M}^{(\mathrm{simpleq})}(2\sqrt{e}\kappa) = \lim_{e\to 0} \rho\mathcal{M}^{(\mathrm{Bogolyubov})}(2\sqrt{e}\kappa) = -\frac{1}{2}\left(1 - \frac{\kappa^2 + 1}{\sqrt{(\kappa^2+1)^2 - 1}}\right). \tag{2.46}$$

The rotation invariance of v is presumably not necessary. However, the proof of this theorem is based on [CJL21], where rotational symmetry was assumed for convenience.

3. The Simplified approach without translation invariance, proof of Theorem 2.2

3.1. Factorization

We will first compute f_i and u_i in Assumption 2.1.

3.1.1. Factorization of g_2

We start by considering g_2.

Lemma 3.1

Assumption 2.1 with $i = 2$ and (2.13)-(2.14) imply that

$$g_2(x, y) = g_1(x)g_1(y)(1 - u(x, y))(1 + O(V^{-2})). \tag{3.1}$$

Proof: Assumption 2.1 implies

$$g_2(x, y) = f_2(x)f_2(y)(1 - u_2(x, y)). \tag{3.2}$$

and by (2.14),

$$g_1(x) = f_2(x) \int \frac{dy}{V} f_2(y)(1 - u_2(x, y)). \tag{3.3}$$

1 - Let us first take an expansion to order V^{-1}. By (2.11)

$$\int \frac{dy}{V}\ f_2(y)u_2(x, y) = O(V^{-1}) \tag{3.4}$$

and so

$$g_1(x) = f_2(x) \left(\int \frac{dy}{V} f_2(y) + O(V^{-1}) \right). \tag{3.5}$$

Applying $\int \frac{dx}{V} \cdot$ to both sides of (3.5), we find that

$$\int \frac{dy}{V} f_2(y) = 1 + O(V^{-1}) \tag{3.6}$$

so (3.5) yields

$$f_2(x) = g_1(x)(1 + O(V^{-1})). \tag{3.7}$$

2 - We now push the expansion to order V^{-2}. Inserting (3.7) into (3.3),

$$g_1(x) = f_2(x) \int \frac{dy}{V}\ f_2(y) - g_1(x) \left(\int \frac{dy}{V}\ g_1(y)u_2(x, y) + O(V^{-2}) \right). \tag{3.8}$$

However, by (2.14),

$$g_1(x) \int \frac{dy}{V}\ g_1(y)(1 - u_2(x, y)) = g_1(x) \tag{3.9}$$

so, by (2.13),

$$\int dy\ g_1(y)u_2(x, y) = 0 \tag{3.10}$$

and

$$g_1(x)(1 + O(V^{-2})) = f_2(x) \int \frac{dy}{V}\ f_2(y). \tag{3.11}$$

Taking $\int \frac{dx}{V} \cdot$ on both sides, we find that

$$f_2(x) = g_1(x)(1 + O(V^{-2})). \tag{3.12}$$

□

Remark: Note that this proof can easily be generalized to show that $f_2 = g_1(1 + O(V^{-n}))$ for any n.

3.1.2. Factorization of g_3

We now turn to g_3.

Lemma 3.2

Assumption 2.1 with $i = 2, 3$ and (2.13)-(2.15) imply that

$$g_3(x, y, z) = g_1(x)g_1(y)g_1(z)(1 - u_3(x, y))(1 - u_3(x, z))(1 - u_3(y, z))(1 + O(V^{-2})) \tag{3.13}$$

with

$$u_3(x, y) := u_2(x, y) + \frac{w_3(x, y)}{V} \tag{3.14}$$

$$w_3(x, y) := (1 - u_2(x, y)) \int dz\ g_1(z)u_2(x, z)u_2(y, z). \tag{3.15}$$

Proof: Using (2.15) in (2.9),

$$g_2(x_1, x_2) = W_3(x_1, x_2) \int \frac{dx_3}{V}\ W_3(x_1, x_3)W_3(x_2, x_3). \tag{3.16}$$

1 - We first expand to order V^{-1}. By (2.11),

$$\int \frac{dz}{V} f_3^2(z)u_3(x, z) = O(V^{-1}) \tag{3.17}$$

so, by (2.10),

$$g_2(x, y) = f_3^2(x)f_3^2(y)(1 - u_3(x, y)) \left(\int \frac{dz}{V}\ f_3^2(z) + O(V^{-1}) \right). \tag{3.18}$$

By Lemma 3.1,

$$g_1(x)g_1(y)(1 - u_2(x, y)) = f_3^2(x)f_3^2(y)(1 - u_3(x, y)) \left(\int \frac{dz}{V}\ f_3^2(z) + O(V^{-1}) \right). \tag{3.19}$$

We take $\int \frac{dy}{V} \cdot$ on both sides of this equation. By (3.10) and (3.17),

$$g_1(x) = f_3^2(x) \left(\left(\int \frac{dy}{V} f_3^2(y) \right)^2 + O(V^{-1}) \right) \tag{3.20}$$

and, integrating once more implies that $\int \frac{dy}{V} f_3^2(y) = 1 + O(V^{-1})$. Therefore,

$$f_3^2(x) = g_1(x)(1 + O(V^{-1})) \tag{3.21}$$

and

$$u_3(x, y) = u_2(x, y)(1 + O(V^{-1})). \tag{3.22}$$

2 - We push the expansion to order V^{-2}: (3.16) is

$$g_2(x,y) = f_3^2(x) f_3^2(y)(1-u_3(x,y)) \int \frac{dz}{V} f_3^2(z) \left(1-u_3(x,z)-u_3(y,z)+u_3(x,z)u_3(y,z)\right). \tag{3.23}$$

