On the convolution inequality $f \geqslant f \star f$

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Abstract

We consider the inequality $f \geqslant f \star f$ for real functions in $L^1(\mathbb{R}^d)$ where $f \star f$ denotes the convolution of f with itself. We show that all such functions f are non-negative, which is not the case for the same inequality in L^p for any $1 < p \leqslant 2$, for which the convolution is defined. We also show that all solutions in $L^1(\mathbb{R}^d)$ satisfy $\int_{\mathbb{R}^d} f(x)dx \leqslant \frac{1}{2}$. Moreover, if $\int_{\mathbb{R}^d} f(x)dx = \frac{1}{2}$, then f must decay fairly slowly: $\int_{\mathbb{R}^d} |x|f(x)dx = \infty$, and this is sharp since for all $r < 1$, there are solutions with $\int_{\mathbb{R}^d} f(x)dx = \frac{1}{2}$ and $\int_{\mathbb{R}^d} |x|^r f(x)dx < \infty$. However, if $\int_{\mathbb{R}^d} f(x)dx =: a < \frac{1}{2}$, the decay at infinity can be much more rapid: we show that for all $a < \frac{1}{2}$, there are solutions such that for some $\epsilon > 0$, $\int_{\mathbb{R}^d} e^{\epsilon|x|} f(x)dx < \infty$.

Our subject is the set of real, integrable solutions of the inequality

$$f(x) \geqslant f \star f(x), \quad \forall x \in \mathbb{R}^d \ , \tag{1}$$

where $f \star f(x)$ denotes the convolution $f \star f(x) = \int_{\mathbb{R}^d} f(x-y)f(y)dy$. By Young's inequality [LL96, Theorem 4.2], for all $1 \leqslant p \leqslant 2$ and all $f \in L^p(\mathbb{R}^d)$, $f \star f$ is well defined as an element of $L^{p/(2-p)}(\mathbb{R}^d)$. Thus, one may consider this inequality in $L^p(\mathbb{R}^d)$ for all $1 \leqslant p \leqslant 2$, but the case $p = 1$ is special: the solution set of (1) is restricted in a number of surprising ways. Integrating both sides of (1), one sees immediately that $\int_{\mathbb{R}^d} f(x)dx \leqslant 1$. We prove that, in fact, all integrable solutions satisfy $\int_{\mathbb{R}^d} f(x)dx \leqslant \frac{1}{2}$, and this upper bound is sharp.
Perhaps even more surprising, we prove that all integrable solutions of (1) are non-negative. This is *not true* for solutions in $L^p(\mathbb{R}^d)$, $1 < p \leqslant 2$. For $f \in L^p(\mathbb{R}^d)$, $1 \leqslant p \leqslant 2$, the Fourier transform $\widehat{f}(k) = \int_{\mathbb{R}^d} e^{-i2\pi k\cdot x} f(x)dx$ is well defined as an element of $L^{p/(p-1)}(\mathbb{R}^d)$. If f solves the equation $f = f \star f$, then $\widehat{f} = \widehat{f}^2$, and hence $\widehat{f}$ is the indicator function of a measurable set. By the Riemann-Lebesgue Theorem, if $f \in L^1(\mathbb{R}^d)$, then $\widehat{f}$ is continuous and vanishes at infinity, and the only such indicator function is the indicator function of the empty set. Hence the only integrable solution of $f = f \star f$ is the trivial solution $f = 0$. However, for $1 < p \leqslant 2$, solutions abound: take $d = 1$ and define g to be the indicator function of the interval $[-a, a]$. Define

$$f(x) = \int_{\mathbb{R}} e^{-i2\pi kx} g(k)dk = \frac{\sin 2\pi xa}{\pi x} \ , \tag{2}$$

which is not integrable, but which belongs to $L^p(\mathbb{R})$ for all $p > 1$. By the Fourier Inversion Theorem $\widehat{f} = g$. Taking products, one gets examples in any dimension.
To construct a family of solutions to (1), fix $a, t > 0$, and define $g_{a,t}(k) = ae^{-2\pi|k|t}$. By [SW71, Theorem 1.14],

$$f_{a,t}(x) = \int_{\mathbb{R}^d} e^{-i2\pi kx} g_{a,t}(k)dk = a\Gamma((d+1)/2)\pi^{-(d+1)/2} \frac{t}{(t^2 + x^2)^{(d+1)/2}} \ .$$

