A Theorem on Ellipses, an Integrable System and a Theorem of Boltzmann

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Abstract

We study a mechanical system that was considered by Boltzmann in 1868 in the context of the derivation of the canonical and microcanonical ensembles. This system was introduced as an example of ergodic dynamics, which was central to Boltzmann's derivation. It consists of a single particle in two dimensions, which is subjected to a gravitational attraction to a fixed center. In addition, an infinite plane is fixed at some finite distance from the center, which acts as a hard wall on which the particle collides elastically. Finally, an extra centrifugal force is added. We will show that, in the absence of this extra centrifugal force, there are two independent integrals of motion. Therefore the extra centrifugal force is necessary for Boltzmann's claim of ergodicity to hold.

Keywords: Ergodicity, Chaotic hypothesis, Gibbs distributions, Boltzmann, Integrable systems

In 1868, Boltzmann [1868a] laid the foundations for our modern understanding of the behavior of many-particle systems by introducing the "microcanonical ensemble" (for more details on this history, see Gallavotti [2016]). The principal idea behind this ensemble is that one can achieve a good understanding of many-particle systems by focusing not on the dynamics of each individual particle, but on the statistical properties of the whole. More precisely, the state of the system becomes a random variable, chosen according to a probability distribution on phase space, which came to be called the "microcanonical ensemble". An important assumption that was made implicitly by Boltzmann is that the dynamics of the system be ergodic. In this case, time-averages of the dynamics can be rewritten as averages over phase space, and the qualitative properties of the dynamics can be formulated as statistical properties of the microcanonical ensemble.

To support this assumption, Boltzmann presented a mechanical system that very same year (Boltzmann [1868b]) as an example of an ergodic system. This system consists of a particle in two dimensions that is attracted to a fixed center via a gravitational potential $-\frac{\alpha}{2r}$. In addition, he added an extra centrifugal potential $\frac{g}{2r^2}$. As was known since at least the times of Kepler, this system is subjected to a central force, and is therefore integrable. In order to break the integrability, Boltzmann added an extra ingredient: a rigid infinite planar wall, located a finite distance away from the center (see figure 1). Whenever the particle hits the wall, it undergoes an elastic collision and is reflected back. Boltzmann's argument was, roughly, that in the absence of the wall, the dynamics is quasi-periodic, so the particle should intersect the plane of the wall at points which should fill up a segment of the wall densely as the dynamics evolves, and concluded that the region of phase space in which the energy is constant must also be filled densely. As we will show, this is not the whole story; following a conjectured integrability for $g = 0$, [Gallavotti, 2014, p.150], and first tests in [Gallavotti, 2016, p.225–228], we have found that, in the absence of the centrifugal term $g = 0$, the dynamics (which has two degrees of freedom) still admits two constants of motion even in presence of the hard wall. This suggests that, if a suitable KAM analysis could be carried out, the system would not be ergodic for small values of g.

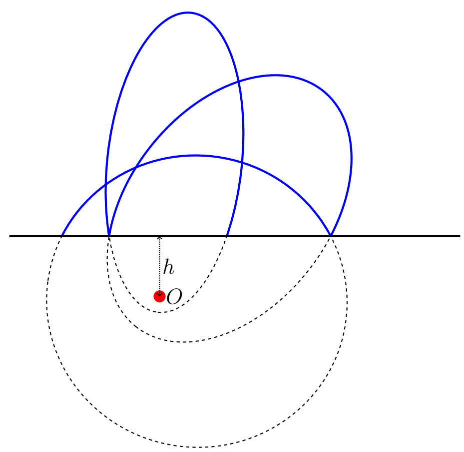

Figure 1: A trajectory. The large dot is the attraction center O, and the line is the hard wall $\mathcal{L}$. In between collisions, the trajectories are ellipses. The ellipses are drawn in full, but the part that is not covered by the particle is dashed.

