The renormalization group, in the weak- and strong-coupling regimes

a PhD thesis by

Ian Jauslin

2015

Supervised by **Alessandro Giuliani**

Table of contents:

Acknowledgements

in order of appearance

I started my PhD in the fall of 2012, at the University of Rome "Sapienza". Over the past three years, I have had the good fortune to meet a bunch of wonderful people (throughout this document, the word "bunch" will be used to mean "more than one can count on one's fingers, provided one does not have too many fingers") who have helped me, each in their own way, take these first few steps in the world of academic research. I would like to take a moment to give thanks to all of these, in my own way.

Let me start with my parents, Hans-Rudolf Jauslin and Lucy Moser-Jauslin (in alphabetical order), who have consistently supported me through all of my endeavors. It may sound easy but it really isn't. They have provided advice at every fork in the road, all the while allowing me to make my own path. Always available for discussion, be it about my latest obsession with Joan of Arc, my repeated failures at flipping an omelette without getting any egg on the ceiling, algebraic field extensions of $\mathbb{Q}$, or the absence of joint probabilities in quantum mechanics, they have provided me with a sympathetic ear and an enlightened mouth.

My brother and sister, Andreas and Tania (in alphabetical order) have also been of great help. Both have a fantastically open, imaginative and bright mind, and have made every family reunion a wonderful experience (in the literal sense, as in "full of wonders").

I'm not going to go through the entire clique of cousins, uncles and grand-parents, because I have to stop at some point. I would however like to mention my cousin Madi, with whom I have had countless late night talks, about everything and anything. Her extremely inquisitive mind, producing a seemingly unending stream of usually unexpected and often wise ideas, has lead me to as much introspection, outrospection, and every other kind of spection I could possibly need.

I would also like to acknowledge the unwavering support I have received from my lifelong friend, Arthur, whom I met in kindergarten, and has been by my side ever since (if you're not impressed, note that that includes adolescence).

I've often been asked why I came to Rome for my PhD. I met Giovanni Gallavotti in Paris, at a series of lectures he was giving on applying renormalization group techniques to classical mechanics problems, based on a reinterpretation of KAM theory. His outlook on science immediately resonated with me: studying real, physical phenomena, while understanding the underlying mathematical structure precisely and thoroughly. It was exactly what I was looking for. I came to Rome for the first six months of 2011, and worked with Alessandro Giuliani on the model of bilayer graphene discussed below. I adored the city, with its many ruins, delicious food and great (though capricious at times) weather. But most of all, I reveled in the methodology of the Roman Renormalization Group Group (I don't think anyone has ever called it that, but I like it!) (at the time, composed of Giuseppe Benfatto, Giovanni Gallavotti, Guido Gentile, Alessandro Giuliani, Vieri Mastropietro). Physics with convergent expansions! Endless, but explicit computations that often bleed over from late afternoon into late night. Thorough discussions at the blackboard where the titles of "student" and "professor" dissolve into a flurry of ideas. And trees, so many trees!

When time came to decide on a PhD advisor, I asked Alessandro, and definitively moved to Rome. I have never looked back.

Alessandro has been an outstanding PhD advisor. He has always been extremely nice and patient, even when I would make the most stupid mistakes. He has introduced me to several very interesting problems, and, by his extensive knowledge of the subject and an explicit and personal understanding of the methods involved, has been a wonderful teacher.

When I first arrived in Rome in 2011, Alessandro had a PhD student, Serena Cenatiempo, and a postdoc, Rafael Greenblatt. Both were extremely helpful in my introduction to mathematical physics, and making me feel welcome in what was, after all, a foreign place. To this day, every one of our meetings turns into an interesting and lively discussion. Around the start of my PhD, Michele Correggi joined Alessandro's group, bringing his good humor and understanding of the inner workings of quantum mechanics to the lunch table. Later on, Niels Benedikter and Martin Lohmann took short term positions in the group. We had many conversations on the meaning of the renormalization group, mathematical physics, and research at large, which have helped me keep my ideas straight. Of particular mention, is Giovanni Antinucci, another PhD student of Alessandro's, who started a year after I did. Always eager to talk about mathematical physics, he often asks for advice on his work and offers advice on mine. Since I am the type of person that needs someone to bounce ideas off of in order to keep my error rate at acceptable levels, I have benefited a lot from having Giovanni around.

About half-way through my PhD, I started to branch out into other areas of mathematical physics. Margherita Disertori visited Alessandro and me in Rome several times, to discuss a model of hard rods in three dimensions. We kicked ideas around for a few months, and had a great time at it.

In June 2014, I started a collaboration with Giuseppe Benfatto and Giovanni Gallavotti, the results of which are discussed in this thesis. Since we started collaborating, I have had the opportunity to work rather intensely with Giovanni Gallavotti, which has been extremely enriching and enjoyable, and am very grateful for it.

In the fall of 2014, I was invited by Giovanni Gallavotti to spend six weeks at Rutgers University. There, I met Joel Lebowitz, and started discussing phase transitions in systems of hard objects with him. My time in Rutgers was idyllic, largely due to Joel's friendliness, as well as that of the wonderful mathematical physicists of Rutgers.

In June 2015, Elliott Lieb visited Alessandro in Rome, and we started to work on a problem involving expressing monomer-dimer partition functions as Pfaffians. In August, Elliott invited me to Princeton and I had a very productive, and enjoyable time. I am very thankful to both Elliott and Joel for being so welcoming, and the many scientific discussions we have had.

When I arrived in Rome, in 2011, I did not know anyone, and did not speak a word of Italian. Since then, I made some great friends, who have helped me keep my sanity for the past three years.

I would first like to mention the PhD students at the mathematics department of La Sapienza, Emanuela, Giulia, Lorenzo and Lucrezia. I often burst into the PhD office, spewing twelve of the worst puns that have ever been uttered in an academic institution, every minute, and they refrain from throwing things at me, which shows admirable restraint. Despite their lack of knowledge about French pastries, we have shared many pleasant moments.

When I first arrived in Rome, my mother told me of family friends that lived there. I wrote them, and was invited to dinner. I was greeted by Wendy and Bruno, and their children Lily and Leo. Throughout my stay in Rome, a deep friendship developed between us. Bruno, very unfortunately, died in February 2014. We spent a lot of quality time together, enjoying extraordinary home cooked meals, discussing art and life, traveling. But most of all, with another close friend, Benoît, we played a lot of music. Chamber music. Wendy would play the violin, Benoît the cello, and I the guitar (find the odd element in this trio...), Lily would occasionally join on the violin, playing Vivaldi solos faster than I could listen to them. We would then spend long evenings, Benoît, Wendy and I, talking about mostly anything. If my stay in Rome was pleasant, they made it wonderful.

The structure of this document is roughly the following. It starts out with an introductory part, which is a fairly informal discussion, meant to convey the basic concepts that the renormalization group technique is based on, and how the specific problems considered in this thesis fit into this big picture. The main body of work of this thesis is contained in the two following parts, in which two systems, specifically a weakly interacting electronic model of bilayer graphene, and a hierarchical s-d model, are studied in full detail, using renormalization group techniques. In the introductory part, the discourse has been kept light and accessible, at the occasional expense of precision, and the two main parts of the thesis are much more explicit and precise, although, somewhat technical.

Introduction

The renormalization group

In this introductory part, we will describe the renormalization group technique in general terms, focusing on the ideas it relies on. The range of applications of renormalization group methods is broad, covering quantum field theory [Wi65], equilibrium statistical mechanics, classical [DJ69] and quantum [BG90], classical mechanics [Ga94] and beyond... In order not to get entangled in an overly general discussion, we will mostly focus on applications to quantum statistical mechanics, and, when needed, specialize the discussion to many-Fermion lattice systems, even though most underlying ideas apply to all of the previously mentioned fields.

Before we begin our discussion, a comment on the vocabulary involved in renormalization group methods is in order. The maturation of the renormalization group technique was a long and involved process, and the vernacular that was developed does not always reflect the modern point of view.

The term "renormalization group" itself is a good example of this fact: in the following, there will be no talk of "normalizations" and little mention of "groups". According to K. Wilson - [Wi83], it was first introduced by E.C.G. Stueckelberg and A. Petermann [SP53] in the context of Quantum ElectroDynamics. While it had already been established [To48, Sc49, Fe49, Dy49] that by correcting the values of the charge e and mass m of the electron, the pesky divergences that appeared in the theory could be removed, E.C.G. Stueckelberg and A. Petermann noticed that there is actually a continuum of possible values for e and m that accomplish this regularization, related to each other by a group of rescaling transformations. They called these groups "groupes de normalisation". M. Gell-Mann and F.E. Low [GL54] independently understood that these different values are related to the behavior of the system at different energy scales. The point of view that will be adopted in this document is close to that developed by K. Wilson [Wi65], in which the "renormalization" is understood as a map from the *effective Hamiltonian* on a given energy scale to that on another.

A term that will be used extensively below is "non-perturbative", which can be found to mean two different (related) things in the literature. In renormalization group methods, one typically makes use of power series, which we will call "perturbative expansions". When these series are shown to be convergent, the discussion is said to be "non-perturbative", in the sense that it goes beyond a perturbative discussion. This is the sense in which the term is to be understood in this document. However, "non-perturbative" is also used to mean "beyond the radius of convergence" of these power series. In order to avoid confusion, we will call such a regime of parameters the *strong-coupling* regime.

The renormalization group has been extensively studied since the late 1940's, and has spawned several variants, which are not always aptly named. The point of view we have adopted in this document, is sometimes called the "Wilsonian" renormalization group. A central goal of the discussion below is to be *non-perturbative*, and rigorously prove the convergence of the power series we manipulate. This has led us to use the qualifiers "rigorous", "non-perturbative" and even "exact" to describe our take on the renormalization group. However, there is another renormalization group-based method, called the "functional renormalization group" [Ro12] that also uses the "non-perturbative" and "exact" adjectives. Their use of "non-perturbative" corresponds to our "strong-coupling".

We are now ready to delve into the discussion of the core concepts of the renormalization group technique.

The starting point of the discussion is a Hamiltonian, H, which can be written as $H = H_0 + V$, where H_0 is *integrable*. In the context we are focusing on, H describes the dynamics of a many-electron system, and is an operator on a Fermionic Fock space, H_0 is typically the kinetic term and V an interaction between electrons. The concept of *integrability*, in this case, stands for the fact that the moments of the *free Gibbs measure*

$$\langle\cdot\rangle_0 := \frac{\mathrm{Tr}(e^{-\beta H_0}\cdot)}{\mathrm{Tr}(e^{-\beta H_0})} \tag{0.0.1}$$

are known explicitly, in terms of integrals, for every $\beta > 0$.