Since $g_{a,t}^2(k) = g_{a^2,2t}$, $f_{a,t} \star f_{a,t} = f_{a^2,2t}$, Thus, $f_{a,t} \geqslant f_{a,t} \star f_{a,t}$ reduces to

$$\frac{t}{(t^2 + x^2)^{(d+1)/2}} \geqslant \frac{2at}{(4t^2 + x^2)^{(d+1)/2}}$$

which is satisfied for all $a \leqslant 1/2$. Since $\int_{\mathbb{R}^d} f_{a,t}(x)dx = a$, this provides a class of solutions of (1) that are non-negative and satisfy

$$\int_{\mathbb{R}^d} f(x)dx \leqslant \frac{1}{2} \ , \tag{3}$$

all of which have fairly slow decay at infinity, so that in every case,

$$\int_{\mathbb{R}^d} |x| f(x)dx = \infty \ . \tag{4}$$

Our results show that this class of examples of integrable solutions of (1) is surprisingly typical of *all* integrable solutions: every real integrable solution f of (1) is positive, satisfies (3), and if there is equality in (3), f also satisfies (4). The positivity of all real solutions of (1) in $L^1(\mathbb{R}^d)$ may be considered surprising since it is false in $L^p(\mathbb{R}^d)$ for all $p > 1$, as the example (2) shows. We also show that when strict inequality holds in (3) for a solution f of (1), it is possible for f to have rather fast decay; we construct examples such that $\int_{\mathbb{R}^d} e^{\epsilon|x|} f(x)dx < \infty$ for some $\epsilon > 0$. The conjecture that integrable solutions of (1) are necessarily positive was motivated by recent work [CJL20, CJL20b] on a partial differential equation involving a quadratic nonlinearity of $f \star f$ type, and the result proved here is the key to the proof of positivity for solutions of this partial differential equations; see [CJL20]. Autoconvolutions $f \star f$ have been studied extensively; see [MV10] and the work quoted there. However, the questions investigated by these authors are quite different from those considered here.

Theorem 1

Let f be a real valued function in $L^1(\mathbb{R}^d)$ such that

$$f(x) - f \star f(x) =: u(x) \geqslant 0 \tag{5}$$

for all x. Then $\int_{\mathbb{R}^d} f(x)\, dx \leqslant \frac{1}{2}$, and f is given by the series

$$f(x) = \frac{1}{2} \sum_{n=1}^{\infty} c_n 4^n (\star^n u)(x) \tag{6}$$

which converges in $L^1(\mathbb{R}^d)$, and where the $c_n \geqslant 0$ are the Taylor coefficients in the expansion of $\sqrt{1-x}$

$$\sqrt{1-x} = 1 - \sum_{n=1}^{\infty} c_n x^n, \quad c_n = \frac{(2n-3)!!}{2^n n!} \sim n^{-3/2} \tag{7}$$

In particular, f is positive. Moreover, if $u \geqslant 0$ is any integrable function with $\int_{\mathbb{R}^d} u(x)dx \leqslant \frac{1}{4}$, then the sum on the right in (6) defines an integrable function f that satisfies (5), and $\int_{\mathbb{R}^d} f(x)dx = \frac{1}{2}$ if and only if $\int_{\mathbb{R}^d} u(x)dx = \frac{1}{4}$.