1 Definition of the model and main result

Let us now specify the model formally, and state our main result more precisely. We fix the gravitational center to the origin of the x, y-plane and let $\mathcal{L}$ be the line $y = h$. The Hamiltonian for the system in between collisions is

$$H = \frac{p_x^2 + p_y^2}{2} - \frac{\alpha}{2r} + \frac{g}{2r^2} \tag{1.1}$$

where $\alpha > 0, g \geq 0, r = \sqrt{x^2 + y^2}$ and the particle moves following Hamilton's equations as long as it stays away from the obstacle $\mathcal{L}$. When an encounter with

$\mathcal{L}$ occurs the particle is reflected elastically and continues on.

Boltzmann [1868b], considered this system on the hyper-surface $A = \mathbf{p}^2 - \frac{\alpha}{r} + \frac{g}{r^2}$. The intersection of this hyper-surface with $y = h$ is the region $\mathcal{F}_A$ enclosed within the curves

$$\pm\sqrt{(A - \frac{g}{x^2+h^2} + \frac{\alpha}{\sqrt{x^2+h^2}})}, \qquad x_{min} < x < x_{max} \tag{1.2}$$

with x_{min} and $x_{\max}$ the roots of $A = \frac{g}{x^2+h^2} - \frac{\alpha}{\sqrt{x^2+h^2}}$. He argued that all motions (with few exceptions) would cover densely the surfaces of constant $A < 0$ if $\alpha, g > 0$.

From now on, unless it is explicitly stated otherwise, we will assume that $g = 0$.

In this case, the motion between collisions takes place at constant energy $\frac{1}{2}A$ and constant angular momentum a, and traces out an ellipse. One of the foci of the ellipse is located at the origin, and we will denote the angle that the aphelion of the ellipse makes with the x-axis by θ_0. Thus, the ellipse is entirely determined by the triplet (A, a, θ_0). When a collision occurs, A remains unchanged, but a and θ_0 change discontinuously to values $(a', \theta_0') = \mathcal{F}(a, \theta_0)$, and thus the Kepler ellipse of the trajectory changes. In addition, the semi-major axis a_M of the ellipse is also fixed to $a_M = -\frac{\alpha}{2A}$ (Kepler's law): so the successive ellipses have the same semi-major axis, while the eccentricity varies because at each collision the angular momentum changes: $e^2 = 1 + \frac{4Aa^2}{\alpha^2}$. Thus, the motion will take place on arcs of various ellipses $\mathcal{E}$, which all share the same focus and the same semi-major axis, but whose angle and eccentricity changes at each collision.

Our main result is that the (canonical) map $(a', \varphi_0') = \mathcal{F}(a, \theta_0)$, which maps the angular momentum and angle of the aphelion before a collision to their values after the collision, admits a constant of motion. This follows from the following geometric lemma about ellipses.

Lemma 1: *Given an ellipse $\mathcal{E}$ with a focus at O that intersects $\mathcal{L}$ at a point P. Let Q denote the orthogonal projection of O onto $\mathcal{L}$ (see figure 2). The distance R_0 between Q and the center of $\mathcal{E}$ depends solely on the semi-major axis a_M, the distance r from O to P, and $\cos(2\lambda)$ where λ is the angle between the tangent of the ellipse at P and $\mathcal{L}$ (to define the direction of the tangent, we parametrize the ellipse in the counter-clockwise direction):*

$$R_0 = \sqrt{\frac{1}{4}r^2 + \frac{1}{4}(2a_M - r)^2 + \frac{1}{2}r(2a_M - r)\cos(2\lambda)}. \tag{1.3}$$

Proof: We switch to polar coordinates $p = (r\cos\varphi, r\sin\varphi)$.

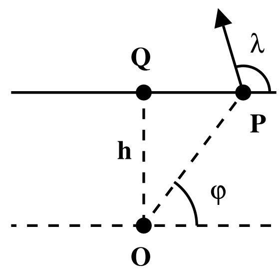

Figure 2: The attractive center is O, hence it is the focus of the ellipse in absence of centrifugal force $g = 0$. Q is the projection of O on the line $\mathcal{L}$ and P is a collision point. The arrow represents the velocity of the particle after the collision.