For the sake of definiteness, let us focus on one physical observable, the partition function Z of the system:

$$Z := \mathrm{Tr}(e^{-\beta H}) \equiv \mathrm{Tr}(e^{-\beta(H_0+V)}). \tag{0.0.2}$$

Formally, one can express Z using (0.0.1): indeed, one readily checks that (at least formally)

$$Z = \mathrm{Tr}(e^{-\beta H_0}) \sum_{N=0}^{\infty} (-1)^N \int_0^\beta dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{N-1}} dt_N \ \langle V(t_1)\cdots V(t_N)\rangle_0 \tag{0.0.3}$$

where

$$V(t) := e^{tH_0} V e^{-tH_0}. \tag{0.0.4}$$

Using the integrability of H_0, one can express the right side of (0.0.3) explicitly, which, *provided the formal expansion in* (0.0.3) *converges*, yields an adequate expression for the partition function Z. This is where things get tricky: in a number of models of physical interest, not only is the formal expansion in (0.0.3) divergent, but every one of its terms is formally infinite.

The fundamental question, which the renormalization group technique was devised to answer, is whether this formal divergence is an artifact of the method, or whether it carries physical importance. This is, by no means, an innocent question. Infinities, or rather, singularities, *do* exist and are, in some cases, fundamental to understand the physical phenomena in which they show up. The vortices one observes in turbulent flows of fluids are an obvious example. Phase transitions in statistical mechanics are singular events, whose roughness is directly observable, perhaps most strikingly in the so-called critical opalescence one observes when liquid and gas lose their distinction. The long debate over the stability of the Solar System provides a more subtle example of the importance of taking divergences seriously in physics. By neglecting the divergence of the series they were manipulating, J.L. Lagrange and P.S. Laplace were led to believe that the Solar System is stable, overlooking the singular effects brought forth by near-resonances, which were later shown [La89] to destabilize the system.

How the importance of the divergence of these series was realized is an edifying story (see [Ba94] for details and references). In 1884, king Oscar II of Sweden appointed G. Mittag-Leffler to organize a prize that would reward an important mathematical discovery. After some debate, Mittag-Leffler invited C. Hermite and K. Weierstrass to propose four problems that would be worth the prize. The first of these was to find a convergent series expansion for the trajectories of the n-body problem (i.e. n point masses interacting via Newton's law of gravitation). A young H. Poincaré took home the prize in January 1889 for solving this problem in a simplified setting, namely the *restricted* 3-body problem. Shortly after the prize was announced, H. Gyldén, an established astronomer at the Royal Swedish Academy, claimed that he had proved Poincaré's result two years earlier, which set off a bitter dispute between the Academy, which sided with Gyldén, and Mittag-Leffler, who persisted in upholding the originality of Poincaré's work. In an ironic twist, in June 1889, while the debate over who had priority over this result raged on, L.E. Phragmén, the editor in charge of Poincaré's memoir, which was to be published in the Acta Mathematica, discovered an error in the proof. Upon re-inspection, Poincaré understood that this error was indeed deep, and that the series expansions he had computed for the restricted 3-body problem, for which Gyldén was fighting so hard to be credited, are divergent! In the subsequent memoir he published, Poincaré described the implications of this fact: of the trajectories he had constructed, some (the perturbations of *homoclinic* orbits), which would have been very regular had the series converged, were actually quite erratic. This was the first construction of a chaotic trajectory in a Hamiltonian system, and laid the first stone on top of which Chaos Theory was to be built.

A fundamental idea of the renormalization group is that (0.0.3) is not the right expansion to look at. Essentially, the terms in (0.0.3) diverge because of singularities of $\langle\cdot\rangle_0$ in the long- and short-distance regimes, and V describes the interaction between fields on *all* length scales at once. Instead, the idea of the renormalization group, at least in K. Wilson's formulation [Wi65], is to view the system as a *sequence* of *effective* Hamiltonians $H^{(h)} = H_0^{(h)} + V^{(h)}$ for $h \in \mathbb{Z}$, which *approximate* the behavior of the system on distances of the order of 2^{-h}. The effective Hamiltonians $H^{(h)}$ are computed by iterating a map called the *beta function*, which maps $H^{(h)}$ to $H^{(h-1)}$.

The beta function is *expressed as a power series* in $V^{(h)}$. Since $V^{(h)}$ only describes the interaction on scale 2^{-h}, the resulting formal expansion is well defined, and, provided $V^{(h)}$ is small enough in an appropriate norm, is either convergent (as in the purely Fermionic cases studied in this thesis), or an asymptotic series. The main issue to address is to ensure that $V^{(h)}$ is indeed *small enough*. This task may seem daunting a priori: after all, in most cases, the effective potentials $V^{(h)}$ are parametrized by an infinite number of parameters (called *running coupling constants*), so that controlling the iterates of the beta function involves controlling the flow of an infinite-dimensional dynamical system, which, to top it all off, is defined by series. However, by computing the effect of a *rescaling* transformation $x \mapsto \lambda x$ on the free Gibbs measure $\langle\cdot\rangle_0$, it is possible to bound some of the terms of $V^{(h)}$ by simple dimensional arguments. Such terms have been given the name *irrelevant* (which is a rather unfortunate denomination: the fact that these terms are well-behaved a priori by no means affects their physical relevance, and their effects *must* be taken into account to get good quantitative predictions). If the number of terms in $V^{(h)}$ that are *not* irrelevant is *finite*, then the system is said to be *renormalizable*. In this case, the size of the effective potential $V^{(h)}$ can be controlled by studying the flow of a *finite-dimensional* dynamical system, which can be constructed from the beta function by neglecting the irrelevant terms. It may be worth insisting that this does *not* mean that the irrelevant terms are negligible, but merely that their contributions do not need to be checked explicitly in order to control the flow of the running coupling constants.

The point of this technique is that, for systems that are amenable to it, it provides a robust method to compute physical observables as convergent or asymptotic series. The method is *robust* in the sense that if small irrelevant terms are added to V, the analysis can essentially be carried out in the same way, and only the numerical values one can compute for the physical observables will change.

It, however, has some important limitations, which have, so far, prevented it from being used successfully to study some interesting phenomena, such as high-T_c superconductivity, or quantum gravity. First, for systems that are not renormalizable, renormalization group techniques are extremely hard to make sense of. Second, the approach described above is based on perturbative expansions, which, if $V^{(h)}$ is not small enough, breaks down completely. There are commonplace models in which $V^{(h)}$ is not small, even if the interaction V is. For example, in the case of a 2- or 3-dimensional electron gas, it is expected that the Coulomb interaction plays a substantial role in the large-distance behavior of the system, so much so that the very concept of an individual electron loses its meaning, and should be replaced by *Cooper pairs*. Such considerations (which, it may be worth noting, have so far not been proved rigorously) are at the basis of the celebrated BCS theory of superconductivity. In such cases, the system is said to be in the *strong-coupling* regime, and it is not yet clear how to replace the perturbative expansions that usually appear in renormalization group techniques.

The system that is studied in detail in part I is in the *weak-coupling* regime, and provides an explicit example in which the ideas sketched above are worked out, and yield rigorous results. The discussion in part II is a first step towards a possible approach to study systems in the *strong-coupling* regime using renormalization group techniques.

In part I, we study a model for the electrons of bilayer graphene at half-filling, which is a 2-dimensional crystal of carbon atoms, in which we consider a short-range (exponentially decaying) interaction between electrons, which is thought to correspond to a screened Coulomb interaction. A more thorough introduction to this system will be given below. Here, we will merely discuss some key properties of the model, and the difficulties and simplicities that result from them.

First, the particles that are modeled are electrons, which satisfy Fermionic statistics. There is a well-established formalism to study interacting Fermionic field theories by renormalization group techniques, which was developed by G. Benfatto and G. Gallavotti [BG90] (and is built on a solid base of seminal works in constructive quantum field theory). The Fermionic nature of the model will turn out to be a great advantage, in that the power series that we will introduce below as part of the renormalization group technique, are *convergent*. In interacting Bosonic field theories, the terms of the power series produced by renormalization group techniques typically grow as the factorial of the power, and may only, in the best of cases, be shown to be asymptotic series (as in [EMS75, MS77]). Instead, in interacting Fermionic theories, due to the signs that appear when Fermionic fields are exchanged, the terms of the power series can be bounded more efficiently. Technically, this comes from the fact that while multi-dimensional Gaussian integrals yield permanents, Gaussian Grassmann integrals yield determinants.

The model which we will study is a lattice model. The physical interpretation underlying this assumption is that the electrons are bound to an atom of the crystal, and occasionally tunnel to a neighboring atom. This fact, which is not an artifact of the model, but, rather, is rooted in the physics of the system, substantially simplifies the renormalization group treatment. Indeed, the lattice structure regularizes the singularities due to short-distance effects, since there effectively is *no* distance shorter than the inter-atom spacing in the model, so we only need to worry about the long-distance behavior. In the renormalization group vernacular, this means that there is *no ultraviolet problem*. This is a common property of problems in condensed matter physics, which have an underlying physical lattice regularization.

After all is said and done, the model is *superrenormalizable*. This statement requires some clarification. While there is no ultraviolet problem, there is an *infrared problem*: the non-interacting Hamiltonian H_0 can be diagonalized in Fourier space, and it turns out that there are 8 momenta at which an eigenvalue vanishes. These momenta, called *Fermi points*, generate singularities in the free Gibbs measure $\langle\cdot\rangle_0$, which are called *infrared* singularities. The scaling properties of H_0, which, as was mentioned above, are crucial to the renormalization group analysis, depend on the way that these eigenvalues approach 0 as the momentum is varied. As it turns out, in bilayer graphene, this behavior changes as the momentum gets closer to the Fermi points: there are 3 regimes (actually, 4, but this will not be discussed here) (see figure I.1.4 for a graphical representation): a first one, far from the Fermi points, in which the eigenvalues decrease linearly, followed by another, closer to the Fermi points, in which they behave quadratically, and, finally, a third regime, that extends all the way down to the Fermi points, and in which, as in the first, they behave linearly. In the first and third regimes, every term of the effective potentials $V^{(h)}$ is *irrelevant*. This implies that the size of the running coupling constants can be controlled by purely dimensional arguments. In this case, the system is called *superrenormalizable*. The real challenge in this model is to control the running coupling constants in the second regime, in which the scaling behavior of H_0 is such that the quartic terms in $V^{(h)}$ are *marginal* (not irrelevant). A priori, there could be a risk that the system slip into the strong-coupling regime because of the marginal couplings, which is a concern that can even be backed up by a heuristic computation [Va10]. However, and this is our main result, one can show that by optimizing the treatment of the first superrenormalizable regime, the running couplings can be controlled in the marginal regime as well, under a specific set of hypotheses, see theorem I.1.1. In order to arrive at this conclusion, one needs to maintain control over the crossovers between these different regimes, for which our approach to the renormalization group has proved to be particularly well suited.