Proof: Note that u is integrable. Let $a := \int_{\mathbb{R}^d} f(x)dx$ and $b := \int_{\mathbb{R}^d} u(x)dx \geqslant 0$. Fourier transforming, (5) becomes

$$\widehat{f}(k) = \widehat{f}(k)^2 + \widehat{u}(k)\ . \tag{8}$$

At $k = 0$, $a^2 - a = -b$, so that $\left(a - \frac{1}{2}\right)^2 = \frac{1}{4} - b$. Thus $0 \leqslant b \leqslant \frac{1}{4}$. Furthermore, since $u \geqslant 0$,

$$|\widehat{u}(k)| \leqslant \widehat{u}(0) \leqslant \tfrac{1}{4} \tag{9}$$

and the first inequality is strict for $k \neq 0$. Hence for $k \neq 0$, $\sqrt{1 - 4\widehat{u}(k)} \neq 0$. By the Riemann-Lebesgue Theorem, $\widehat{f}(k)$ and $\widehat{u}(k)$ are both continuous and vanish at infinity, and hence we must have that

$$\widehat{f}(k) = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - 4\widehat{u}(k)} \tag{10}$$

for all sufficiently large k, and in any case $\widehat{f}(k) = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4\widehat{u}(k)}$. But by continuity and the fact that $\sqrt{1 - 4\widehat{u}(k)} \neq 0$ for any $k \neq 0$, the sign cannot switch. Hence (10) is valid for all k, including $k = 0$, again by continuity. At $k = 0$, $a = \frac{1}{2} - \sqrt{1 - 4b}$, which proves (3). The fact that c_n as specified in (7) satisfies $c_n \sim n^{-3/2}$ is a simple application of Stirling's formula, and it shows that the power series for $\sqrt{1-z}$ converges absolutely and uniformly everywhere on the closed unit disc. Since $|4\widehat{u}(k)| \leqslant 1$, $\sqrt{1 - 4\widehat{u}(k)} = 1 - \sum_{n=1}^{\infty} c_n (4\widehat{u}(k))^n$. Inverting the Fourier transform, yields (6), and since $\int_{\mathbb{R}^d} 4^n \star^n u(x)dx \leqslant 1$, the convergence of the sum in $L^1(\mathbb{R}^d)$ follows from the convergence of $\sum_{n=1}^{\infty} c_n$. The final statement follows from the fact that if f is defined in terms of u in this manner, then (10) is valid, and then (8) and (5) are satisfied. □

Theorem 2

Let $f \in L^1(\mathbb{R}^d)$ satisfy (1) and $\int_{\mathbb{R}^d} f(x)\, dx = \frac{1}{2}$. Then $\int_{\mathbb{R}^d} |x| f(x)\, dx = \infty$.

Proof: If $\int_{\mathbb{R}^d} f(x)\, dx = \frac{1}{2}$, $\int_{\mathbb{R}^d} 4u(x)\, dx = 1$, then $w(x) = 4u(x)$ is a probability density, and we can write $f(x) = \frac{1}{2}\sum_{n=0}^{\infty} \star^n w$. Aiming for a contradiction, suppose that $|x|f(x)$ is integrable. Then $|x|w(x)$ is integrable. Let $m := \int_{\mathbb{R}^d} xw(x)dx$. Since first moments add under convolution, the trivial inequality $|m||x| \geqslant m \cdot x$ yields

$$|m| \int_{\mathbb{R}^d} |x| \star^n w(x)dx \geqslant \int_{\mathbb{R}^d} m \cdot x \star^n w(x)dx = n|m|^2\ .$$

It follows that $\int_{\mathbb{R}^d} |x| f(x)dx \geqslant \frac{|m|}{2} \sum_{n=1}^{\infty} nc_n = \infty$. Hence $m = 0$.
Suppose temporarily that in addition, $|x|^2 w(x)$ is integrable. Let σ^2 be the variance of w; i.e., $\sigma^2 = \int_{\mathbb{R}^d} |x|^2 w(x)dx$ Define the function $\varphi(x) = \min\{1, |x|\}$. Then

$$\int_{\mathbb{R}^d} |x| \star^n w(x)dx = \int_{\mathbb{R}^d} |n^{1/2}x| \star^n w(n^{1/2}x)n^{d/2}dx \geqslant n^{1/2} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x)n^{d/2}dx.$$