By (3.21)-(3.22) and Lemma 3.1,

$$\begin{aligned} f_3^2(x) f_3^2(y)(1-u_3(x,y)) \int \frac{dz}{V} f_3^2(z) = g_1(x)g_1(y)(1-u_2(x,y)) \cdot \\ \cdot \left(1 + \int \frac{dz}{V} \left(g_1(z)(u_2(x,z)+u_2(y,z)-u_2(x,z)u_2(y,z)\right)) + O(V^{-2})\right). \end{aligned} \tag{3.24}$$

Therefore, by (3.10),

$$\begin{aligned} f_3^2(x) f_3^2(y)(1-u_3(x,y)) \int \frac{dz}{V} f_3^2(z) = g_1(x)g_1(y)(1-u_2(x,y)) \cdot \\ \cdot \left(1 - \int \frac{dz}{V} g_1(z)u_2(x,z)u_2(y,z) + O(V^{-2})\right). \end{aligned} \tag{3.25}$$

Now, let us apply $\int \frac{dy}{V} \cdot$ to both sides of the equation. Note that, by (2.11),

$$\int \frac{dy}{V} \, g_1(y)u_2(x,y) \int \frac{dz}{V} g_1(z)u_2(x,z)u_2(y,z) = O(V^{-2}). \tag{3.26}$$

Furthermore, by (3.10),

$$\int \frac{dy}{V} \, g_1(y)u_2(x,y) = 0, \quad \int \frac{dy}{V} \, g_1(y) \int \frac{dz}{V} \, g_1(z)u_2(x,z)u_2(y,z) = 0 \tag{3.27}$$

and by (3.21) and (3.22),

$$\int \frac{dy}{V} \, f_3^2(y)u_3(x,y) = \int \frac{dy}{V} \, g_1(y)u_2(x,y) + O(V^{-2}) = O(V^{-2}). \tag{3.28}$$

We are thus left with

$$f_3^2(x) \left(\int \frac{dy}{V} \, f_3^2(y)\right)^2 = g_1(x)(1+O(V^{-2})). \tag{3.29}$$

Taking $\int \frac{dx}{V} \cdot$, we thus find that

$$\left(\int \frac{dx}{V} f_3^2(x)\right)^3 = 1 + O(V^{-2}) \tag{3.30}$$

and

$$f_3^2(x) = g_1(x)(1+O(V^{-2})). \tag{3.31}$$

Therefore,

$$1-u_3(x,y) = (1-u_2(x,y)) \left(1 - \frac{1}{V} \int dz \; g_1(z)u_2(x,z)u_2(y,z) + O(V^{-2})\right). \tag{3.32}$$

□

3.1.3. Factorization of g_4

Lemma 3.3

Assumption 2.1 and (2.13)-(2.16) imply that

$$g_4(x_1, x_2, x_3, x_2) = g_1(x_1)g_1(x_2)g_1(x_3)g_1(x_4)\left(\prod_{i<j}(1 - u_4(x_i, x_j))\right)(1 + O(V^{-2})) \tag{3.33}$$

with

$$u_4(x, y) := u_2(x, y) + \frac{2w_3(x, y)}{V} \tag{3.34}$$

where w_3 is the same as in Lemma 3.2.

Proof: Using (2.16) in (2.9),

$$g_2(x_1, g_2) = W_4(x_1, x_2)\int \frac{dx_3dx_4}{V^2}\ W_4(x_1, x_3)W_4(x_1, x_4)W_4(x_2, x_3)W_4(x_2, x_4)W_4(x_3, x_4). \tag{3.35}$$

1 - We expand to order V^{-1}. By (2.11),

$$\int \frac{dz}{V} f_4^3(z)u_4(x, z) = O(V^{-1}) \tag{3.36}$$

so by (2.10),

$$g_2(x, y) = f_4^3(x)f_4^3(y)(1 - u_4(x, y))\left(\int \frac{dzdt}{V^2} f_4^3(z)f_4^3(t) + O(V^{-1})\right). \tag{3.37}$$

By Lemma 3.1,

$$g_1(x)g_1(y)(1 - u_2(x, y)) = f_4^3(x)f_4^3(y)(1 - u_4(x, y))\left(\left(\int \frac{dz}{V} f_4^3(z)\right)^2 + O(V^{-1})\right). \tag{3.38}$$

Applying $\int \frac{dy}{V}\cdot$ to both sides of the equation, using (3.10) and (3.36),

$$g_1(x) = f_4(x)^3\left(\left(\int \frac{dy}{V}\ f_4^3(y)\right)^3 + O(V^{-1})\right). \tag{3.39}$$

Integrating once more, we have $\int \frac{dy}{V} f_4^3(z) = 1 + O(V^{-1})$ and

$$f_4^3(x) = g_1(x)(1 + O(V^{-1})). \tag{3.40}$$

Therefore,

$$u_4(x, y) = u_2(x, y)(1 + O(V^{-1})). \tag{3.41}$$

2 - We push the expansion to order V^{-2}: by (2.11),

$$\int \frac{dzdt}{V^2} u_4(x,z)u_4(y,t) = O(V^{-2}), \quad \int \frac{dzdt}{V^2} u_4(x,z)u_4(z,t) = O(V^{-2}) \tag{3.42}$$

$$\int \frac{dzdt}{V^2} u_4(x,z)u_4(x,t) = O(V^{-2}) \tag{3.43}$$

so

$$\begin{aligned} g_2(x,y) = f_4^3(x)f_4^3(y)(1-u_4(x,y)) \Bigg(\int \frac{dzdt}{V^2} f_4^3(z)f_4^3(t) + \\ + \int \frac{dzdt}{V^2} g_1(z)g_1(t)(-2u_2(x,z) - 2u_2(y,z) - u_2(z,t) + 2u_2(x,z)u_2(y,z)) + O(V^{-2}) \Bigg). \end{aligned} \tag{3.44}$$