The final ingredient of the model is that the Fermi points mentioned above are indeed *points*. Generically, in 2-dimensional systems, the infrared singularities would form a Fermi *curve*, which adds difficulties to the renormalization group analysis (see e.g. [BGM06] or [FKT04]). In the case of bilayer graphene (at half-filling, which is the regime which we are studying here), this degeneracy of the Fermi curve is enforced by symmetries. As a consequence, since the symmetries are preserved by the beta function, the infrared singularities of $H_0^{(h)}$ remain point-like for all h. However, this is only true because we are neglecting some extra complexity in our model: as was mentioned earlier the electrons are assumed to tunnel from one atom to its neighbors, but if one considers tunnelings to non-nearest neighbor atoms, then one of the symmetries of our model would be violated, and the Fermi curve would no longer be degenerate. If one were to include these extra tunnelings, one would then have a fourth regime, beyond the third, in which the 1-dimensional nature of 6 of the 8 Fermi points would become apparent. However, this effect is thought to only manifest itself at very low temperatures, so our model should adequately represent the system down to these low temperatures, and in particular, our result on the control of the marginal regime is in no way diminished by this fact.

To sum up, the ingredients of our bilayer graphene model that simplify its renormalization group treatment are the Fermionic statistics, the lattice structure, the superrenormalizability of the first and third regimes, and the degeneracy of the Fermi curve. The main difficulty of the analysis lies in the control of the marginal running coupling constants in the second regime, which we are able to control because of optimal control over the superrenormalizable regimes.

In part II, we lay the groundwork for a technique that might be of some use to study strongly-coupled systems using rigorous renormalization group techniques.

In the 1990's, a variant of the renormalization group technique , called the *functional renormalization group*, and based on a formulation by J. Polchinski [Po84] and, independently, G. - Gallavotti [Ga85], was developed to study systems in the strong-coupling regime, see [Ro12] for a review. Essentially, the functional renormalization group technique relies on the same perturbative expansions as in standard implementations of the renormalization group, but these series are truncated in a particular way. It was found that this method yields predictions that are in good agreement with experiments, even when applied to strongly-coupled systems, most notably quantum chromodynamics [Br09]. However, as of now, it is not understood why this truncation yields good results: in the strong-coupling regime, the parameters appearing in these series are far beyond their radius of convergence, which makes it difficult to control the remainder of the truncation.

In part II, we investigate an alternative approach, based on studying *hierarchical* approximations of strongly-coupled Fermionic models. Hierarchical models have been used to prove results on their non-hierarchical counterparts in many other settings, for instance in the Ising model with long-range interactions [Dy69], the φ^4 model in 2 and 3 dimensions [BCe78, BCe80], and the $\nabla\varphi^4$ model in $\geqslant 2$ dimensions [GK81, GK82]. Their use in Fermionic models is limited to the Gross-Neveu model [Do91] in the weak-coupling regime, which had previously been understood using non-perturbative renormalization group methods [GK85]. So far, to our knowledge, an investigation of strongly-coupled Fermionic hierarchical models has not been attempted.

Without going into details (for which we refer the reader to part II), the *hierarchical approximation* of a model is an alternative model that shares the same *scaling* properties as the original model. As was already stated above, the scaling properties are essential to renormalization group treatments, and the rough idea of the hierarchical approximation is that since the hierarchical model has the same scaling behavior as its non-hierarchical counterpart, its renormalization group analysis will be closely related to that of the non-hierarchical model. In addition, the ingredients that are neglected in the hierarchical approximation are, at every length scale 2^h, the correlations over larger distances than 2^h and fluctuations over shorter lengths than 2^h. These are, in some sense, expected to be small.

One of the motivations for considering Fermionic hierarchical models to study strong-coupling effects, is that they are *exactly solvable* from a renormalization group point of view: the beta function can be computed *exactly* and *explicitly*, without any series expansions. This means that Fermionic hierarchical models can be studied rigorously using renormalization group techniques, whether they are in the strong- or weak-coupling regime.

In part II, we study a hierarchical approximation of the $s-d$ model (also called the Kondo model). This model was introduced to study the effect of magnetic impurities in conductors, namely to understand why, at low temperatures, the resistivity of magnetically polluted metals increases as the temperature decreases. The $s-d$ model describes a collection of non-interacting electrons, living on a 1-dimensional discrete chain of atoms, interacting with a single magnetic impurity localized at one of the sites of the chain. By carrying out computations using a Born approximation (which is not controlled), J. Kondo showed [Ko64] that the interaction between the impurity and the electrons could qualitatively change the conducting behavior of the electrons. Our aim is to study this effect using renormalization group methods. The range of temperatures we are interested in is that in which the interaction between the impurity and the electrons qualitatively changes the behavior of the electrons, in other words, we are interested in the strong-coupling regime.

The $s-d$ model was studied using renormalization group methods by P. Anderson [An70] and K. Wilson [Wi75], though the methods used in these references are perturbative or numerical. The endgame of our analysis is to study the $s-d$ model analytically and non-perturbatively, although, so far, we have only studied a hierarchical version of the model. We have investigated the strong-coupling regime of the hierarchical $s-d$ model by iterating an explicit beta function, expressed in terms of rational functions, and found that the qualitative behavior of the hierarchical model matches up with that of the non-hierarchical $s-d$ model.

The methods we used to perform this analysis are the following. The *exact* computation of the beta function was carried out with the help of a computer program, *meankondo* [Ja15], written for the purpose of this computation. *Meankondo* takes the expression of the *propagator* of the hierarchical $s-d$ model as input, and outputs the beta function, computed symbolically. It is a rather flexible tool, and can be used to compute the beta function for a wide variety of Fermionic hierarchical model. The beta function was then iterated numerically. Note that there is a fundamental difference between the numerical treatment in [Wi75] and the one carried out here: in [Wi75], the beta function itself is computed numerically, whereas here, it is merely the iteration of the beta function that is carried out numerically. Studying this iteration means studying a finite-dimensional dynamical system, involving only non-singular rational functions. The results presented in part II could probably be proved analytically, or, as an alternative, a computer assisted proof could easily be worked out. By "computer assisted proof", we mean that the truncation errors made by the numerical iteration could easily be controlled. We have not carried this out, because it seemed more cumbersome than necessary, given the clarity of the numerical results.

This result on the hierarchical $s-d$ model is quite intriguing: this model can be studied exactly by a renormalization group method in the strong-coupling regime and seems to reproduce the same qualitative behavior of the non-hierarchical $s-d$ model. An enticing question is whether the $s-d$ model can be seen as a perturbation of this, or another, hierarchical approximation. We are currently investigating this direction, as well as considering other strongly-coupled Fermionic models which could be studied via hierarchical approximations.

Before moving on to the main body of this thesis, let me make one last remark on the motivations for turning heuristic arguments into mathematically precise statements.

The argument has already been made that important physical phenomena can be overlooked by neglecting convergence issues. This is a pragmatic argument, that new and interesting physics can come from rigorous mathematical analysis. There also is a deeper, more fundamental motivation: a rigorous mathematical description of natural phenomena is an integral part of the scientific method. The reasoning is simple: if one accepts that the goal of physics is to model natural phenomena as mathematical objects that can be manipulated in order to produce measurable predictions, then mathematical rigor is clearly part of the process. Of course, foregoing rigor can be very useful to save time and to keep focus on the truly physical aspects of a problem, but making results mathematically precise remains an important step that cannot be cast aside.

References

[An70] P.W. Anderson - *A poor man's derivation of scaling laws for the Kondo problem*, Journal of Physics C: Solid State Physics, Vol. 3, p. 2436, 1970, doi:10.1088/0022-3719/3/12/008.

[Ba94] J. Barrow-Green - *Oscar II's prize competition and the error in Poincaré's memoir on the three body problem*, Archive for History of Exact Sciences, Vol. 48, n. 2, p. 107-131, 1994, doi:10.1007/BF00374436.

[BCe78] G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicolò, E. Olivieri, E. Presutti, E. Scacciatelli - *Some probabilistic techniques in field theory*, Communications in Mathematical Physics, Vol. 59, n. 2, p. 143-166, 1978, doi:10.1007/BF01614247.

[BCe80] G. Benfatto, M. Cassandro, G. Gallavotti, F. Nicolò, E. Olivieri, E. Presutti, E. Scacciatelli - *Ultraviolet stability in Euclidean scalar field theories*, Communications in Mathematical Physics, Vol. 71, n. 2, p. 95-130, 1980, doi:10.1007/BF01197916.

[BG90] G. Benfatto, G. Gallavotti - *Perturbation theory of the Fermi surface in a quantum liquid - a general quasiparticle formalism and one-dimensional systems*, Journal of Statistical Physics, Vol. 59, n. 3-4, p. 541-664, 1990, doi:10.1007/BF01025844.

[BGM06] G. Benfatto, A. Giuliani, V. Mastropietro - *Fermi liquid behavior in the 2D Hubbard model*, Annales Henri Poincaré, Vol. 7, p. 809-898, 2006, doi:10.1007/s00023-006-0270-z.

[Br09] J. Braun - *The QCD phase boundary from quark-gluon dynamics*, European Physics Journal C, Vol. 64, n. 3, p. 459-482, 2009, doi:10.1140/epjc/s10052-009-1136-6.

[DJ69] C. Di Castro, G. Jona-Lasinio - *On the microscopic foundation of scaling laws*, Physics Letters A, Vol. 29, n. 6, p. 322-323, 1969, doi:10.1016/0375-9601(69)90148-0.

[Do91] T.C. Dorlas - *Renormalization group analysis of a simple hierarchical fermion model*, Communications in Mathematical Physics, Vol. 136, p. 169-194, 1991, doi:10.1007/BF02096796.

[Dy49] F.J. Dyson - *The radiation theories of Tomonaga, Schwinger and Feynman*, Physical Review, Vol. 75, n. 3, p. 486-502, 1949, doi:10.1103/PhysRev.75.486.

[Dy69] F.J. Dyson - *Existence of a phase-transition in a one-dimensional Ising ferromagnet*, Communications in Mathematical Physics, Vol. 12, p. 91-107, 1969, doi:10.1007/BF01645907.

[EMS75] J.P. Eckmann, J. Magnen, R. Sénéor - *Decay properties and borel summability for the Schwinger functions in $P(\phi)_2$ theories*, Communications in Mathematical Physics, Vol. 39, n. 4, p. 251-271, 1975, doi:10.1007/BF01705374.

[FKT04] J. Feldman, H. Knörrer, E. Trubowitz - *A two dimensional Fermi liquid. Part 1: Overview*, Communications in Mathematical Physics, Vol. 247, n. 1, p. 1-47, 2004, doi:10.1007/s00220-003-0996-0.