By the Central Limit Theorem, since φ is bounded and continuous,

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \varphi(x) \star^n w(n^{1/2}x)n^{d/2}dx = \int_{\mathbb{R}^d} \varphi(x)\gamma(x)dx =: C > 0 \tag{11}$$

where $\gamma(x)$ is a centered Gaussian probability density with variance σ^2.
This shows that there is a $\delta > 0$ such that for all sufficiently large n, $\int_{\mathbb{R}^d} |x| \star^n w(x)dx \geqslant \sqrt{n}\delta$, and then since $c_n \sim n^{-3/2}$, $\sum_{n=1}^{\infty} c_n \int_{\mathbb{R}^d} |x| \star^n w(x)dx = \infty$.
To remove the hypothesis that w has finite variance, note that if w is a probability density with zero mean and infinite variance, $\star^n w(n^{1/2}x)n^{d/2}$ is "trying" to converge to a "Gaussian of infinite variance". In particular, one would expect that for all $R > 0$,

$$\lim_{n\to\infty} \int_{|x|\leqslant R} \star^n w(n^{1/2}x)n^{d/2}dx = 0 \ , \tag{12}$$

so that the limit in (11) has the value 1. The proof then proceeds as above. The fact that (12) is valid is a consequence of Lemma 4 below, which is closely based on the proof of [CGR08, Corollary 1]. □

Theorem 3

Let $f \in L^1(\mathbb{R}^d)$ satisfy (1), $\int_{\mathbf{R}^d} xu(x)dx = 0$, and $\int_{\mathbb{R}^d} |x|^2 u(x)\ dx < \infty$, then, for all $0 \leqslant p < 1$,

$$\int_{\mathbb{R}^d} |x|^p f(x)\ dx < \infty. \tag{13}$$

Proof: We may suppose that f is not identically 0. Let $t := 4\int_{\mathbb{R}^d} u(x)dx \leqslant 1$. Then $t > 0$. Define $w := t^{-1}4u$; w is a probability density and

$$f(x) = \frac{1}{2}\sum_{n=1}^{\infty} c_n t^n \star^n w(x) \ . \tag{14}$$

By hypothesis, w has a zero mean and variance $\sigma^2 = \int_{\mathbb{R}^d} |x|^2 w(x)dx < \infty$. Since variance is additive under convolution,

$$\int_{\mathbb{R}^d} |x|^2 \star^n w(x)dx = n\sigma^2 \ .$$

By Hölder's inequality, for all $0 < p < 2$, $\int_{\mathbb{R}^d} |x|^p \star^n w(x)dx \leqslant (n\sigma^2)^{p/2}$. It follows that for $0 < p < 1$,

$$\int_{\mathbb{R}^d} |x|^p f(x)dx \leqslant \frac{1}{2}(\sigma^2)^{p/2} \sum_{n=1}^{\infty} n^{p/2} c_n < \infty \ ,$$

again using the fact that $c_n \sim n^{-3/2}$. □

Remark

In the subcritical case $\int_{\mathbb{R}^d} f(x)dx < \frac{1}{2}$, the hypothesis that $\int_{\mathbb{R}^d} xu(x)dx = 0$ is superfluous, and one can conclude more. In this case the quantity t in (14) satisfies $0 < t < 1$, and if we let m denote the mean of w, $\int_{\mathbb{R}^d} |x|^2 \star^n w(x)dx = n^2|m|^2 + n\sigma^2$. For $0 < t < 1$, $\sum_{n=1}^{\infty} n^2 c_n t^n < \infty$

and we conclude that $\int_{\mathbb{R}^d} |x|^2 f(x)dx < \infty$. Finally, the final statement of Theorem 1 shows that critical case functions f satisfying the hypotheses of Theorem 2 are readily constructed.

