

On the convolution inequality $f \geq f \star f$

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Abstract

We consider the inequality $f \geq f \star f$ for real functions in $L^1(\mathbb{R}^d)$ where $f \star f$ denotes the convolution of f with itself. We show that all such functions f are non-negative, which is not the case for the same inequality in L^p for any $1 < p \leq 2$, for which the convolution is defined. We also show that all solutions in $L^1(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} f(x) dx \leq \frac{1}{2}$. Moreover, if $\int_{\mathbb{R}^d} f(x) dx = \frac{1}{2}$, then f must decay fairly slowly: $\int_{\mathbb{R}^d} |x| f(x) dx = \infty$, and this is sharp since for all $r < 1$, there are solutions with $\int_{\mathbb{R}^d} f(x) dx = \frac{1}{2}$ and $\int_{\mathbb{R}^d} |x|^r f(x) dx < \infty$. However, if $\int_{\mathbb{R}^d} f(x) dx =: a < \frac{1}{2}$, the decay at infinity can be much more rapid: we show that for all $a < \frac{1}{2}$, there are solutions such that for some $\epsilon > 0$, $\int_{\mathbb{R}^d} e^{\epsilon|x|} f(x) dx < \infty$.

Our subject is the set of real, integrable solutions of the inequality

$$f(x) \geq f \star f(x), \quad \forall x \in \mathbb{R}^d, \quad (1)$$

where $f \star f(x)$ denotes the convolution $f \star f(x) = \int_{\mathbb{R}^d} f(x-y)f(y)dy$, which by Young's inequality [LL96, Theorem 4.2] is well defined as an element of $L^{p/(2-p)}(\mathbb{R}^d)$ for all $1 \leq p \leq 2$. Thus, one may consider this inequality in $L^p(\mathbb{R}^d)$ for all $1 \leq p \leq 2$, but the case $p = 1$ is special: the solution set of (1) is restricted in a number of surprising ways. Integrating both sides of (1), one sees immediately that $\int_{\mathbb{R}^d} f(x)dx \leq 1$. We prove that, in fact, all integrable solutions satisfy $\int_{\mathbb{R}^d} f(x)dx \leq \frac{1}{2}$, and this upper bound is sharp.

Perhaps even more surprising, we prove that all integrable solutions of (1) are non-negative. This is *not true* for solutions in $L^p(\mathbb{R}^d)$, $1 < p \leq 2$. For $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq 2$, the Fourier transform $\widehat{f}(k) = \int_{\mathbb{R}^d} e^{-i2\pi k \cdot x} f(x)dx$ is well defined as an element of $L^{p/(p-1)}(\mathbb{R}^d)$. If f solves the equation $f = f \star f$, then $\widehat{f} = \widehat{f}^2$, and hence \widehat{f} is the indicator function of a measurable set. By the Riemann-Lebesgue Theorem, if $f \in L^1(\mathbb{R}^d)$, then \widehat{f} is continuous and vanishes at infinity, and the only such indicator function is the indicator function of the empty set. Hence the only integrable solution of $f = f \star f$ is the trivial solution $f = 0$. However, for $1 < p \leq 2$, solutions abound: take $d = 1$ and define g to be the indicator function of the interval $[-a, a]$. Define

$$f(x) = \int_{\mathbb{R}} e^{-i2\pi kx} g(k)dk = \frac{\sin 2\pi xa}{\pi x}, \quad (2)$$

which is not integrable, but which belongs to $L^p(\mathbb{R})$ for all $p > 1$. By the Fourier Inversion Theorem $\widehat{f} = g$. Taking products, one gets examples in any dimension.

To construct a family of solutions to (1), fix $a, t > 0$, and define $g_{a,t}(k) = ae^{-2\pi|k|t}$. By [SW71, Theorem 1.14],

$$f_{a,t}(x) = \int_{\mathbb{R}^d} e^{-i2\pi kx} g_{a,t}(k)dk = a\Gamma((d+1)/2)\pi^{-(d+1)/2} \frac{t}{(t^2 + x^2)^{(d+1)/2}}.$$