[Fe49] R.P. Feynman - *Space-time approach to Quantum Electrodynamics*, Physical Review, Vol. 76, n. 6, p. 769-789, 1949, doi:10.1103/PhysRev.76.769.

[Ga85] G. Gallavotti - *Renormalization theory and ultraviolet stability for scalar fields via renormalization group methods*, Reviews of Modern Physics, Vol. 57, p. 471-562, 1985, doi:10.1103/RevModPhys.57.471.

[Ga94] G. Gallavotti - *Twistless KAM tori, quasi flat homoclinic intersections, and other cancellations in the perturbation series of certain completely integrable hamiltonian systems: A review*, Reviews in Mathematical Physics, Vol. 6, n. 3, p. 343, 1994, doi:10.1142/S0129055X9400016X.

[GK81] K. Gawedzki, A. Kupiainen - *Renormalization group study of a critical lattice model I: Convergence to the line of fixed points*, Communications in Mathematical Physics, Vol. 82, n. 3, p. 407-433, 1981, doi:10.1007/BF01237048.

[GK82] K. Gawedzki, A. Kupiainen - *Renormalization group study of a critical lattice model II: The correlation functions*, Communications in Mathematical Physics, Vol. 83, n. 4, p. 469-492, 1982, doi:10.1007/BF01208712.

[GK85] K. Gawędzki, A. Kupiainen - *Gross-Neveu model through convergent perturbation expansions*, Communications in Mathematical Physics, Vol. 102, p. 1-30, 1985, doi:10.1007/BF01208817.

[GL54] M. Gell-Mann, F.E. Low - *Quantum electrodynamics at small distances*, Physical Review, Vol. 95, n. 5, p. 1300-1312, 1954, doi:10.1103/PhysRev.95.1300.

[Ja15] meankondo, developped by I. Jauslin, http://ian.jauslin.org/software/meankondo

[Ko64] J. Kondo - *Resistance minimum in dilute magnetic alloys*, Progress of Theoretical Physics, Vol. 32, n. 1, 1964, doi:10.1143/PTP.32.37.

[La89] J. Laskar - *A numerical experiment on the chaotic behavior of the Solar System*, Nature, Vol. 338, p. 237-238, 1989, doi:10.1038/338237a0.

[MS77] J. Magnen, R. Sénéor - *Phase space cell expansion and Borel summability for the Euclidean φ^4_3 theory*, Communications in Mathematical Physics, Vol. 56, n. 3, p. 237-276, 1977, doi:10.1007/BF01614211.

[Po84] J. Polchinski - *Renormalization and effective lagrangians*, Nuclear Physics B, Vol. 231, n. 2, p. 269-295, 1984, doi:10.1016/0550-3213(84)90287-6.

[Ro12] O.J. Rosten - *Fundamentals of the exact renormalization group*, Physics Reports, Vol. 511, n. 4, p. 177-272, 2012, doi:10.1016/j.physrep.2011.12.003.

[Sc49] J. Schwinger - *Quantum Electrodynamics III. The electromagnetic properties of the electron-radiative corrections to scattering*, Physical Review, Vol. 76, n. 6, p. 790-817, 1949, doi:10.1103/PhysRev.75.651.

[SP53] E.C.G. Stueckelberg, A. Petermann - *La normalisation des constantes dans la theorie des quanta*, Helvetica Physica Acta, Vol. 26, n. 5, p. 499-520, 1953.

[To48] S. Tomonaga - *On infinite field reactions in quantum field theory*, Physical Review, Vol. 74, p. 224-225, 1948, doi:10.1103/PhysRev.74.224.

[Va10] O. Vafek - *Interacting Fermions on the honeycomb bilayer: from weak to strong coupling*, Physical Review B, Vol. 82, 205106, 2010, doi:10.1103/PhysRevB.82.205106.

[Wi65] K.G. Wilson - *Model Hamiltonians for Local Quantum Field Theory*, Physical Review, Vol. 140, n. 2B, p. B445-B457, 1965, doi:10.1103/PhysRev.140.B445.

[Wi75] K.G. Wilson - *The renormalization group: Critical phenomena and the Kondo problem*, Reviews of Modern Physics, Vol. 47, n. 4, 1975, doi:10.1103/RevModPhys.47.773.

[Wi83] K.G. Wilson - *The renormalization group and critical phenomena*, Reviews of Modern Physics, Vol. 55, n. 3, p. 583-600, 1983, doi:10.1103/RevModPhys.55.583.

Part I

The ground state construction of bilayer graphene

We will now turn to the first model studied in this thesis, namely a model for the electrons of bilayer-graphene at half filling, with a weak short-range interaction. The discussion in this part is taken from [GJ15], and is the fruit of a collaboration with Alessandro Giuliani.

I.1. Introduction

Graphene, a one-atom thick layer of graphite, has captivated a large part of the scientific community for the past decade. With good reason: as was shown by A. Geim's team, graphene is a stable two-dimensional crystal with very peculiar electronic properties [NGe04]. The mere fact that a two-dimensional crystal can be synthesized, and manipulated, at room temperature without working inside a vacuum [Ge11] is, in and of itself, quite surprising. But the most interesting features of graphene lay within its electronic properties. Indeed, electrons in graphene were found to have an extremely high mobility [NGe04], which could make it a good candidate to replace silicon in microelectronics; and they were later found to behave like massless Dirac Fermions [NGe05, ZTe05], which is of great interest for the study of fundamental Quantum Electro-Dynamics. These are but a few of the intriguing features [GN07] that have prompted a lively response from the scientific community.

These peculiar electronic properties stem from the particular energy structure of graphene. It consists of two energy bands, that meet at exactly two points, called the *Fermi points* [Wa47]. Graphene is thus classified as a *semi-metal*: it is not a *semi-conductor* because there is no gap between its energy bands, nor is it a *metal* either since the bands do not overlap, so that the density of charge carriers vanishes at the Fermi points. Furthermore, the bands around the Fermi points are approximately conical [Wa47], which explains the masslessness of the electrons in graphene, and in turn their high mobility.

Graphene is also interesting for the mathematical physics community: its free energy and correlation functions, in particular its conductivity, can be computed non-perturbatively using constructive Renormalization Group (RG) techniques [GM10, GMP11, GMP12], at least if it is at *half-filling*, the interaction is *short-range* and its strength is *small enough*. This is made possible, again, by the special energy structure of graphene. Indeed, since the *propagator* (in the quantum field theory formalism) diverges at the Fermi points, the fact that there are only two such singularities in graphene instead of a whole line of them (which is what one usually finds in two-dimensional theories), greatly simplifies the RG analysis. Furthermore, the fact that the bands are approximately conical around the Fermi points, implies that a short-range interaction between electrons is *irrelevant* in the RG sense, which means that one need only worry about the renormalization of the propagator, which can be controlled.

Using these facts, the formalism developed in [BG90] has been applied in [GM10, GMP12] to express the free energy and correlation functions as convergent series.

Let us mention that the case of Coulomb interactions is more difficult, in that the effective interaction is marginal in an RG sense. In this case, the theory has been constructed at all orders in renormalized perturbation theory [GMP10, GMP11b], but a non-perturbative construction is still lacking.

In the present work, we shall extend the results of [GM10] by performing an RG analysis of half-filled *bilayer* graphene with short-range interactions. Bilayer graphene consists of two layers of graphene in so-called *Bernal* or *AB* stacking (see below). Before the works of A. Geim et al. [NGe04], graphene was mostly studied in order to understand the properties of graphite, so it was natural to investigate the properties of multiple layers of graphene, starting with the bilayer [Wa47, SW58, Mc57]. A common model for hopping electrons on graphene bilayers is the so-called *Slonczewski-Weiss-McClure* model, which is usually studied by retaining only certain hopping terms, depending on the energy regime one is interested in: including more hopping terms corresponds to probing the system at lower energies. The fine structure of the Fermi surface and the behavior of the dispersion relation around it depends on which hoppings are considered or, equivalently, on the range of energies under inspection.

In a first approximation, the energy structure of bilayer graphene is similar to that of the monolayer: there are only two Fermi points, and the dispersion relation is approximately conical around them. This picture is valid for energy scales larger than the transverse hopping between the two layers, referred to in the following as the *first regime*. At lower energies, the effective dispersion relation around the two Fermi points appears to be approximately *parabolic*, instead of conical. This implies that the effective mass of the electrons in bilayer graphene does not vanish, unlike those in the monolayer, which has been observed experimentally [NMe06].

From an RG point of view, the parabolicity implies that the electron interactions are *marginal* in bilayer graphene, thus making the RG analysis non-trivial. The flow of the effective couplings has been studied by O. Vafek [Va10, VY10], who has found that it diverges logarithmically, and has identified the most divergent channels, thus singling out which of the possible quantum instabilities are dominant (see also [TV12]). However, as was mentioned earlier, the assumption of parabolic dispersion relation is only an approximation, valid in a range of energies between the scale of the transverse hopping and a second threshold, proportional to the cube of the transverse hopping (asymptotically, as this hopping goes to zero). This range will be called the *second regime*.

By studying the smaller energies in more detail, one finds [MF06] that around each of the Fermi points, there are three extra Fermi points, forming a tiny equilateral triangle around the original ones. This is referred to in the literature as *trigonal warping*. Furthermore, around each of the now eight Fermi points, the energy bands are approximately conical. This means that, from an RG perspective, the logarithmic divergence studied in [Va10] is cut off at the energy scale where the conical nature of the eight Fermi points becomes observable (i.e. at the end of the second regime). At lower energies the electron interaction is irrelevant in the RG sense, which implies that the flow of the effective interactions remains bounded at low energies. Therefore, the analysis of [Va10] is meaningful only if the flow of the effective constants has grown significantly in the second regime.

However, our analysis shows that the flow of the effective couplings in this regime does not grow at all, due to their smallness after integration over the first regime, which we quantify in terms both of the bare couplings and of the transverse hopping. This puts into question the physical relevance of the "instabilities" coming from the logarithmic divergence in the second regime, at least in the case we are treating, namely small interaction strength and small interlayer hopping.

The transition from a normal phase to one with broken symmetry as the interaction strength is increased from small to intermediate values was studied in [CTV12] at second order in perturbation theory. Therein, it was found that while at small bare couplings the infrared flow is convergent, at larger couplings it tends to increase, indicating a transition towards an *electronic nematic state*.

Let us also mention that the third regime is not believed to give an adequate description of the system at arbitrarily small energies: at energies smaller than a third threshold (proportional to the fourth power of the transverse hopping) one finds [PP06] that the six extra Fermi points around the two original ones, are actually microscopic ellipses. The analysis of the thermodynamic properties of the system in this regime (to be called the fourth regime) requires new ideas and techniques, due to the extended nature of the singularity, and goes beyond the scope of this paper. It may be possible to adapt the ideas of [BGM06] to this regime, and we hope to come back to this issue in a future publication.