Theorem 2 implies that when $\int f = \frac{1}{2}$, f cannot decay faster than $|x|^{-(d+1)}$. However, integrable solutions f of (1) such that $\int_{\mathbb{R}^d} f(x)dx < \frac{1}{2}$ can decay more rapidly, as indicated in the previous remark. In fact, they may even have finite exponential moments, as we now show.
Consider a non-negative, integrable function u, which integrates to $r < \frac{1}{4}$, and satisfies

$$\int_{\mathbb{R}^d} u(x)e^{\lambda|x|}dx < \infty \tag{15}$$

for some $\lambda > 0$. The Laplace transform of u is $\widetilde{u}(p) := \int e^{-px}u(x)\ dx$ which is analytic for $|p| < \lambda$, and $\widetilde{u}(0) < \frac{1}{4}$. Therefore, there exists $0 < \lambda_0 \leqslant \lambda$ such that, for all $|p| \leqslant \lambda_0$, $\widetilde{u}(p) < \frac{1}{4}$. By Theorem 1, $f(x) := \dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty} 4^n c_n(\star^n u)(x)$ is an integrable solution of (1). For $|p| \leqslant \lambda_0$, it has a well-defined Laplace transform $\widetilde{f}(p)$ given by

$$\widetilde{f}(p) = \int e^{-px}f(x)\ dx = \frac{1}{2}(1 - \sqrt{1 - 4\widetilde{u}(p)}) \tag{16}$$

which is analytic for $|p| \leqslant \lambda_0$. Note that $e^{s|x|} \leqslant \displaystyle\prod_{j=1}^{d} e^{|sx_j|} \leqslant \frac{1}{d}\sum_{j=1}^{d} e^{d|sx_j|} \leqslant \frac{2}{d}\sum_{j=1}^{d}\cosh(dsx_j)$. Thus, for $|s| < \delta := \lambda_0/d$, $\int_{\mathbb{R}^d}\cosh(dsx_j)f(x)dx < \infty$ for each j, and hence $|s| < \delta$, $\int_{\mathbb{R}^d} e^{s|x|}f(x)dx < \infty$.
However, there are no integrable solutions of (1) that have compact support: We have seen that all solutions of (1) are non-negative, and if A is the support of a non-negative integrable function, the Minkowski sum $A + A$ is the support of $f \star f$.

Remark

One might also consider the inequality $f \leqslant f \star f$ in $L^1(\mathbb{R}^d)$, but it is simple to construct solutions that have both signs. Consider any radial Gaussian probability density g, Then $g \star g(x) \geqslant g(x)$ for all sufficiently large $|x|$, and taking $f := ag$ for a sufficiently large, we obtain $f < f \star f$ everywhere. Now on a small neighborhood of the origin, replace the value of f by -1. If the region is taken small enough, the new function f will still satisfy $f < f \star f$ everywhere.

We close with a lemma validating (12) that is closely based on a construction in [CGR08].

Lemma 4

Let w be a mean zero, infinite variance probability density on $\mathbb{R}^d$. Then for all $R > 0$, (12) is valid.

Proof: Let $X_1, \ldots, X_n$ be n independent samples from the density w, and let B_R denote the centered ball of radius R. The quantity in (12) is $p_{n,R} := \mathbb{P}(n^{-1/2}\sum_{j=1}^n X_j \in B_R)$. Let $\widetilde{X}_1, \ldots, \widetilde{X}_n$ be another n independent samples from the density w, independent of the first n. Then also $p_{n,R} := \mathbb{P}(-n^{-1/2}\sum_{j=1}^n \widetilde{X}_j \in B_R)$. By the independence and the triangle inequality,

$$p_{n,R}^2 \leqslant \mathbb{P}(n^{-1/2}\sum_{j=1}^{n}(X_j - \widetilde{X}_j) \in B_{2R})\ .$$