By (3.40), (3.41), and Lemma 3.1,

$$\begin{aligned} f_4^3(x)f_4^3(x)(1-u_4(x,y)) \left(\int \frac{dz}{V}\ f_4^3(z) \right)^2 = g_1(x)g_1(y)(1-u_2(x,y)) \cdot \\ \cdot \left(1 + \int \frac{dzdt}{V^2}\ g_1(z)g_1(t)(2u_2(x,z) + 2u_2(y,z) + u_2(z,t) - 2u_2(x,z)u_2(y,z)) + O(V^{-2}) \right). \end{aligned} \tag{3.45}$$

By (3.10),

$$\begin{aligned} f_4^3(x)f_4^3(y)(1-u_4(x,y)) \left(\int \frac{dz}{V}\ f_4^3(z) \right)^2 = \\ = g_1(x)g_1(y)(1-u_2(x,y)) \left(1 - 2\int \frac{dz}{V} g_1(z)u_2(x,z)u_2(y,z) + O(V^{-2}) \right). \end{aligned} \tag{3.46}$$

We apply $\int \frac{dy}{V} \cdot$ to both sides of the equation. By (3.26)-(3.28), we find

$$f_4^3(x) \left(\int \frac{dy}{V} f_4^3(z) \right)^3 = g_1(x)(1 + O(V^{-2})). \tag{3.47}$$

Taking $\int \frac{dx}{V} \cdot$, we find that

$$f_4(x) = 1 + O(V^{-2}) \tag{3.48}$$

and

$$f_4^3(x) = g_1(x)(1 + O(V^{-2})). \tag{3.49}$$

Therefore,

$$1 - u_4(x,y) = (1 - u_2(x,y)) \left(1 - \frac{2}{V} \int dz\ g_1(z)u_2(x,z)u_2(y,z) + O(V^{-2}) \right). \tag{3.50}$$

□

3.2. Consequences of the factorization

1 - We first rewrite (2.4) as a family of equations for g_i.

1-1 - Integrating (2.4) with respect to $x_1, \cdots, x_N$, we find that

$$E_0 = G_0^{(2)} + F_0^{(1)} + B_0 \tag{3.51}$$

with

$$G_0^{(2)} := \frac{N(N-1)}{2V^2} \int dxdy\ v(x,y) g_2(x,y) \tag{3.52}$$

$$F_0^{(1)} := \frac{N}{V} \int dx\ \varpi g_1(x) \tag{3.53}$$

and B_0 is a boundary term:

$$B_0 = -\frac{N}{2V} \int dx\ \Delta g_1(x). \tag{3.54}$$

1-2 - If, now, we integrate (2.4) with respect to $x_2, \cdots, x_N$, we find

$$-\frac{\Delta}{2} g_1(x) + \varpi g_1(x) + G_1^{(2)}(x) + G_1^{(3)}(x) + F_1^{(2)}(x) + B_1(x) = E_0 g_1(x) \tag{3.55}$$

with

$$G_1^{(2)}(x) := \frac{N-1}{V} \int dy\ v(x,y) g_2(x,y) \tag{3.56}$$

$$G_1^{(3)}(x) := \frac{(N-1)(N-2)}{2V^2} \int dydz\ v(y,z) g_3(x,y,z) \tag{3.57}$$

$$F_1^{(2)}(x) := \frac{N-1}{V} \int dy\ \varpi_y g_2(x,y) \tag{3.58}$$

in which we use the notation ϖ_y to indicate that ϖ applies to $y \mapsto g_2(x,y)$, and B_1 is a boundary term

$$B_1(x) := -\frac{N-1}{2V} \int dy\ \Delta_y g_2(x,y). \tag{3.59}$$

1-3 - If we integrate with respect to $x_3, \cdots, x_N$, we find

$$\begin{aligned}-\frac{1}{2}(\Delta_x + \Delta_y) g_2(x,y) + v(x,y) g_2(x,y) + (\varpi_y + \varpi_x) g_2(x,y) + \\ + G_2^{(3)}(x,y) + G_2^{(4)}(x,y) + F_2^{(3)}(x,y) + B_2(x,y) = E_0 g_2(x,y)\end{aligned} \tag{3.60}$$

where, here again, ϖ_y indicates that ϖ applies to the y-degree of freedom, whereas ϖ_x applies to x, with

$$G_2^{(3)}(x,y) := \frac{N-2}{V} \int dz\ (v(x,z) + v(y,z)) g_3(x,y,z) \tag{3.61}$$

$$G_2^{(4)}(x,y) := \frac{(N-2)(N-3)}{2V^2} \int dzdt\ v(z,t) g_4(x,y,z,t) \tag{3.62}$$

$$F_2^{(3)}(x,y) := \frac{N-2}{V} \int dz\ \varpi_z g_3(x,y,z) \tag{3.63}$$

and B_2 is a boundary term

$$B_2(x) := -\frac{N-2}{2V} \int dz\ \Delta_z g_3(x,y,z). \tag{3.64}$$

2 - We rewrite (3.51), (3.55) and (3.60) using Lemmas 3.1, 3.2 and 3.3.