Let O' denote the other focus of the ellipse, and C denote its center. The first step is to compute the vector $\overrightarrow{O'P}$, which in polar coordinates is

$$\overrightarrow{O'P} = ((2a_M - r)\cos\varphi', (2a_M - r)\sin\varphi') \tag{1.4}$$

Let $\psi := \pi + \varphi - \lambda$ denote the angle between the tangent of the ellipse at P and the vector $\overrightarrow{PO}$ (see figure 3), and $\psi' := \pi + \varphi' - \lambda$ denote the angle between the tangent of the ellipse at P and the vector $\overrightarrow{PO'}$.

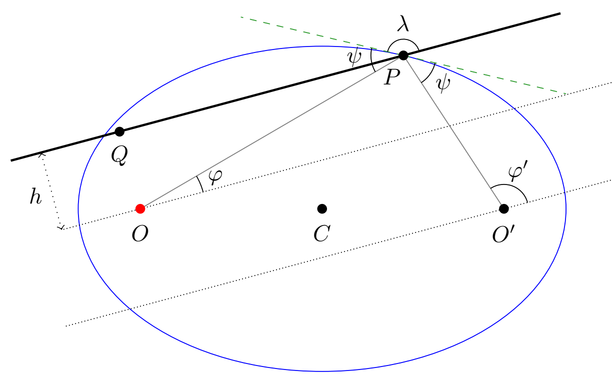

Figure 3: An ellipse with foci O and O' and center C. The thick line is $\mathcal{L}$, which intersects the ellipse at P, and Q is the projection of O onto $\mathcal{L}$. The dashed line is the tangent at P. λ is the angle between $\mathcal{L}$ and the tangent, φ is the polar coordinate, φ' is the angle between $\mathcal{L}$ and $\overrightarrow{O'P}$. ψ is the angle between the tangent and $\overrightarrow{PO}$, which is equal to the angle between the tangent and $\overrightarrow{PO'}$. R_0 is the distance between Q and C.

By the focus-to-focus reflection property of ellipses, we have $\psi' = \pi - \psi$. Thus $\varphi' = 2\lambda - \pi - \varphi$ and we find;

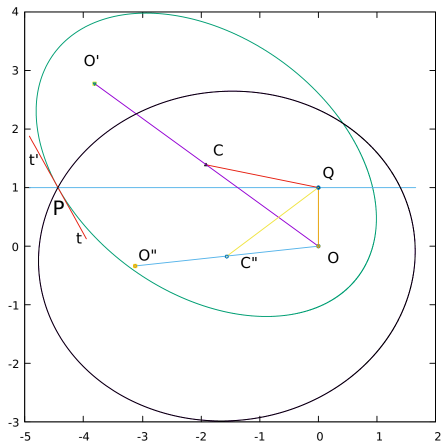

Figure 4: Two ellipses, before and after a collision. The collision line $\mathcal{L}$ is the line at $y = 1$, P is the collision point; Q is the projection of O onto $\mathcal{L}$; the two ellipses $\mathcal{E}$ and $\mathcal{E}'$ have a common focus O, and O, O' are the foci of $\mathcal{E}$, whereas O, O'' are the foci of $\mathcal{E}'$; C and C'' are the centers of $\mathcal{E}$ and $\mathcal{E}'$ respectively; the ellipses are drawn completely although the trajectory is restricted to the parts above $y = h = 1$. The distance from C'' to Q is the same as that from C to Q. The upper ellipse $\mathcal{E}$ contains the trajectory that starts at the collision point P following the other ellipse $\mathcal{E}'$ which has undergone reflection.

$$R_0^2 = |Q - C|^2 = \frac{1}{4}\left(r^2 + (2a_M - r)^2 + 2r(2a_M - r)\cos(2\lambda)\right). \tag{1.5}$$

See figures 3 and 4. □

Theorem 1: *The quantity*

$$R = a^2 + h\alpha e \sin\theta_0 \equiv \frac{\alpha}{2a_M}(h^2 + a_M^2 - R_0^2) \tag{1.6}$$

where e is the eccentricity $e = \sqrt{1 + \frac{4Aa^2}{\alpha^2}}$, is a constant of motion.