To summarize, at weak coupling and small transverse hopping, we can distinguish four energy regimes: in the first, the system behaves like two uncoupled monolayers, in the second, the energy bands are approximately parabolic, in the third, the trigonal warping is taken into account and the bands are approximately conical, and in the fourth, six of the Fermi points become small curves. We shall treat the first, second and third regimes, which corresponds to retaining only the three dominant Slonczewski-Weiss-McClure hopping parameters. Informally, we will prove that *the interacting half-filled system is analytically close to the non-interacting one* in these regimes, and that the effect of the interaction is merely to renormalize the hopping parameters. The proof depends on a sharp multiscale control of the crossover regions separating one regime from the next.

We will now give a quick description of the model, and a precise statement of the main result of the present work, followed by a brief outline of its proof.

I.1.1. Definition of the model

We shall consider a crystal of bilayer graphene, which is made of two honeycomb lattices in *Bernal* or AB stacking, as shown in figure I.1.1. We can identify four inequivalent types of sites in the lattice, which we denote by a, $\tilde{b}$, $\tilde{a}$ and b (we choose this peculiar order for practical reasons which will become apparent in the following).

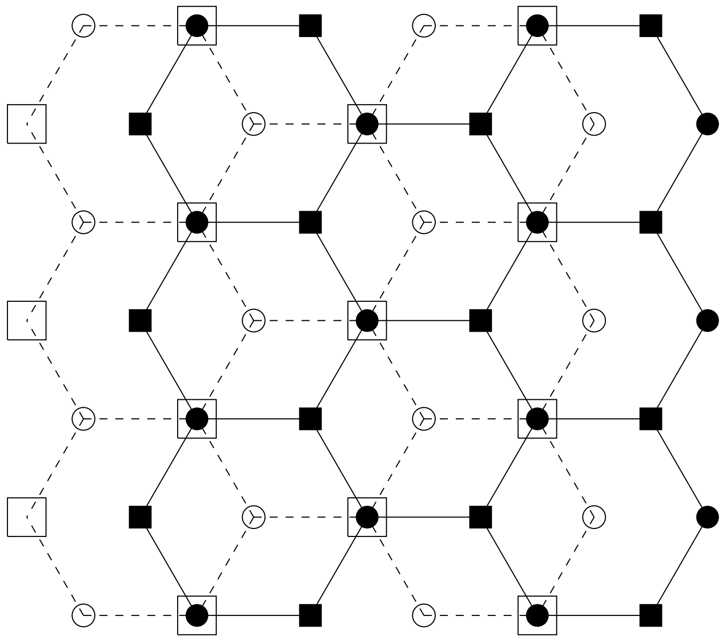

fig I.1.1: ● and ■ represent atoms of type a and b on the lower layer and ○ and □ represent atoms of type $\tilde a$ and $\tilde b$ on the upper layer. Full lines join nearest neighbors within the lower layer and dashed lines join nearest neighbors within the upper layer.

We consider a Hamiltonian of the form

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \tag{I.1.1}$$

where the *free Hamiltonian* $\mathcal{H}_0$ plays the role of a kinetic energy for the electrons, and the *interaction Hamiltonian* $\mathcal{H}_I$ describes the interaction between electrons.

$\mathcal{H}_0$ is given by a *tight-binding* approximation, which models the movement of electrons in terms of *hoppings* from one atom to the next. There are four inequivalent types of hoppings which we shall consider here, each of which will be associated a different *hopping strength* γ_i. Namely, the hoppings between neighbors of type a and b, as well as $\tilde a$ and $\tilde b$ will be associated a hopping strength γ_0; a and $\tilde b$ a strength γ_1; $\tilde a$ and b a strength γ_3; $\tilde a$ and a, and $\tilde b$ and b a strength γ_4 (see figure I.1.2). We can thus express H_0 in *second quantized* form in *momentum space* at *zero chemical potential* as [Wa47, SW58, Mc57]

$$\mathcal{H}_0 = \frac{1}{|\hat\Lambda|} \sum_{k\in\hat\Lambda} \hat A_k^\dagger H_0(k) \hat A_k \tag{I.1.2}$$

$$\hat A_k := \begin{pmatrix} \hat a_k \\ \hat{\tilde b}_k \\ \hat{\tilde a}_k \\ \hat b_k \end{pmatrix} \text{ and } H_0(k) := -\begin{pmatrix} \Delta & \gamma_1 & \gamma_4\Omega(k) & \gamma_0\Omega^*(k) \\ \gamma_1 & \Delta & \gamma_0\Omega(k) & \gamma_4\Omega^*(k) \\ \gamma_4\Omega^*(k) & \gamma_0\Omega^*(k) & 0 & \gamma_3\Omega(k)e^{3ik_x} \\ \gamma_0\Omega(k) & \gamma_4\Omega(k) & \gamma_3\Omega^*(k)e^{-3ik_x} & 0 \end{pmatrix} \tag{I.1.3}$$

in which $\hat a_k$, $\hat{\tilde b}_k$, $\hat{\tilde a}_k$ and $\hat b_k$ are *annihilation operators* associated to atoms of type a, $\tilde b$, $\tilde a$ and b, $k \equiv (k_x, k_y)$, $\hat\Lambda$ is the *first Brillouin zone*, and $\Omega(k) := 1 + 2e^{-i\frac{3}{2}k_x}\cos\left(\frac{\sqrt3}{2}k_y\right)$. These objects will be properly defined in section I.2.1. The Δ parameter in H_0 models a shift in the chemical potential around atoms of type a and $\tilde b$ [SW58, Mc57]. We choose the energy unit in such a way that $\gamma_0 = 1$. The hopping strengths have been measured experimentally in graphite [DD02, TDD77, MMD79, DDe79] and in bilayer graphene samples [ZLe08, MNe07]; their values are given in the following table:

	bilayer graphene [MNe07]	graphite [DD02]
γ_1	0.10	0.12
γ_3	0.034	0.10
γ_4	0.041	0.014
Δ	0.006 [ZLe08]	-0.003

(I.1.4)

We notice that the relative order of magnitude of γ_3 and γ_4 is quite different in graphite and in bilayer graphene. In the latter, γ_1 is somewhat small, and γ_3 and γ_4 are of the same order, whereas Δ is of the order of γ_1^2. We will take advantage of the smallness of the hopping strengths and treat $\gamma_1 =: \epsilon$ as a small parameter: we fix

$$\frac{\gamma_1}{\epsilon} = 1, \quad \frac{\gamma_3}{\epsilon} = 0.33, \quad \frac{\gamma_4}{\epsilon} = 0.40, \quad \frac{\Delta}{\epsilon^2} = 0.58 \tag{I.1.5}$$

and assume that ϵ is as small as needed.

Remark: The symbols used for the hopping parameters are standard. The reason why γ_2 was omitted is that it refers to next-to-nearest layer hopping in graphite. In addition, for simplicity, we have neglected the intra-layer next-to-nearest neighbor hopping $\gamma_0' \approx 0.1\gamma_1$, which is known to play an analogous role to γ_4 and Δ [ZLe08].

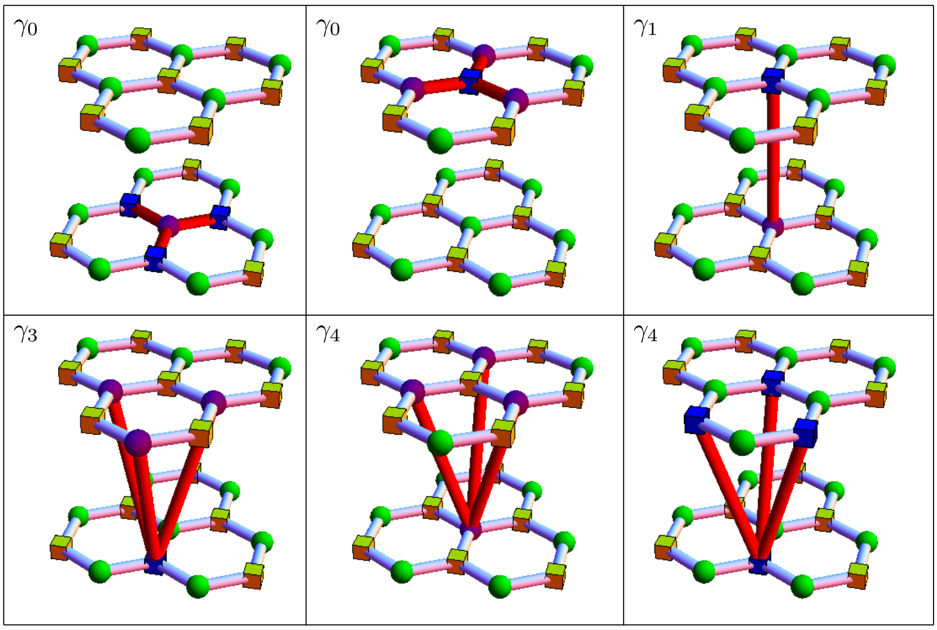

fig I.1.2: The different types of hopping. From top-left to bottom-right: $a \leftrightarrow b$, $\tilde{a} \leftrightarrow \tilde{b}$, $a \leftrightarrow \tilde{b}$, $b \leftrightarrow \tilde{a}$, $a \leftrightarrow \tilde{a}$, $b \leftrightarrow \tilde{b}$. Atoms of type a and $\tilde{a}$ are represented by spheres and those of type b and $\tilde{b}$ by cubes; the interaction is represented by solid red (color online) cylinders; the interacting atoms are displayed either in purple or in blue.

The interactions between electrons will be taken to be of *extended Hubbard* form, i.e.

$$\mathcal{H}_I := U \sum_{(x,y)} v(x-y) \left(n_x - \frac{1}{2}\right)\left(n_y - \frac{1}{2}\right) \tag{I.1.6}$$

where $n_x := \alpha_x^\dagger \alpha_x$ in which α_x is one of the annihilation operators a_x, $\tilde{b}_x$, $\tilde{a}_x$ or b_x; the sum over (x,y) runs over all pairs of atoms in the lattice; v is a short range interaction potential (exponentially decaying); U is the *interaction strength* which we will assume to be small.