2-1 - We start with (3.51): by (2.5) and Lemma 3.1,

$$G_0^{(2)} = \frac{N(N-1)}{2V^2} \int dxdy\ v(x,y) g_1(x) g_1(y)(1-u_2(x,y)) + O(V^{-1}) \tag{3.65}$$

so

$$E_0 = \frac{N(N-1)}{2V^2} \int dxdy\ v(x,y) g_1(x) g_1(y)(1-u_2(x,y)) + \frac{N}{V}\int dx\ \varpi g_1(x) + B_0 + O(V^{-1}). \tag{3.66}$$

2-2 - We now turn to (3.55): by (2.5) and Lemma 3.1,

$$G_1^{(2)}(x) = \frac{N}{V} g_1(x) \left(\int dy\ v(x,y) g_1(y)(1-u_2(x,y)) + O(V^{-2}) \right) \tag{3.67}$$

and by Lemma 3.2,

$$\begin{aligned} G_1^{(3)}(x) = g_1(x) \bigg(\frac{N^2}{2V^2} \int dydz\ v(y,z) g_1(y) g_1(z)(1-u_2(x,y))(1-u_2(x,z))(1-u_3(y,z)) - \\ - \frac{3N}{2V^2} \int dydz\ v(y,z) g_1(y) g_1(z)(1-u_2(y,z)) + O(V^{-1}) \bigg) \end{aligned} \tag{3.68}$$

(we used (3.14) to write $u_3 = u_2 + O(V^{-1})$; this works fine for $u_3(x,y)$ and $u_3(x,z)$ because the integrals over y and z are controlled by $v(y,z)w_3(x,y)$ and $v(y,z)w_3(x,z)$ using (2.5) and (2.11); in the first term, it does not work for $u_3(y,z)$, as $v(y,z)w_3(y,z)$ can only control one of the integrals, and not both; the second term has an extra V^{-1} that lets us replace u_3 by u_2) and by (2.11) and (2.6),

$$F_1^{(2)}(x) = g_1(x) \left(\frac{N}{V} \int dy\ \varpi_y (g_1(y)(1-u_2(x,y))) - \frac{1}{V}\int dy\ \varpi g_1(y) + O(V^{-1}) \right). \tag{3.69}$$

The first term in $G_1^{(3)}$ is of order V:

$$\begin{aligned} &\frac{N^2}{2V^2} \int dydz\ v(y,z) g_1(y) g_1(z)(1-u_2(x,y))(1-u_2(x,z))(1-u_3(y,z)) = \\ &\quad = \frac{N^2}{2V^2} \int dydz\ v(y,z) g_1(y) g_1(z)(1-u_2(y,z)) - \frac{N^2}{2V^3}\int dydz\ v(y,z) g_1(y) g_1(z) w_3(y,z) + \\ &+ \frac{N^2}{2V^2} \int dydz\ v(y,z) g_1(y) g_1(z)(1-u_2(y,z))(-u_2(x,y) - u_2(x,z) + u_2(x,y)u_2(x,z)) + O(V^{-1}) \end{aligned} \tag{3.70}$$

in which the only term of order V is the first one, and is equal to the first term of order V in E_0, and thus cancels out. There is a similar cancellation between the second term of order V in $F_1^{(2)}$

and E_0. All in all,

$$\left(-\frac{\Delta}{2}+\varpi+\bar{G}_1^{(2)}(x)+\bar{G}_1^{(3)}(x)+\bar{F}_1^{(2)}(x)+\bar{E}_0-B_0\right)g_1(x)+B_1(x)=g_1(x)O(V^{-1}) \tag{3.71}$$

with, recalling $\rho := N/V$,

$$\bar{G}_1^{(2)}(x) := \rho\int dy\ v(x,y)g_1(y)(1-u_2(x,y)) \tag{3.72}$$

and using (3.15),

$$\begin{aligned}\bar{G}_1^{(3)}(x) := -\frac{\rho}{2}\int\frac{dydz}{V}\ v(y,z)g_1(y)g_1(z)(1-u_2(y,z))\left(3+\rho\int dt\ g_1(t)u_2(y,t)u_2(z,t)\right)+\\ +\frac{\rho^2}{2}\int dydz\ v(y,z)g_1(y)g_1(z)(1-u_2(y,z))(-u_2(x,y)-u_2(x,z)+u_2(x,y)u_2(x,z))\end{aligned} \tag{3.73}$$

$$\bar{F}_1^{(2)}(x) := -\rho\int dy\ \varpi_y(g_1(y)u_2(x,y))-\int\frac{dy}{V}\ \varpi g_1(y) \tag{3.74}$$

$$\bar{E}_0 := \frac{\rho}{2}\int\frac{dxdy}{V}\ v(x,y)g_1(x)g_1(y)(1-u_2(x,y)). \tag{3.75}$$

Rewriting this using (2.20)-(2.23), we find (2.18) with

$$\Sigma_1(x) := B_1(x)-B_0g_1(x)+O(V^{-1}). \tag{3.76}$$

2-3 - Finally, we rewrite (3.60): by (2.5) and Lemma 3.2,

$$\begin{aligned}G_2^{(3)}(x,y) = \frac{N}{V}g_1(x)g_1(y)(1-u_2(x,y))\cdot\\ \cdot\left(\int dz\ (v(x,z)+v(y,z))g_1(z)(1-u_2(x,z))(1-u_2(y,z))+O(V^{-1})\right)\end{aligned} \tag{3.77}$$

and by Lemma 3.3,

$$\begin{aligned}G_2^{(4)}(x,y) = g_1(x)g_1(y)\left(\frac{N^2}{2V^2}(1-u_4(x,y))\int dzdt\ v(z,t)g_1(z)g_1(t)(1-u_4(z,t))\Pi(x,y,z,t)-\right.\\ \left.-\frac{5N}{2V^2}(1-u_2(x,y))\int dzdt\ v(z,t)g_1(z)g_1(t)(1-u_2(z,t))+O(V^{-1})\right)\end{aligned} \tag{3.78}$$