Proof: During a collision, the value of λ changes from λ to $\pi - \lambda$, while r and a_M stay the same. By lemma 1, this implies that the distance R_0 between Q and the center of the ellipse is preserved during a collision. Furthermore, the position of the center C of the ellipse is given by $C = a_M e(\cos\theta_0, \sin\theta_0)$ so

$$R_0^2 = |Q - C|^2 = a_M^2 e^2 - 2a_M eh \sin\theta_0 + h^2. \tag{1.7}$$

Furthermore, the angular momentum is equal to $a^2 = \frac{1}{2}a_M\alpha(1 - e^2)$ so

$$-R_0^2 + h^2 + a_M^2 = \frac{2a_M}{\alpha}(a^2 + e\alpha h \sin\theta_0) \tag{1.8}$$

is a conserved quantity. □

Remark: Some useful inequalities are

$$\begin{aligned} &r_{max} < 2a_M;\; x_{max} = \sqrt{r_{max}^2 - h^2};\; R_0^2 \in ((a_M - r)^2, a_M^2); \\ &\frac{\alpha h^2}{2a_M} \,<\, R \,<\, (1 + \frac{a_M^2}{h^2} - \Big(\frac{a_M}{h} - \frac{r}{h}\Big)^2)\frac{\alpha h^2}{2a_M} \end{aligned} \tag{1.9}$$

hence in the plane (x, λ) the rectangle $(-x_{max}, x_{max}) \times (0, \pi)$ (recall that x_{max} is the largest x accessible at energy $\frac{1}{2}A$) is the surface of energy $\frac{1}{2}A$ and the trajectories are the curves of constant R inside this rectangle.

2 Conjectures on action angle variables

In the previous section, we exhibited a constant of motion, which, along with the conservation of energy, brings the number of independent conserved quantities to two. In a continuous Hamiltonian system, this would imply the existence of action-angle variables, which are canonically conjugate to the position and momentum of the particle, in terms of which the dynamics reduces to a linear evolution on a torus. In this case, the collision with the wall introduces some discreteness into the problem, and the existence of the action angle variables is not guaranteed by standard theorems. Indeed, in the presence of the collisions, we no longer have a Hamiltonian system, but rather a discrete symplectic map (or a non-differentiable Hamiltonian), which describes the change in the state of the particle during a collision. In this section, we present some conjectures pertaining to the existence of action angle variables for this problem.

The first step is to change to variables which are action-angle variables for the motion in between collision. We choose the *Delaunay* variables, whose angles are the argument of the aphelion θ_0 defined above, the *mean anomaly* M, and whose actions are the angular momentum a, and another momentum usually denoted by L and related to the semi-major axis a_M and to the energy $E = \frac{1}{2}A$:

$$L := -\sqrt{\frac{\alpha}{2}a_M}, \quad a_M := -\frac{\alpha}{2A}, \quad A := p^2 + \frac{a^2}{r^2} - \frac{\alpha}{r} \equiv -\frac{\alpha^2}{4L^2} \tag{2.1}$$

It is well known that this change of variables is canonical. In between collisions, the dynamics of the particle in the variables $(M, \theta_0; L, a)$ is, simply,

$$\dot{M} = \frac{\alpha^2}{4L^3}, \quad \dot{\theta}_0 = 0, \quad \dot{L} = 0, \quad \dot{a} = 0. \tag{2.2}$$

These variables are thus action-angle variables in between collisions, but when a collision occurs, θ_0 and a will change.

The following conjecture states that there exists an action-angle variable during the collisions.

Conjecture 1: *There exists a variable γ and an integer k such that, every k collisions, the change in γ is*

$$\gamma' = \gamma + \omega(L, R) \tag{2.3}$$

in which case γ is an angle that rotates on a circle of radius depending on L, R. The function $\omega(L, R)$ has a non zero derivative with respect to R at constant L, i.e. the motion on the energy surface is quasi periodic and anisochronous.

We will now sketch a construction of this variable γ, which we obtain using a generating function $F(L, R, M, \theta_0)$.