We then define the *Gibbs average* as

$$\langle \cdot \rangle := \frac{1}{Z} \mathrm{Tr}\left(e^{-\beta\mathcal{H}} \cdot\right)$$

where

$$Z := \mathrm{Tr}\left(e^{-\beta\mathcal{H}}\right) =: e^{-\beta|\Lambda|f}.$$

The physical quantities we will study here are the *free energy* f, and the *two-point Schwinger function* defined as the 4×4 matrix

$$\check{s}_2(\mathbf{x}_1 - \mathbf{x}_2) := \left(\left\langle \mathbf{T}(\alpha'_{\mathbf{x}_1} \alpha^\dagger_{\mathbf{x}_2})\right\rangle\right)_{(\alpha',\alpha)\in\{a,\tilde{b},\tilde{a},b\}^2}, \tag{I.1.7}$$

where $\mathbf{x}_1 := (t_1, x_1)$ and $\mathbf{x}_2 := (t_2, x_2)$ includes an extra *imaginary time* component, $t_{1,2} \in [0, \beta)$, which is introduced in order to compute Z and Gibbs averages,

$$\alpha_{t,x} := e^{\mathcal{H}_0 t} \alpha_x e^{-\mathcal{H}_0 t} \quad \text{for } \alpha \in \{a, \tilde{b}, \tilde{a}, b\}$$

and $\mathbf{T}$ is the *Fermionic time ordering operator*:

$$\mathbf{T}(\alpha'_{t_1,x_1} \alpha^\dagger_{t_2,x_2}) = \begin{cases} \alpha'_{t_1,x_1} \alpha^\dagger_{t_2,x_2} & \text{if } t_1 > t_2 \\ -\alpha^\dagger_{t_2,x_2} \alpha'_{t_1,x_1} & \text{if } t_1 \leq t_2 \end{cases}. \tag{I.1.8}$$

We denote the Fourier transform of $\check{s}_2(\mathbf{x})$ (or rather of its anti-periodic extension in imaginary time for t's beyond $[0, \beta)$) by $s_2(\mathbf{k})$ where $\mathbf{k} := (k_0, k)$, and $k_0 \in \frac{2\pi}{\beta}(\mathbb{Z} + \frac{1}{2})$.

I.1.2. Non-interacting system

In order to state our main results on the interacting two-point Schwinger function, it is useful to first review the scaling properties of the non-interacting one,

$$s_2^{(0)}(\mathbf{k}) = -(ik_0 \mathbb{1} + H_0(k))^{-1},$$

including a discussion of the structure of its singularities in momentum space.

1 - Non-interacting Fermi surface. If $H_0(k)$ is not invertible, then $s_2^{(0)}(0, k)$ is divergent. The set of quasi-momenta $\mathcal{F}_0 := \{k,\ \det H_0(k) = 0\}$ is called the non-interacting *Fermi surface* at zero chemical potential, which has the following structure: it contains two isolated points located at

$$p^\omega_{F,0} := \left(\frac{2\pi}{3}, \omega\frac{2\pi}{3\sqrt{3}}\right),\ \omega \in \{-1, +1\} \tag{I.1.9}$$

around each of which there are three very small curves that are approximately elliptic (see figure I.1.3). The whole singular set $\mathcal{F}_0$ is contained within two small circles (of radius $O(\epsilon^2)$), so that on scales larger than ϵ^2, $\mathcal{F}_0$ can be approximated by just two points, $\{p^\pm_{F,0}\}$, see figure I.1.3. As we zoom in, looking at smaller scales, we realize that each small circle contains four Fermi points: the central one, and three secondary points around it, called $\{p^\pm_{F,j},\ j \in \{1, 2, 3\}\}$. A finer zoom around the secondary points reveals that they are actually curves of size ϵ^3.

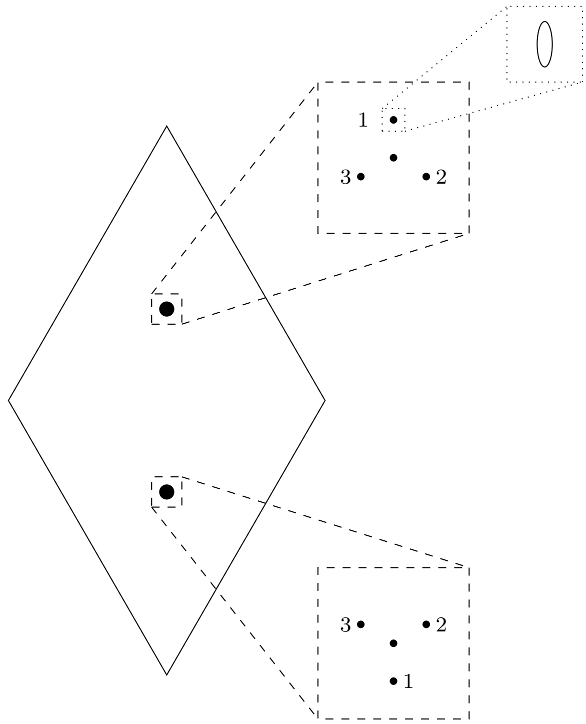

fig I.1.3: Schematic representation of the Fermi points. Each dotted square represents a zoom into the finer structure of the Fermi points. The secondary Fermi points are labeled as indicated in the figure. In order not to clutter the drawing, only one of the zooms around the secondary Fermi point was represented.

2 - Non-interacting Schwinger function. We will now make the statements about approximating the Fermi surface more precise, and discuss the behavior of $s_2^{(0)}$ around its singularities. We will identify four regimes in which the Schwinger function behaves differently.

2-1 - First regime. One can show that, if $\mathbf{p}_{F,0}^{\pm} := (0, p_{F,0}^{\pm})$, and

$$\|(k_0, k'_x, k'_y)\|_{\mathrm{I}} := \sqrt{k_0^2 + (k'_x)^2 + (k'_y)^2}$$

then

$$s_2^{(0)}(\mathbf{p}_{F,0}^{\pm} + \mathbf{k}') = \left(\mathfrak{L}_{\mathrm{I}}\hat{A}(\mathbf{p}_{F,0}^{\pm} + \mathbf{k}')\right)^{-1} \left(\mathbb{1} + O(\|\mathbf{k}'\|_{\mathrm{I}}, \epsilon\|\mathbf{k}'\|_{\mathrm{I}}^{-1})\right) \tag{I.1.10}$$

in which $\mathfrak{L}_{\mathrm{I}}\hat{A}$ is a matrix, independent of γ_1, γ_3, γ_4 and Δ, whose eigenvalues vanish *linearly* around $\mathbf{p}_{F,0}^{\pm}$ (see figure I.1.4b). We thus identify a *first regime*:

$$\epsilon \ll \|\mathbf{k}'\|_{\mathrm{I}} \ll 1$$

in which the error term in (I.1.10) is *small*. In this first regime, γ_1, γ_3, γ_4 and Δ are negligible, and the Fermi surface is approximated by $\{p_{F,0}^{\pm}\}$, around which the Schwinger function diverges *linearly*.

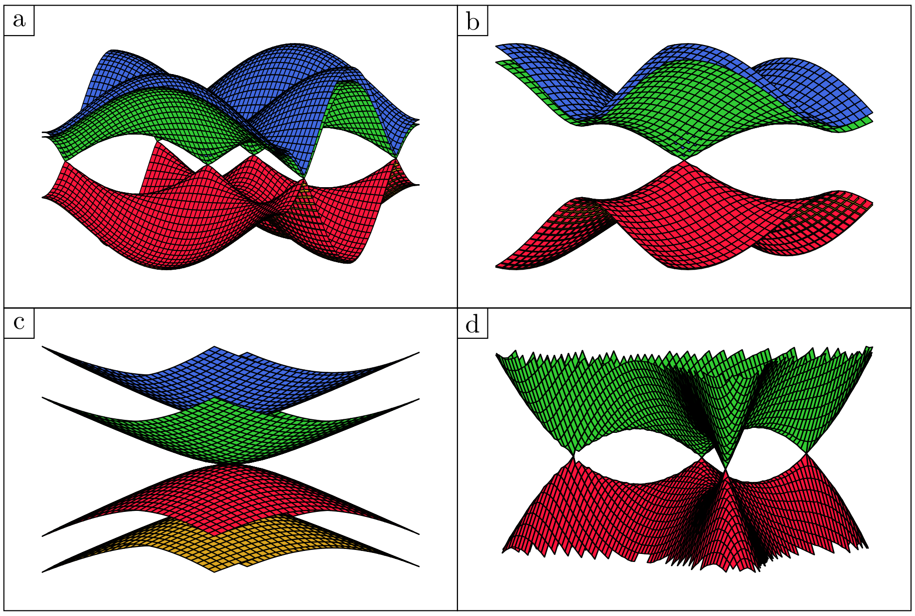

fig I.1.4: Eigenvalues of $H_0(k)$. The sub-figures b,c,d are finer and finer zooms around one of the Fermi points.

2-2 - Second regime. Now, if

$$\|(k_0,k'_x,k'_y)\|_{\mathrm{II}}:=\sqrt{k_0^2+\frac{(k'_x)^4}{\gamma_1^2}+\frac{(k'_y)^4}{\gamma_1^2}}$$

then

$$s_2^{(0)}(\mathbf p_{F,0}^\pm+\mathbf k')=\left(\mathfrak L_{\mathrm{II}}\hat A(\mathbf p_{F,0}^\pm+\mathbf k')\right)^{-1}\left(\mathbb 1+O(\epsilon^{-1}\|\mathbf k'\|_{\mathrm{II}},\epsilon^{3/2}\|\mathbf k'\|_{\mathrm{II}}^{-1/2})\right)\qquad\text{(I.1.11)}$$

in which $\mathfrak L_{\mathrm{II}}\hat A$ is a matrix, independent of γ_3, γ_4 and Δ. Two of its eigenvalues vanish *quadratically* around $\mathbf p_{F,0}^\pm$ (see figure I.1.4c) and two are bounded away from 0. The latter correspond to *massive* modes, whereas the former to *massless* modes. We thus identify a *second regime*:

$$\epsilon^3\ll\|\mathbf k'\|_{\mathrm{II}}\ll\epsilon$$

in which γ_3, γ_4 and Δ are negligible, and the Fermi surface is approximated by $\{p_{F,0}^\pm\}$, around which the Schwinger function diverges *quadratically*.