$$\Pi(x,y,z,t) := (1-u_2(x,z))(1-u_2(x,t))(1-u_2(y,z))(1-u_2(y,t)) \tag{3.79}$$

and by (2.11) and (2.6),

$$\begin{aligned}F_2^{(3)}(x,y) = g_1(x)g_1(y)\left(\frac{N}{V}(1-u_3(x,y))\int dz\ \varpi_z(g_1(z)(1-u_2(x,z))(1-u_2(y,z)))-\right.\\ \left.-\frac{2}{V}(1-u_2(x,y))\int dz\ \varpi g_1(z)+O(V^{-1})\right).\end{aligned} \tag{3.80}$$

The first term in $G_2^{(4)}$ is of order V: by (3.34),

$$\frac{N^2}{2V^2}(1-u_4(x,y))\int dzdt\ v(z,t)g_1(z)g_1(t)(1-u_4(z,t))\Pi(x,y,z,y) =$$

$$\begin{aligned} &= \frac{N^2}{2V^2}(1-u_2(x,y))\int dzdt\ v(z,t)g_1(z)g_1(t)(1-u_2(z,t))- \\ &-\frac{N^2}{V^3}w_3(x,y)\int dzdt\ v(z,t)g_1(z)g_1(y)(1-u_2(z,t))- \\ &-\frac{N^2}{V^3}(1-u_2(x,y))\int dzdt\ v(z,t)g_1(z)g_1(t)w_3(z,t)+ \\ &+\frac{N^2}{2V^2}(1-u_2(x,y))\int dzdt\ v(z,t)g_1(z)g_1(t)(1-u_2(z,t))\left(\Pi(x,y,z,t)-1\right)+O(V^{-1}) \end{aligned} \tag{3.81}$$

in which the only term of order V is the first one, and is equal to the term of order V in E_0, and thus cancels out. There is a similar cancellation between the term of order V in $F_2^{(3)}$ and E_0. All in all,

$$\begin{aligned} &\left(-\frac{1}{2}(\Delta_x+\Delta_y)+v(x,y)+\varpi_x+\varpi_y+\bar G_2^{(3)}(x,y)+\bar G_2^{(4)}(x,y)+\bar F_2^{(3)}(x,y)+\bar E_0-B_0\right)\cdot \\ &\qquad\cdot g_1(x)g_1(y)(1-u_2(x,y))+B_2(x,y)=g_1(x)g_1(y)O(V^{-1}) \end{aligned} \tag{3.82}$$

with

$$\bar G_2^{(3)}(x,y) := \rho\int dz\ (v(x,z)+v(y,z))g_1(z)(1-u_2(x,z))(1-u_2(y,z)) \tag{3.83}$$

and by (3.15),

$$\begin{aligned} \bar G_2^{(4)}(x,y) := &-\frac{\rho}{2}\left(5+2\rho\int dr\ g_1(r)u_2(x,r)u_2(y,r)\right)\int \frac{dzdt}{V}\ v(z,t)g_1(z)g_1(t)(1-u_2(z,t))- \\ &-\rho^2\int\frac{dzdt}{V}\ v(z,t)g_1(z)g_1(t)(1-u_2(z,t))\int dr\ g_1(r)u_2(z,r)u_2(t,r)+ \\ &+\frac{\rho^2}{2}\int dzdt\ v(z,t)g_1(z)g_1(t)(1-u_2(z,t))\left(\Pi(x,y,z,t)-1\right) \end{aligned} \tag{3.84}$$

$$\begin{aligned} \bar F_2^{(3)}(x,y) := \rho\int dz\ \varpi_z(g_1(z)(-u_2(x,z)-u_2(y,z)+u_2(x,z)u_2(y,z)))- \\ -\left(2+\rho\int dr\ g_1(r)u_2(x,r)u_2(y,r)\right)\int\frac{dz}{V}\ \varpi g_1(z) \end{aligned} \tag{3.85}$$

$$\bar E_0 = \frac{\rho}{2}\int\frac{dxdy}{V}\ v(x,y)g_1(x)g_1(y)(1-u_2(x,y)). \tag{3.86}$$

2-4 - Expanding out Π, see (3.79), we find (2.19) with

$$\begin{aligned} \bar R_2(x,y) := &\rho\int dz\ g_1(z)\left(\bar S(x,z)+\bar S(y,z)-2\int\frac{dt}{V}\ g_1(t)\bar S(t,z)\right)+ \\ &+\frac{\rho^2}{2}\left(\bar S\bar{*}u_2\bar{*}u_2(x,x)+\bar S\bar{*}u_2\bar{*}u_2(y,y)-2\int\frac{dt}{V}\ g_1(t)\bar S\bar{*}u_2\bar{*}u_2(t,t)\right)+ \\ &+\rho^2\int dzdt\ g_1(z)g_1(t)u_2(x,z)u_2(y,z)\left(\bar S(z,t)-\int\frac{dr}{V}\ g_1(r)\bar S(z,r)\right)- \\ &-\rho^2\int dt\ g_1(t)(\bar S\bar{*}u_2(x,t)+\bar S\bar{*}u_2(y,t))+\bar F_2^{(3)}(x,y)+\varpi_x+\varpi_y \end{aligned} \tag{3.87}$$

and

$$\Sigma_2(x,y) := B_2(x,y)-B_0g_1(x)g_1(y)(1-u_2(x,y))+O(V^{-1}). \tag{3.88}$$

Using (2.22) and (2.23), (3.87) becomes (2.26).