First of all, by theorem 1, the angular momentum $a(\theta_0)$ is a solution of

$$a^2 = R - h\alpha \sin\theta_0 \sqrt{1 - \frac{a^2}{L^2}} \tag{2.4}$$

that is, if $\varepsilon = \pm$,

$$a^2 = R - \frac{h^2\alpha^2}{2L^2}\sin^2\theta_0 + \varepsilon\sqrt{\frac{h^4\alpha^2}{4L^4}\sin^4\theta_0 + h^2\alpha^2\sin^2\theta_0 - \frac{R\alpha^2 h^2}{L^2}\sin^2\theta_0} \tag{2.5}$$

and $a = \eta\sqrt{a^2}$, so that there may be four possibilities for the value of a denoted $a = a_{\varepsilon,\eta}(\theta_0, R, L)$ with $\varepsilon = \pm, \eta = \pm$. The choice of the signs $\varepsilon = \pm 1$, and η must be examined carefully.

We then define the generating function

$$F(L, R, M, \theta_0) = LM + \int_0^{\theta_0} a(L, R, \psi) d\psi \tag{2.6}$$

which yields the following canonical transformation:

$$\begin{aligned} \gamma =& \partial_R \int_0^{\theta_0} a_{\varepsilon,\eta}(L, R, \psi)\, d\psi \\ M' =& M + \partial_L \int_0^{\theta_0} a_{\varepsilon,\eta}(L, R, \psi)\, d\psi \end{aligned} \tag{2.7}$$

It is natural, if Boltzmann's system is integrable (at $g = 0$), that the new variables are its action angle variables and M', γ rotate uniformly in spite of the collisions.

However, in this case, the signs ε and η may change from one collision to the next, complicating the situation. A careful numerical study of the system has led us to the following conjecture (see figure 5).

Conjecture 2: *If $R > h\alpha$ (which is the case in which the circle, of radius R_0, of the centers encloses the focus O), when the motion collides for the n-th time,*

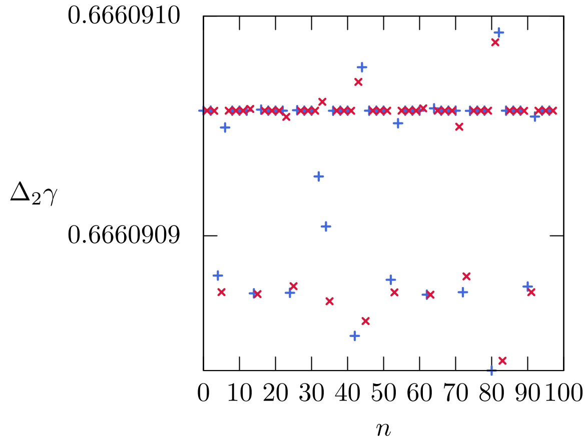

Figure 5: A plot of the increment in γ between the n-th and the $n+2$-nd collision as a function of n. The blue '+' signs correspond to even n, and the red '×' to odd n. The variation of $\Delta_2\gamma$ is as small as 1 part per million, thus supporting conjecture 2.

the angular momentum is proportional to $(-1)^n$, *and, thus,* $\epsilon = (-1)^n$. *The sign* η *is fixed to* $+$. *The increment* $\Delta_2\gamma$ *in* γ *between the* n*-th and the* $n+2$*-th collision is independent of* n.

Remark: The change of variables over the variables a, θ_0 to R, γ at fixed L is *remarkably* essentially the same as the one (*a priori* unrelated) to find action-angle variable for the auxiliary Hamiltonian $R = R(a, \theta_0)$. This might remain true even when $R < h\alpha$: interpretable as a kind of auxiliary pendulum motion.

At the time of publication, it has been brought to our attention that G. Felder has proved that the orbits are all either periodic or quasi-periodic, which would be implied from conjecture 1.

3 Conclusion and outlook

In this brief note, we have shown that the system considered by Boltzmann in 1868, in the case $g = 0$, admits two independent constants of motion. This indicates that it should be possible to compute action angle variables for this system, which is not entirely trivial because of the discontinuous nature of the collision process. If such a construction could be brought to its conclusion, then it would show that the trajectories are either periodic or quasi-periodic, a fact which is consistent with the numerical simulations we have run.

This is not a contradiction of Boltzmann's claim that this model is ergodic, since Boltzmann considered the model at $g \neq 0$. However, we expect that a

KAM-type argument can be set up for this model, to show that the system cannot be ergodic, even if $g > 0$, provided g is sufficiently small. However it may still have invariant regions of positive volume where the motion is ergodic.

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