Since $g_{a,t}^2(k) = g_{a^2,2t}$, $f_{a,t} \star f_{a,t} = f_{a^2,2t}$. Thus, $f_{a,t} \geq f_{a,t} \star f_{a,t}$ reduces to

$$\frac{t}{(t^2 + x^2)^{(d+1)/2}} \geq \frac{2at}{(4t^2 + x^2)^{(d+1)/2}}$$

which is satisfied for all $a \leq 1/2$. Since $\int_{\mathbb{R}^d} f_{a,t}(x)dx = a$, this provides a class of solutions of (1) that are non-negative and satisfy

$$\int_{\mathbb{R}^d} f(x)dx \leq \frac{1}{2}, \quad (3)$$

all of which have fairly slow decay at infinity, so that in every case,

$$\int_{\mathbb{R}^d} |x|f(x)dx = \infty. \quad (4)$$

Our results show that this class of examples of integrable solutions of (1) is surprisingly typical of *all* integrable solutions: every real integrable solution f of (1) is positive, satisfies (3), and if there is equality in (3), f also satisfies (4). The positivity of all real solutions of (1) in $L^1(\mathbb{R}^d)$ may be considered surprising since it is false in $L^p(\mathbb{R}^d)$ for all $p > 1$, as the example (2) shows. We also show that when strict inequality holds in (3) for a solution f of (1), it is possible for f to have rather fast decay; we construct examples such that $\int_{\mathbb{R}^d} e^{\epsilon|x|} f(x)dx < \infty$ for some $\epsilon > 0$. The conjecture that integrable solutions of (1) are necessarily positive was motivated by recent work [CJL19] on a partial differential equation involving a quadratic nonlinearity of $f \star f$ type, and the result proved here is the key to the proof of positivity for solutions of this partial differential equations; see [CJL19].

Theorem 1

If f is a real valued function in $L^1(\mathbb{R}^d)$ such that

$$f(x) - f \star f(x) =: u(x) \geq 0 \quad (5)$$

for all x . Then $\int_{\mathbb{R}^d} f(x) dx \leq \frac{1}{2}$, and f is given by the convergent series

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x) \quad (6)$$

where the $c_n \geq 0$ are the Taylor coefficients in the expansion of $\sqrt{1-x}$

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} c_n x^n, \quad c_n = \frac{(2n-3)!!}{2^n n!} \sim n^{-3/2} \quad (7)$$

In particular, f is positive. Moreover, if $u \geq 0$ is any integrable function with $\int_{\mathbb{R}^d} u(x) dx \leq \frac{1}{4}$, then the sum on the right in (6) defines an integrable function f that satisfies (5).

Proof: Note that u is integrable. Let $a := \int_{\mathbb{R}^d} f(x) dx$ and $b := \int_{\mathbb{R}^d} u(x) dx \geq 0$. Fourier transforming, (5) becomes

$$\widehat{f}(k) = \widehat{f}(k)^2 + \widehat{u}(k). \quad (8)$$

At $k = 0$, $a^2 - a = -b$, so that $(a - \frac{1}{2})^2 = \frac{1}{4} - b$. Thus $0 \leq b \leq \frac{1}{4}$. Furthermore, since $u \geq 0$,

$$|\widehat{u}(k)| \leq \widehat{u}(0) \leq \frac{1}{4} \quad (9)$$

and the first inequality is strict for $k \neq 0$. Hence for $k \neq 0$, $\sqrt{1 - 4\widehat{u}(k)} \neq 0$. By the Riemann-Lebesgue Theorem, $\widehat{f}(k)$ and $\widehat{u}(k)$ are both continuous and vanish at infinity, and hence we must have that

$$\widehat{f}(k) = \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4\widehat{u}(k)} \quad (10)$$

for all sufficiently large k , and in any case $\widehat{f}(k) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4\widehat{u}(k)}$. But by continuity and the fact that $\sqrt{1 - 4\widehat{u}(k)} \neq 0$ for any $k \neq 0$, the sign cannot switch. Hence (10) is valid for all k , including $k = 0$, again by continuity. At $k = 0$, $a = \frac{1}{2} - \sqrt{1 - 4b}$, which proves (3). The fact that c_n as specified in (7) satisfies $c_n \sim n^{-3/2}$ is a simple application of Stirling's formula, and it shows that the power series for $\sqrt{1-z}$ converges absolutely and uniformly everywhere on the closed unit disc. Since $|4\widehat{u}(k)| \leq 1$, $\sqrt{1 - 4\widehat{u}(k)} = 1 - \sum_{n=1}^{\infty} c_n (4\widehat{u}(k))^n$. Inverting the Fourier transform, yields (6), and since $\int_{\mathbb{R}^d} 4^n \star^n u(x) dx \leq 1$, the convergence of the sum in $L^1(\mathbb{R}^d)$ follows from the convergence of $\sum_{n=1}^{\infty} c_n$. The final statement follows from the fact that if f is defined in terms of u in this manner, (10) is valid, and then (8) and (5) are satisfied. \square

Theorem 2

Let $f \in L^1(\mathbb{R}^d)$ satisfy (1) and $\int_{\mathbb{R}^d} f(x) dx = \frac{1}{2}$. Then $\int_{\mathbb{R}^d} |x| f(x) dx = \infty$.