3 - Finally, (2.28) follows from (3.51) with

$$\Sigma_0 := B_0 + O(V^{-1}). \tag{3.89}$$

□

3.3. Sanity check, proof of Corollary 2.3

Assuming the translation invariance of the solution, $g_1(x)$ is constant. By (2.13),

$$g_1(x) = 1. \tag{3.90}$$

Furthermore, $\varpi \equiv 0$. We then have

$$\bar S(x,y) = S(x-y), \quad \bar K(x,y) = K(x-y), \quad \bar L(x,y) = L(x-y) \tag{3.91}$$

(see (2.30)-(2.31)). Furthermore,

$$\mathcal{E}(x) \equiv \mathcal{E}(y) \equiv \langle \mathcal{E} \rangle = \frac{\rho}{2} \int dy\ S(y) \tag{3.92}$$

$$\bar A(x) \equiv \bar A(y) \equiv \langle \bar A \rangle = \rho^2 S * u * u(0) \tag{3.93}$$

$$\bar C(x) \equiv \bar C_2(y) = 2\rho^2 \int dz\ u(z) \int dt\ S(t) \tag{3.94}$$

which vanishes by (2.14). Thus,

$$\bar R_2(x,y) \equiv 0. \tag{3.95}$$

We conclude by taking the thermodynamic limit. □

4. The momentum distribution

4.1. Computation of the momentum distribution, proof of Theorem 2.4

We use Theorem 2.2 with ϖ as in (2.33). Note that, by (2.33),

$$\int dx\ \varpi f(x) = 0 \tag{4.1}$$

which trivially satisfies (2.6).

1 - We change variables in (2.19) to

$$\xi = \frac{x+y}{2}, \quad \zeta = x - y \tag{4.2}$$

and find

$$\left(-\frac{1}{4}\Delta_\xi - \Delta_\zeta + v(\zeta) - 2\rho \bar K(\xi+\tfrac{\zeta}{2}, \xi-\tfrac{\zeta}{2}) + \rho^2 \bar L(\xi+\tfrac{\zeta}{2}, \xi-\tfrac{\zeta}{2}) + \bar R_2(\xi+\tfrac{\zeta}{2}, \xi-\tfrac{\zeta}{2})\right) \cdot \\ \cdot g_1(\xi+\tfrac{\zeta}{2}) g_1(\xi-\tfrac{\zeta}{2})(1 - u_2(\xi+\tfrac{\zeta}{2}, \xi-\tfrac{\zeta}{2})) = -\Sigma_2. \tag{4.3}$$

In addition, by (2.28),

$$e = \frac{\rho}{2} \int \frac{d\xi d\zeta}{V}\ g_1(\xi+\tfrac{\zeta}{2}) g_1(\xi-\tfrac{\zeta}{2}) v(\zeta)(1 - u_2(\xi+\tfrac{\zeta}{2}, \xi-\tfrac{\zeta}{2})) + \int \frac{dx}{V}\ \varpi g_1(x) + \Sigma_1. \tag{4.4}$$

We expand in powers of ϵ:

$$g_1(x) = 1 + \epsilon g_1^{(1)}(x) + O(\epsilon^2), \quad u_2(\xi+\tfrac{\zeta}{2}, \xi-\tfrac{\zeta}{2}) = u_2^{(0)}(\zeta) + \epsilon u_2^{(1)}(\xi+\tfrac{\zeta}{2}, \xi-\tfrac{\zeta}{2}) + O(\epsilon^2) \tag{4.5}$$

in which we used the fact that, at $\epsilon = 0$, $g_1(x)|_{\epsilon=0} = 1$, see (3.90). In particular, the terms of order 0 in ϵ are independent of ξ. Note, in addition, that, by (2.13),

$$\int \frac{dx}{V}\ g_1^{(1)}(x) = 0. \tag{4.6}$$

2 - The trick of this proof is to take the average with respect to ξ on both sides of (4.3). Since we take periodic boundary conditions, the Δ_ξ term drops out. We will only focus on the first order contribution in ϵ, and, as was mentioned above, terms of order 0 are independent of ξ. Thus, the average over ξ will always apply to a single term, either $g_1^{(1)}$ or $u_2^{(1)}$. By (2.13), the terms involving $g_1^{(1)}$ have zero average. We can therefore replace $g_1^{(1)}$ by 1. (The previous argument does not apply to the terms in which Δ_ζ acts on g_1, but these terms have a vanishing average as well because of the periodic boundary conditions.) In particular, by (2.14) and Lemma 3.1,

$$\int \frac{d\xi}{V}\ (1 - u_2^{(1)}(\xi + \tfrac{\zeta}{2}, \xi - \tfrac{\zeta}{2})) = 1 \tag{4.7}$$

so

$$\int \frac{d\xi}{V}\ u_2^{(1)}(\xi + \tfrac{\zeta}{2}, \xi - \tfrac{\zeta}{2}) = 0 \tag{4.8}$$

and thus, we can replace u_2 with $u_2^{(0)}$. Thus, using the translation invariant computation detailed in Section 3.3, we find that the average of (4.3) is

$$(-\Delta + v(\zeta) - 2\rho K(\zeta) + \rho^2 L(\zeta))(1 - u_2^{(0)}(\zeta)) + \epsilon F(\zeta) + O(\epsilon^2) + \Sigma_2 = 0 \tag{4.9}$$

where K and L are defined in (2.30) and (2.31) and F comes from the contribution to $\bar R_2$ of ϖ, see (2.26):

$$\begin{aligned}F(\zeta) := \epsilon^{-1} \int \frac{d\xi}{V}\ \Big(&\varpi_x + \varpi_y - 2\langle\varpi\rangle + \rho \int dz\ \varpi_z (u_2^{(0)}(\xi + \tfrac{\zeta}{2} - z) u_2^{(0)}(\xi - \tfrac{\zeta}{2} - z)) - \\ &-\rho \int dz\ \varpi_z u_2^{(0)}(\xi + \tfrac{\zeta}{2} - z) - \rho \int dz\ \varpi_z u_2^{(0)}(\xi - \tfrac{\zeta}{2} - z)\Big)(1 - u_2^{(0)}(\zeta)).\end{aligned} \tag{4.10}$$