2-3 - Third regime. If $\mathbf p_{F,j}^\pm:=(0,p_{F,j}^\pm)$, $j=0,1,2,3$, and

$$\|(k_0,k'_{j,x},k'_{j,y})\|_{\mathrm{III}}:=\sqrt{k_0^2+\gamma_3^2(k'_{j,x})^2+\gamma_3^2(k'_{j,y})^2}$$

then

$$s_2^{(0)}(\mathbf p_{F,j}^\pm+\mathbf k'_j)=\left(\mathfrak L_{\mathrm{III},j}\hat A(\mathbf p_{F,j}^\pm+\mathbf k'_j)\right)^{-1}\left(\mathbb 1+O(\epsilon^{-3}\|\mathbf k'_j\|_{\mathrm{III}},\epsilon^{4}\|\mathbf k'_j\|_{\mathrm{III}}^{-1})\right)\qquad\text{(I.1.12)}$$

in which $\mathfrak L_{\mathrm{III},j}\hat A$ is a matrix, independent of γ_4 and Δ, two of whose eigenvalues vanish *linearly* around $\mathbf p_{F,j}^\pm:=(0,p_{F,j}^\pm)$ (see figure I.1.4d) and two are bounded away from 0. We thus identify a *third regime*:

$$\epsilon^4\ll\|\mathbf k'_j\|_{\mathrm{III}}\ll\epsilon^3$$

in which γ_4 and Δ are negligible, and the Fermi surface is approximated by $\{p_{F,j}^\pm\}_{j\in\{0,1,2,3\}}$, around which the Schwinger function diverges *linearly*.

Remark: If $\gamma_4=\Delta=0$, then the error term $O(\epsilon^4\|\mathbf k'_j\|_{\mathrm{III}}^{-1})$ in (I.1.12) vanishes identically, which allows us to extend the third regime to all momenta satisfying

$$\|\mathbf k'_j\|_{\mathrm{III}}\ll\epsilon^3.$$

I.1.3. Main theorem

We now state the main theorem, whose proof will occupy the rest of the paper. Roughly, our result is that as long as $|U|$ and ϵ are small enough and $\gamma_4 = \Delta = 0$ (see the remarks following the statement for an explanation of why this is assumed), the free energy and the two-point Schwinger function are well defined in the thermodynamic and zero-temperature limit $|\Lambda|, \beta \to \infty$, and that the two-point Schwinger function is analytically close to that with $U = 0$. The effect of the interaction is shown to merely *renormalize* the constants of the non-interacting Schwinger function.

We define

$$\mathcal{B}_\infty := \mathbb{R} \times \left(\mathbb{R}^2/(\mathbb{Z}G_1 + \mathbb{Z}G_2)\right), \quad G_1 := \left(\frac{2\pi}{3}, \frac{2\pi}{\sqrt{3}}\right), \ G_2 := \left(\frac{2\pi}{3}, -\frac{2\pi}{\sqrt{3}}\right),$$

where the physical meaning of $\mathbb{R}^2/(\mathbb{Z}G_1 + \mathbb{Z}G_2)$ is that of the *first Brillouin zone*, and $G_{1,2}$ are the generators of the dual lattice.

Theorem I.1.1
(*Main theorem*)

If $\gamma_4 = \Delta = 0$, then there exists $U_0 > 0$ and $\epsilon_0 > 0$ such that for all $|U| < U_0$ and $\epsilon < \epsilon_0$, the specific ground state energy

$$e_0 := -\lim_{\beta\to\infty} \lim_{|\Lambda|\to\infty} \frac{1}{\beta|\Lambda|} \log(\mathrm{Tr}(e^{-\beta\mathcal{H}}))$$

exists and is analytic in U. In addition, there exist eight Fermi points $\{\tilde{\mathbf{p}}_{F,j}^\omega\}_{\omega=\pm,j=0,1,2,3}$ such that:

$$\tilde{\mathbf{p}}_{F,0}^\omega = \mathbf{p}_{F,0}^\omega, \qquad |\tilde{\mathbf{p}}_{F,j}^\omega - \mathbf{p}_{F,j}^\omega| \leqslant (\mathrm{const.})\ |U|\epsilon^2,\ j = 1, 2, 3, \tag{I.1.13}$$

and, $\forall \mathbf{k} \in \mathcal{B}_\infty \setminus \{\tilde{\mathbf{p}}_{F,j}^\omega\}_{\omega=\pm,j=0,1,2,3}$, the thermodynamic and zero-temperature limit of the two-point Schwinger function, $\lim_{\beta\to\infty} \lim_{|\Lambda|\to\infty} s_2(\mathbf{k})$, exists and is analytic in U.

Remarks:

- The theorem requires $\gamma_4 = \Delta = 0$. As we saw above, those quantities play a negligible role in the non-interacting theory as long as we do not move beyond the third regime. This suggests that the theorem should hold with $\gamma_4, \Delta \neq 0$ under the condition that β is not too large, i.e., smaller than (const.) ϵ^{-4}. However, that case presents a number of extra technical complications, which we will spare the reader.
- The conditions that $|U| < U_0$ and $\epsilon < \epsilon_0$ are independent, in that we do not require any condition on the relative values of $|U|$ and ϵ. Such a result calls for tight bounds on the integration over the first regime. If we were to assume that $|U| \ll \epsilon$, then the discussion would be greatly simplified, but such a condition would be artificial, and we will not require it be satisfied. L. Lu [Lu13] sketched the proof of a result similar to theorem I.1.1, without discussing the first two regimes, which requires such an artificial condition on U/ϵ. The renormalization of the secondary Fermi points is also ignored in that reference.

In addition to theorem I.1.1, we will prove that the dominating part of the two point Schwinger function is qualitatively the same as the non-interacting one, with renormalized constants. This result is detailed in theorems I.1.2, I.1.3 and I.1.4 below, each of which refers to one of the three regimes.

1 - First regime. Theorem I.1.2 states that in the first regime, the two-point Schwinger function behaves at dominant order like the non-interacting one with renormalized factors.

Theorem I.1.2

Under the assumptions of theorem I.1.1, if $C\epsilon \leqslant \|\mathbf{k}-\mathbf{p}_{F,0}^{\omega}\|_{\mathrm{I}} \leqslant C^{-1}$ for a suitable $C>0$, then, in the thermodynamic and zero-temperature limit,

$$s_2(\mathbf{k}) = -\frac{1}{\bar{k}_0\tilde{k}_0+|\bar{\xi}|^2}\begin{pmatrix} -i\bar{k}_0 & 0 & 0 & \bar{\xi}^* \\ 0 & -i\bar{k}_0 & \bar{\xi} & 0 \\ 0 & \bar{\xi}^* & -i\tilde{k}_0 & 0 \\ \bar{\xi} & 0 & 0 & -i\tilde{k}_0 \end{pmatrix}(\mathbb{1}+r(\mathbf{k})) \tag{I.1.14}$$

where

$$r(\mathbf{p}_{F,0}^{\omega}+\mathbf{k}') = O((1+|U||\log\|\mathbf{k}'\|_{\mathrm{I}}|)\|\mathbf{k}'\|_{\mathrm{I}}, \epsilon\|\mathbf{k}'\|_{\mathrm{I}}), \tag{I.1.15}$$

and, for $(k_0,k'_x,k'_y) := \mathbf{k}-\mathbf{p}_{F,0}^{\omega}$,

$$\bar{k}_0 := z_1 k_0, \quad \tilde{k}_0 := \tilde{z}_1 k_0, \quad \bar{\xi} := \frac{3}{2}v_1(ik'_x+\omega k'_y) \tag{I.1.16}$$

in which $(\tilde{z}_1, z_1, v_1) \in \mathbb{R}^3$ satisfy

$$|1-\tilde{z}_1| \leqslant C_1|U|, \quad |1-z_1| \leqslant C_1|U|, \quad |1-v_1| \leqslant C_1|U| \tag{I.1.17}$$

for some constant $C_1>0$ (independent of U and ϵ).

Remarks:

- The singularities of s_2 are approached linearly in this regime.
- By comparing (I.1.14) with its non-interacting counterpart (I.3.8), we see that the effect of the interaction is to *renormalize* the constants in front of k_0 and ξ in (I.3.8).
- The *inter-layer correlations*, that is the $\{a,b\}\times\{\tilde{a},\tilde{b}\}$ components of the dominating part of $s_2(\mathbf{k})$ vanish. In this regime, the Schwinger function of bilayer graphene behave like that of two independent graphene layers.

2 - Second regime. Theorem I.1.3 states a similar result for the second regime. As was mentioned earlier, two of the components are *massive* in the second (and third) regime, and we first perform a change of variables to isolate them, and state the result on the massive and massless components, which are denoted below by $\bar{s}_2^{(M)}$ and $\bar{s}_2^{(m)}$ respectively.

Theorem I.1.3

Under the assumptions of the theorem I.1.1, if $C\epsilon^3 \leqslant \|\mathbf{k}-\mathbf{p}_{F,0}^{\omega}\|_{\mathrm{II}} \leqslant C^{-1}\epsilon$ for a suitable $C>0$, then, in the thermodynamic and zero-temperature limit,

$$s_2(\mathbf{k}) = \begin{pmatrix} \mathbb{1} & M(\mathbf{k})^\dagger \\ 0 & \mathbb{1} \end{pmatrix}\begin{pmatrix} \bar{s}_2^{(M)} & 0 \\ 0 & \bar{s}_2^{(m)}(\mathbf{k}) \end{pmatrix}\begin{pmatrix} \mathbb{1} & 0 \\ M(\mathbf{k}) & \mathbb{1} \end{pmatrix}(\mathbb{1}+r(\mathbf{k})) \tag{I.1.18}$$

where:

$$r(\mathbf{p}_{F,0}^{\omega}+\mathbf{k}') = O(\epsilon^{-1/2}\|\mathbf{k}'\|_{\mathrm{II}}^{1/2}, \epsilon^{3/2}\|\mathbf{k}'\|_{\mathrm{II}}^{-1/2}, |U|\epsilon\,|\log\epsilon|), \tag{I.1.19}$$

$$\bar{s}_2^{(m)}(\mathbf{k}) = \frac{1}{\bar{\gamma}_1^2\bar{k}_0^2+|\bar{\xi}|^4}\begin{pmatrix} i\bar{\gamma}_1^2\bar{k}_0 & \bar{\gamma}_1(\bar{\xi}^*)^2 \\ \bar{\gamma}_1\bar{\xi}^2 & i\bar{\gamma}_1^2\bar{k}_0 \end{pmatrix}, \quad \bar{s}_2^{(M)} = -\begin{pmatrix} 0 & \bar{\gamma}_1^{-1} \\ \bar{\gamma}_1^{-1} & 0 \end{pmatrix}, \tag{I.1.20}$$

$$M(\mathbf{k}) := -\frac{1}{\bar\gamma_1}\left(\begin{array}{cc}\bar\xi^* & 0\\ 0 & \bar\xi\end{array}\right) \tag{I.1.21}$$

and, for $(k_0, k'_x, k'_y) := \mathbf{k} - \mathbf{p}^\omega_{F,0}$,

$$\bar\gamma_1 := \tilde m_2\gamma_1, \quad \bar k_0 := z_2 k_0, \quad \bar\xi := \frac{3}{2}v_2(ik'_x + \omega k'_y) \tag{I.1.22}$$

in which $(\tilde m_2, z_2, v_2) \in \mathbb{R}^3$ satisfy

$$|1-\tilde m_2| \leqslant C_2|U|, \quad |1-z_2| \leqslant C_2|U|, \quad |1-v_2| \leqslant C_2|U| \tag{I.1.23}$$

for some constant $C_2 > 0$ (independent of U and ϵ).