Proof: If $\int_{\mathbb{R}^d} f(x) dx = \frac{1}{2}$, $\int_{\mathbb{R}^d} 4u(x) dx = 1$, then $w(x) = 4u(x)$ is a probability density, and we can write $f(x) = \sum_{n=0}^{\infty} \star^n w$. Suppose that $|x|f(x)$ is integrable. Then $|x|w(x)$ is integrable. Let $m := \int_{\mathbb{R}^d} xw(x) dx$. Since first moments add under convolution, the trivial inequality $|m||x| \geq m \cdot x$ yields

$$|m| \int_{\mathbb{R}^d} |x| \star^n w(x) dx \geq \int_{\mathbb{R}^d} m \cdot x \star^n w(x) dx = n|m|^2.$$

It follows that $\int_{\mathbb{R}^d} |x| f(x) dx \geq |m| \sum_{n=1}^{\infty} n c_n = \infty$. Hence $m = 0$.

Suppose temporarily that in addition, $|x|^2 w(x)$ is integrable. Let σ^2 be the variance of w ; i.e., $\sigma^2 = \int_{\mathbb{R}^d} |x|^2 w(x) dx$. Define the function $\varphi(x) = \min\{1, |x|\}$. Then

$$\int_{\mathbb{R}^d} |x| \star^n w(x) dx = \int_{\mathbb{R}^d} |n^{1/2}x| \star^n w(n^{1/2}x) n^{d/2} dx \geq n^{1/2} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x) n^{d/2} dx.$$

By the Central Limit Theorem, since φ is bounded and continuous,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x) n^{d/2} dx = \int_{\mathbb{R}^d} \varphi(x) \gamma(x) dx =: C > 0 \quad (11)$$

where $\gamma(x)$ is a centered Gaussian probability density with variance σ^2 .

This shows that there is a $\delta > 0$ for all sufficiently large n , $\int_{\mathbb{R}^d} |x| \star^n w(x) dx \geq \sqrt{n}\delta$, and then since $c_n \sim n^{3/2}$, $\sum_{n=1}^{\infty} c_n \int_{\mathbb{R}^d} |x| \star^n w(x) dx = \infty$.

To remove the hypothesis that w has finite variance, note that if w is a probability density with zero mean and infinite variance, $\star^n w(n^{1/2}x) n^{d/2}$ is “trying” to converge to a Gaussian of infinite variance. In particular, one would expect that for all $R > 0$,

$$\lim_{n \rightarrow \infty} \int_{|x| \leq R} \star^n w(n^{1/2}x) n^{d/2} dx = 0, \quad (12)$$

so that the limit in (11) has the value 1. The proof then proceeds as above. The fact that (12) is valid is proved in [CGR08, Corollary 1]. \square

Remark

For the convenience of the reader, we sketch, at the end of this paper, the part of the argument in [CGR08] that proves (12), since we know of no reference for this simple statement, and the proof in [CGR08] deals with a more complicated variant of this problem.

Theorem 3

If f satisfies (1), and $\int |x|^2 [f(x) - f \star f(x)] dx < \infty$, then, for all $0 \leq p < 1$,

$$\int |x|^p f(x) dx < \infty. \quad (13)$$

Proof: We may suppose that f is not identically 0. Let $u = f - f \star f$ as above. Let $t := 4 \int_{\mathbb{R}^d} u(x) dx \leq 1$. Then $t > 0$. Define $w := t^{-1}4u$; w is a probability density and

$$f(x) = \sum_{n=1}^{\infty} c_n t^n \star^n w(x).$$

By hypothesis, w has a mean $m := \int_{\mathbb{R}^d} x w(x) dx$ and variance $\sigma^2 = \int_{\mathbb{R}^d} |x - m|^2 w(x) dx < \infty$. Since variance is additive under convolution,

$$\int_{\mathbb{R}^d} |x - m|^2 \star^n w(x) dx = n\sigma^2.$$

By Hölder’s inequality, for all $0 < p < 2$, $\int_{\mathbb{R}^d} |x - m|^p \star^n w(x) dx \leq (n\sigma^2)^{p/2}$. It follows that for $0 < p < 1$,

$$\int_{\mathbb{R}^d} |x - m|^p f(x) dx \leq (\sigma^2)^{p/2} \sum_{n=1}^{\infty} n^{p/2} c_n < \infty,$$

again using the fact that $c_n \sim n^{-3/2}$. \square

Theorem 2 implies that when $\int f = \frac{1}{2}$, f cannot decay faster than $|x|^{-(d+1)}$. However, integrable solutions f of (1) such that $\int_{\mathbb{R}^d} f(x)dx < \frac{1}{2}$ can decay quite rapidly, even having finite exponential moments, as we now show.