Similarly, (4.4) is

$$e = \frac{\rho}{2} \int d\zeta\ v(\zeta)(1 - u_2^{(0)}(\zeta)) + \int \frac{dx}{V}\ \varpi g_1(x) + \Sigma_1 + O(\epsilon^2). \tag{4.11}$$

3 - Furthermore, by (2.33),

$$\int dz\ \varpi_z f(z) = 0 \tag{4.12}$$

for any integrable f, so

$$F(\zeta) = \epsilon^{-1} \int \frac{d\xi}{V}\ (\varpi_x + \varpi_y)(1 - u_2^{(0)}(\zeta)) \tag{4.13}$$

and

$$e = \frac{\rho}{2} \int d\zeta\ v(\zeta)(1 - u_2^{(0)}(\zeta)) + \Sigma_1 + O(\epsilon^2). \tag{4.14}$$

Now,

$$\varpi_x f(x-y) = e^{ikx} \int dz\ e^{-ikz} f(z-y) \tag{4.15}$$

so

$$\varpi_x f(\zeta) = \epsilon e^{ik(\xi + \frac{\zeta}{2})} \int dz\ e^{-ik(z + (\xi - \frac{\zeta}{2}))} f(z) = \epsilon e^{ik\zeta} \int dz\ e^{-ikz} f(z) = \epsilon e^{ik\zeta} \hat f(-k). \tag{4.16}$$

Similarly,

$$\varpi_y f(\zeta) = \epsilon e^{-ik\zeta} \hat f(-k). \tag{4.17}$$

Thus

$$F(\zeta) = 2\cos(k\zeta)(\delta(k) - \hat u_2^{(0)}(-k)). \tag{4.18}$$

Since $k \neq 0$, the δ function drops out. We conclude the proof by combining (4.9), (4.14) and (4.18) and taking the thermodynamic limit. □

4.2. The simple equation and Bogolyubov theory, proof of Theorem 2.5

1 - We differentiate (2.44) with respect to ϵ and take $\epsilon = 0$:

$$(-\Delta + v + 4e + 4e\rho u*)\partial_\epsilon u = -4\partial_\epsilon e u + 2\partial_\epsilon e \rho u * u + F. \tag{4.19}$$

Let

$$\mathfrak{K}_e := (-\Delta + v + 4e(1 - \rho u*))^{-1} \tag{4.20}$$

(this operator was introduced and studied in detail in [CJL21]). We apply $\mathfrak{K}_e$ to both sides and take a scalar product with $-\rho v/2$ and find

$$\partial_\epsilon e = \rho \partial_\epsilon e \int dx\ v(x)\mathfrak{K}_e(2u(x) - \rho u * u(x)) - \frac{\rho}{2}\int dx\ v(x)\mathfrak{K}_e F(x) \tag{4.21}$$

and so, using (2.43),

$$\mathcal{M}^{(\mathrm{simpleq})}(k) = \partial_\epsilon e = -\frac{\frac{\rho}{2}\int dx\ v(x)\mathfrak{K}_e F(x)}{1 - \rho\int dx\ v(x)\mathfrak{K}_e(2u(x) - \rho u * u(x))} \tag{4.22}$$

and, by (2.39),

$$\mathcal{M}^{(\mathrm{simpleq})}(k) = \rho\frac{\hat{u}(k)\int dx\ v(x)\mathfrak{K}_e \cos(kx)}{1 - \rho\int dx\ v(x)\mathfrak{K}_e(2u(x) - \rho u * u(x))}. \tag{4.23}$$

Note that

$$\int \frac{dk}{(2\pi)^3}\mathcal{M}^{(\mathrm{simpleq})}(k) = \frac{\rho\int dx\ v(x)\mathfrak{K}_e u(x)}{1 - \rho\int dx\ v(x)\mathfrak{K}_e(2u(x) - \rho u * u(x))} \tag{4.24}$$

which is the expression for the uncondensed fraction for the simple equation [CHe21, (38)].

2 - By [CJL21, (5.8),(5.27)],

$$\mathcal{M}^{(\mathrm{simpleq})}(k) = \rho\left(\hat{u}(k)\int dx\ v(x)\mathfrak{K}_e \cos(k(x))\right)(1 + O(\rho e^{-\frac{1}{2}})). \tag{4.25}$$

Furthermore, by the resolvent identity,

$$\mathfrak{K}_e \cos(kx) = \xi - \mathfrak{K}_e(v\xi), \quad \xi := \mathfrak{Y}_e(\cos(kx)) := (-\Delta + 4e(1 - \rho u*))^{-1}\cos(kx) \tag{4.26}$$

in terms of which, using the self-adjointness of $\mathfrak{K}_e$,

$$\mathcal{M}^{(\mathrm{simpleq})}(k) = \rho\hat{u}(k)\left(\int dx\ v(x)\xi(x) - \int dx\ \mathfrak{K}_e v(x)(v(x)\xi(x))\right). \tag{4.27}$$