The random variable $X_1 - \widetilde{X}_1$ has zero mean and infinite variance and an even density. Therefore, without loss of generality, we may assume that $w(x) = w(-x)$ for all x.
Pick $\epsilon > 0$, and choose a large value σ_0 such that $(2\pi\sigma_0^2)^{-d/2}R^d|B| < \epsilon/3$, where $|B|$ denotes the volume of the unit ball B. The point of this is that if G is a centered Gaussian random variable

with variance *at least* σ_0^2, the probability that G lies in *any* particular translate $B_R + y$ of the ball of radius R is no more than $\epsilon/3$. Let $A \subset \mathbb{R}^d$ be a centered cube such that

$$\int_A |x|^2 w(x)dx =: \sigma^2 \geqslant 2\sigma_0^2 \quad \text{and} \quad \int_A w(x)dx > \frac{3}{4} \ ,$$

and note that since A and w are even, $\displaystyle\int_A xw(x)dx = 0$.
It is then easy to find mutually independent random variables X, Y and α such that X takes values in A and, has zero mean and variance σ^2, α is a Bernoulli variable with success probability $\int_A w(x)dx$, and finally such that $\alpha X + (1-\alpha)Y$ has the probability density w. Taking independent identically distributed (i.i.d.) sequences of such random variables, $w(n^{1/2}x)n^{d/2}$ is the probability density of $W_n := n^{-1/2}\displaystyle\sum_{j=1}^n \alpha_j X_j + n^{-1/2}\sum_{j=1}^n (1-\alpha_j)Y_j$, and we seek to estimate the expectation of $1_{B_R}(W_n)$. We first take the conditional expectation, given the values of the α's and the Y's, and we define $\hat{n} = \sum_{j=1}^n \alpha_j$. These conditional expectations have the form $\mathbb{E}\left[1_{B_R+y}\left(\sum_{j=1}^n n^{-1/2}\alpha_j X_j\right)\right]$ for some translate $B_R + y$ of B_R, the ball of radius R. The sum $n^{-1/2}\sum_{j=1}^n \alpha_j X_j$ is actually the sum of $\hat{n}$ i.i.d. random variables with mean zero and variance σ^2/n. The probability that $\hat{n}$ is significantly less than $\frac{3}{4}n$ is negligible for large n; by classical estimates associated with the Law of Large Numbers, for all n large enough, the probability that $\hat{n} < n/2$ is no more than $\epsilon/3$. Now let Z be a Gaussian random variable with mean zero and variance $\sigma^2\hat{n}/n$ which is at least σ_0^2 when $\hat{n} \geqslant n/2$. Then by the multivariate version [R19] of the Berry-Esseen Theorem [B41, E42], a version of the Central Limit Theorem with rate information, there is a constant K_d depending only on d such that

$$\left|\mathbb{E}\left[1_{B_R+y}\left(\textstyle\sum_{j=1}^n n^{-1/2}\alpha_j X_j\right)\right] - \mathbb{P}\{Z \in B_R + y\}\right| \leqslant K_d \hat{n}\tfrac{\mathbb{E}|X_1|^3}{n^{3/2}} \leqslant K_d \tfrac{\mathbb{E}|X_1|^3}{n^{1/2}} \ .$$

Since A is bounded, $\mathbb{E}|X_1|^3 < \infty$, and hence for all sufficiently large n, when $\hat{n} \geqslant n/2$.

$$\mathbb{E}\left[1_{B_R+y}\left(\textstyle\sum_{j=1}^n n^{-1/2}\alpha_j X_j\right)\right] \leqslant \tfrac{2}{3}\epsilon \ .$$

Since this is uniform in y, we finally obtain $\mathbb{P}(W_n \in B_R) \leqslant \epsilon$ for all sufficiently large n. Since $\epsilon > 0$ is arbitrary, (12) is proved. □

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