Consider a non-negative, integrable function u , which integrates to $r < \frac{1}{4}$, and satisfies

$$\int_{\mathbb{R}^d} u(x)e^{\lambda|x|}dx < \infty \quad (14)$$

for some $\lambda > 0$. The Laplace transform of u is $\tilde{u}(p) := \int e^{-px}u(x) dx$ which is analytic for $|p| < \lambda$, and $\tilde{u}(0) < \frac{1}{4}$. Therefore, there exists $0 < \lambda_0 \leq \lambda$ such that, for all $|p| \leq \lambda_0$, $\tilde{u}(p) < \frac{1}{4}$.

By Theorem 1, $f(x) := \frac{1}{2} \sum_{n=1}^{\infty} 4^n c_n(\star^n u)(x)$ is an integrable solution of (1). For $|p| \leq \lambda_0$, it has a well-defined Laplace transform $\tilde{f}(p)$ given by

$$\tilde{f}(p) = \int e^{-px} f(x) dx = \frac{1}{2}(1 - \sqrt{1 - 4\tilde{u}(p)}) \quad (15)$$

which is analytic for $|p| \leq \lambda_0$. Note that $e^{|s|x|} \leq \prod_{j=1}^d e^{|s x_j|} \leq \frac{1}{d} \sum_{j=1}^d e^{d|s x_j|} \leq \frac{2}{d} \sum_{j=1}^d \cosh(ds x_j)$. What

has just been shown, yields a $\delta > 0$ so that for $|s| < \delta$, $\int_{\mathbb{R}^d} \cosh(ds x_j) f(x) dx < \infty$ for each j . It follows that for $0 < s < \delta$, $\int_{\mathbb{R}^d} e^{s|x|} f(x) dx < \infty$.

Remark

One might also consider the inequality $f \leq f \star f$ in $L^1(\mathbb{R}^d)$, but it is simple to construct solutions that have both signs. Consider any radial Gaussian probability density g , Then $g \star g(x) \geq g(x)$ for all sufficiently large $|x|$, and taking $f := ag$ for a sufficiently large, we obtain $f < f \star f$ everywhere. Now on a small neighborhood of the origin, replace the value of f by -1 . If the region is taken small enough, the new function f will still satisfy $f < f \star f$ everywhere.

We close by sketching a proof of (12) using the construction in [CGR08]. Let w be a mean zero, infinite variance probability density on \mathbb{R}^d . Pick $\epsilon > 0$, and choose a large value σ such that $(2\pi\sigma^2)^{-d/2} R^d |B| < \epsilon/2$, where $|B|$ denotes the volume of the ball. The point of this is that if G is a centered Gaussian random variable with variance σ^2 , the probability is no more than $\epsilon/2$ that G lies in *any* particular translate $B_R + y$ of the ball of radius R .

Let $A \subset \mathbb{R}^d$ be a bounded set so that $\int_A xw(x)dx = 0$ and $\int_A |x|^2 w(x)dx = \sigma^2$. It is then easy to find mutually independent random variables X, Y and α such that X takes values in A and, has zero mean and variance greater than σ^2 , α is a Bernoulli variable with success probability $\int_A w(x)dx$, which we can take arbitrarily close to 1 by increasing the size of A , and finally such that $\alpha X + (1 - \alpha)Y$ has the probability density w . Taking independent i.i.d. sequences of such random variables, $w(n^{1/2}x)n^{d/2}$ is the probability density of $W_n := \frac{1}{\sqrt{n}} \sum_{j=1}^n \alpha_j X_j + \frac{1}{\sqrt{n}} \sum_{j=1}^n (1 - \alpha_j) Y_j$.

We are interested in estimating the expectation of $1_{B_R}(W_n)$. We first take the conditional expectation, given the values of the α 's and the Y 's. These conditional expectations have the form $\mathbb{E} \left[1_{B_R+y} \left(\frac{1}{\sqrt{n}} \sum_{j=1}^n \alpha_j X_j \right) \right]$ for some translate $B_R + y$ of the unit Ball. Since A is bounded, the X_j 's all have the same finite third moment, and now the Berry-Esseen Theorem [B41, E42], a version of the Central Limit Theorem with rate information, allows us to control the error in approximating this expectation by $\mathbb{E}(1_{B_R+y}(G))$ where G is centered Gaussian with variance σ^2 . By the choice of σ , this is no greater than $\epsilon/2$, independent of y . For n large enough, the remaining errors – coming from the small probability that $\sum_{n=1}^n \alpha_j$ is significantly less n , and the error bound provided by the Berry-Esseen Theorem, are readily absorbed into the remaining $\epsilon/2$ for large n . Thus for all sufficiently large n , the integral in (12) is no more than ϵ .

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