3 - Now, taking the Fourier transform,

$$\hat\xi(q)\equiv\int dx\ e^{ikx}\xi(x)=\frac{(2\pi)^3}{2}\frac{\delta(k-q)+\delta(k+q)}{q^2+4e(1-\rho\hat u(q))} \tag{4.28}$$

and so

$$\int dx\ v(x)\xi(x)=\int\frac{dq}{(2\pi)^3}\hat v(q)\hat\xi(q)=\frac{\hat v(k)}{k^2+4e(1-\rho\hat u(k))} \tag{4.29}$$

and thus

$$\rho\hat u(k)\int dx\ v(x)\xi=\rho\hat v(k)\frac{\hat u(k)}{k^2+4e(1-\rho\hat u(k))}. \tag{4.30}$$

We recall [CJL20, (4.25)]:

$$\rho\hat u(k)=\frac{k^2}{4e}+1-\sqrt{\left(\frac{k^2}{4e}+1\right)^2-\hat S(k)} \tag{4.31}$$

and, by [CJL20, (4.24)],

$$\hat S(0)=1. \tag{4.32}$$

Therefore, if we rescale

$$k=2\sqrt{e}\kappa \tag{4.33}$$

we find

$$\rho\hat u(k)\int dx\ v(x)\xi=\frac{\hat v(0)}{4e}\frac{\kappa^2+1-\sqrt{(\kappa^2+1)^2-1}}{\sqrt{(\kappa^2+1)^2-1}}+o(e^{-1}). \tag{4.34}$$

4 - Now,

$$\int dx\ e^{iqx}v(x)\xi(x)=\frac12\frac{1}{k^2+4e(1-\rho\hat u(k))}\int dp\ \hat v(q-p)(\delta(k-p)+\delta(k+p)) \tag{4.35}$$

so

$$\int dx\ e^{iqx}v(x)\xi(x)=\frac12\frac{\hat v(q-k)+\hat v(q+k)}{k^2+4e(1-\rho\hat u(k))}. \tag{4.36}$$

Therefore,

$$\int dx\ \mathfrak{K}_e v(x)(v\xi)=\frac12\frac{1}{k^2+4e(1-\rho\hat u(k))}\int\frac{dq}{(2\pi)^3}\ \widehat{\mathfrak{K}_e v}(q)(\hat v(k-q)+\hat v(k+q)) \tag{4.37}$$

which, using the $q\mapsto -q$ symmetry, is

$$\int dx\ \mathfrak{K}_e v(x)(v\xi)=\frac{1}{k^2+4e(1-\rho\hat u(k))}\int\frac{dq}{(2\pi)^3}\ \widehat{\mathfrak{K}_e v}(q)\hat v(k+q) \tag{4.38}$$

that is,

$$\rho\hat u(k)\int dx\ \mathfrak{K}_e v(x)(v\xi)=\frac{\rho\hat u(k)}{k^2+4e(1-\rho\hat u(k))}\int dx\ e^{-ikx}\mathfrak{K}_e v(x)v(x) \tag{4.39}$$

in which we rescale

$$k=2\sqrt{e}\kappa \tag{4.40}$$

so, by (4.31)-(4.32),

$$\rho\hat u(k)\int dx\ \mathfrak{K}_e v(x)(v\xi)=\frac{\kappa^2+1-\sqrt{(\kappa^2+1)^2-1}}{4e\sqrt{(\kappa^2+1)^2-1}}(1+o(1))\int dx\ e^{-i2\sqrt{e}\kappa x}v(x)\mathfrak{K}_e v(x). \tag{4.41}$$

Therefore, by dominated convergence (using the argument above [CJL21, (5.23)] and the fact that $\mathfrak{K}_e$ is positivity preserving), and by [CJL21, (5.23)-(5.24)],

$$\rho\hat u(k)\int dx\ \mathfrak{K}_e v(x)(v\xi)=\frac{\kappa^2+1-\sqrt{(\kappa^2+1)^2-1}}{4e\sqrt{(\kappa^2+1)^2-1}}(-4\pi a+\hat v(0))+o(e^{-1}). \tag{4.42}$$

5 - Inserting (4.34) and (4.42) into (4.27), we find

$$\mathcal{M}^{(\mathrm{simpleq})}(k) = \frac{\pi a}{e}\frac{\kappa^2+1-\sqrt{(\kappa^2+1)^2-1}}{\sqrt{(\kappa^2+1)^2-1}} + o(e^{-1}). \tag{4.43}$$

Finally, we recall [CJL20, (1.23)]:

$$e = 2\pi\rho a(1+O(\sqrt{\rho})) \tag{4.44}$$

so

$$\mathcal{M}^{(\mathrm{simpleq})}(k) = \frac{1}{2}\frac{\kappa^2+1-\sqrt{(\kappa^2+1)^2-1}}{\sqrt{(\kappa^2+1)^2-1}} + o(e^{-1}). \tag{4.45}$$

6 - Finally, by (2.42)

$$\mathcal{M}^{(\mathrm{Bogolyubov})}(2\sqrt{e}\kappa) = -\frac{1}{2\rho}\left(1-\frac{\frac{4e}{8\pi\rho a}\kappa^2+1}{\sqrt{\frac{e^2}{4\pi^2\rho^2a^2}\kappa^4+\frac{e}{\pi\rho a}\kappa^2}}\right) \tag{4.46}$$

so by (4.44),

$$\mathcal{M}^{(\mathrm{Bogolyubov})}(2\sqrt{e}\kappa) = -\frac{1}{2\rho}\left(1-\frac{\kappa^2+1}{\sqrt{\kappa^4+2\kappa^2}}\right). \tag{4.47}$$

This, together with (4.45), implies (2.46). □

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