Remarks:

- The *massless* components $\{\tilde a, b\}$ are left invariant under the change of basis that block-diagonalizes s_2. Furthermore, M is *small* in the second regime, which implies that the *massive* components are *approximately* $\{a, \tilde b\}$.
- As can be seen from (I.1.20), the *massive* part $\bar s_2^{(M)}$ of s_2 is not singular in the neighborhood of the Fermi points, whereas the *massless* one, i.e. $\bar s_2^{(m)}$, is.
- The massless components of s_2 approach the singularity *quadratically* in the spatial components of $\mathbf{k}$.
- Similarly to the first regime, by comparing (I.1.20) with (I.3.18), we find that the effect of the interaction is to *renormalize* constant factors.

3 - Third regime. Theorem I.1.4 states a similar result as theorem I.1.3 for the third regime, though the discussion is made more involved by the presence of the extra Fermi points.

Theorem I.1.4

For $j = 0, 1$, under the assumptions of theorem I.1.1, if $\|\mathbf{k} - \tilde{\mathbf{p}}^\omega_{F,j}\|_{\mathrm{III}} \leqslant C^{-1}\epsilon^3$ for a suitable $C > 0$, then

$$s_2(\mathbf{k}) = \left(\begin{array}{cc}\mathbb{1} & M(\mathbf{k})^\dagger\\ 0 & \mathbb{1}\end{array}\right)\left(\begin{array}{cc}\bar s_2^{(M)} & 0\\ 0 & \bar s_2^{(m)}(\mathbf{k})\end{array}\right)\left(\begin{array}{cc}\mathbb{1} & 0\\ M(\mathbf{k}) & \mathbb{1}\end{array}\right)(\mathbb{1} + r(\mathbf{k})) \tag{I.1.24}$$

where

$$r(\tilde{\mathbf{p}}^\omega_{F,j} + \mathbf{k}'_j) = O(\epsilon^{-3}\|\mathbf{k}'_j\|_{\mathrm{III}}(1+\epsilon|\log\|\mathbf{k}'_j\|_{\mathrm{III}}||U|), \epsilon(1+|\log\epsilon||U|)) \tag{I.1.25}$$

$$\bar s_2^{(m)}(\mathbf{k}) = \frac{1}{\bar k_{0,j}^2 + \gamma_3^2|\bar x_j|^2}\left(\begin{array}{cc}i\bar k_{0,j} & \gamma_3\bar x_j^*\\ \gamma_3\bar x_j & i\bar k_{0,j}\end{array}\right), \quad \bar s_2^{(M)} = -\left(\begin{array}{cc}0 & \bar\gamma_{1,j}^{-1}\\ \bar\gamma_{1,j}^{-1} & 0\end{array}\right), \tag{I.1.26}$$

$$M(\mathbf{k}) := -\frac{1}{\bar\gamma_{1,j}}\left(\begin{array}{cc}\bar\Xi_j^* & 0\\ 0 & \bar\Xi_j\end{array}\right) \tag{I.1.27}$$

and, for $(k_0, k'_x, k'_y) := \mathbf{k} - \mathbf{p}^\omega_{F,j}$,

$$\begin{array}{c}\bar k_{0,j} := z_{3,j}k_0, \quad \bar\gamma_{1,j} := \tilde m_{3,j}\gamma_1, \quad \bar x_0 := \tilde v_{3,0}\tfrac{3}{2}(ik'_x - \omega k'_y) =: -\bar\Xi_0^*\\ \bar x_1 := \tfrac{3}{2}\left(3\tilde v_{3,1}ik'_x + \tilde w_{3,1}\omega k'_x\right), \quad \bar\Xi_1 := m_{3,1}\gamma_1\gamma_3 + \bar v_{3,1}ik'_x + \bar w_{3,1}k'_y\end{array} \tag{I.1.28}$$

in which $(\tilde{m}_{3,j}, m_{3,j}, z_{3,j}, \bar{v}_{3,j}, \tilde{v}_{3,j}, \bar{w}_{3,j}, \tilde{w}_{3,j}) \in \mathbb{R}^7$ satisfy

$$|m_{3,j} - 1| + |\tilde{m}_{3,j} - 1| \leqslant C_3|U|, \quad |z_{3,j} - 1| \leqslant C_3|U|,$$
$$|\bar{v}_{3,j} - 1| + |\tilde{v}_{3,j} - 1| \leqslant C_3|U|, \quad |\bar{w}_{3,j} - 1| + |\tilde{w}_{3,j} - 1| \leqslant C_3|U| \tag{I.1.29}$$

for some constant $C_3 > 0$ (independent of U and ϵ).

Theorem I.1.4 can be extended to the neighborhoods of $\tilde{\mathbf{p}}^{\omega}_{F,j}$ with $j = 2, 3$, by taking advantage of the symmetry of the system under rotations of angle $2\pi/3$:

Extension to $j = 2, 3$

For $j = 2, 3$, under the assumptions of theorem I.1.1, if $\|\mathbf{k} - \tilde{\mathbf{p}}^{\omega}_{F,j}\|_{\mathrm{III}} \leqslant C^{-1}\epsilon^3$ for a suitable $C > 0$, then

$$s_2(\mathbf{k}'_j + \tilde{\mathbf{p}}^{\omega}_{F,j}) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathcal{T}_{T\mathbf{k}'_j + \tilde{\mathbf{p}}^{\omega}_{F,j-\omega}} \end{pmatrix} s_2(T\mathbf{k}'_j + \tilde{\mathbf{p}}^{\omega}_{F,j-\omega}) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathcal{T}^{\dagger}_{T\mathbf{k}'_j + \tilde{\mathbf{p}}^{\omega}_{F,j-\omega}} \end{pmatrix} \tag{I.1.30}$$

where $T(k_0, k_x, k_y)$ denotes the rotation of the k_x and k_y components by an angle $2\pi/3$, $\mathcal{T}_{(k_0,k_x,k_y)} := e^{-i(\frac{3}{2}k_x - \frac{\sqrt{3}}{2}k_y)\sigma_3}$, and $\tilde{\mathbf{p}}^{-}_{F,4} \equiv \tilde{\mathbf{p}}^{-}_{F,1}$.

Remarks:

- The remarks below theorem I.1.3 regarding the massive and massless fields hold here as well.
- The massless components of s_2 approach the singularities *linearly*.
- By comparing (I.1.24) with (I.3.25) and (I.3.32), we find that the effect of the interaction is to *renormalize* the constant factors.

I.1.4. Sketch of the proof

In this section, we give a short account of the main ideas behind the proof of theorem I.1.1.

1 - Multiscale decomposition. The proof relies on a *multiscale* analysis of the model, in which the free energy and Schwinger function are expressed as successive integrations over individual scales. Each scale is defined as a set of $\mathbf{k}$'s contained inside an annulus at a distance of 2^h for $h \in \mathbb{Z}$ around the singularities located at $\mathbf{p}^{\omega}_{F,j}$. The positive scales correspond to the ultraviolet regime, which we analyze in a multiscale fashion because of the (very mild) singularity of the free propagator at equal imaginary times. It may be possible to avoid the decomposition by employing ideas in the spirit of [SS08]. The negative scales are treated differently, depending on the regimes they belong to (see below), and they contain the essential difficulties of the problem, whose nature is intrinsically infrared.

2 - First regime. In the first regime, i.e. for $-1 \gg h \gg h_\epsilon := \log_2 \epsilon$, the system behaves like two uncoupled graphene layers, so the analysis carried out in [GM10] holds. From a renormalization group perspective, this regime is *super-renormalizable*: the scaling dimension of diagrams with $2l$ external legs is $3-2l$, so that only the two-legged diagrams are relevant whereas all of the others are irrelevant (see section I.5.2 for precise definitions of scaling dimensions, relevance and irrelevance). This allows us to compute a strong bound on four-legged contributions:

$$|\hat{W}^{(h)}_4(\mathbf{k})| \leqslant (\text{const.})\ |U|2^{2h}$$

whereas a naive power counting argument would give $|U|2^h$ (recall that with our conventions h is negative).

The super-renormalizability in the first regime stems from the fact that the Fermi surface is 0-dimensional and that H_0 is linear around the Fermi points. While performing the multiscale integration, we deal with the two-legged terms by incorporating them into H_0, and one must therefore prove that by doing so, the Fermi surface remains 0-dimensional and that the singularity remains linear. This is guaranteed by a symmetry argument, which in particular shows the invariance of the Fermi surface.

3 - Second regime. In the second regime, i.e. for $3h_\epsilon \ll h \ll h_\epsilon$, the singularities of H_0 are quadratic around the Fermi points, which changes the *power counting* of the renormalization group analysis: the scaling dimension of $2l$-legged diagrams becomes $2-l$ so that the two-legged diagrams are still relevant, but the four-legged ones become marginal. One can then check [Va10] that they are actually marginally relevant, which means that their contribution increases proportionally to $|h|$. This turns out not to matter: since the second regime is only valid for $h \gg 3h_\epsilon$, $|\hat{W}_4^{(h)}|$ may only increase by $3|h_\epsilon|$, and since the theory is super-renormalizable in the first regime, there is an extra factor 2^{h_ϵ} in $\hat{W}_4^{(h_\epsilon)}$, so that $\hat{W}_4^{(h)}$ actually increases from 2^{h_ϵ} to $3|h_\epsilon|2^{h_\epsilon}$, that is to say it barely increases at all if ϵ is small enough.

Once this essential fact has been taken into account, the renormalization group analysis can be carried out without major difficulties. As in the first regime, the invariance of the Fermi surface is guaranteed by a symmetry argument.

4 - Third regime. In the third regime, i.e. for $h \ll 3h_\epsilon$, the theory is again super-renormalizable (the scaling dimension is $3-2l$). There is however an extra difficulty with respect to the first regime, in that the Fermi surface is no longer invariant under the renormalization group flow, but one can show that it does remain 0-dimensional, and that the only effect of the multiscale integration is to move $p_{F,j}^{\omega}$ along the line between itself and $p_{F,0}^{\omega}$.

I.1.5. Outline

The rest of this paper is devoted to the proof of theorem I.1.1 and of theorems I.1.2, I.1.3 and I.1.4. The sections are organized as follows.

- In section I.2, we define the model in a more explicit way than what has been done so far; then we show how to compute the free energy and Schwinger function using a Fermionic path integral formulation and a *determinant expansion*, due to Battle, Brydges and Federbush [BF78, BF84], see also [BK87, AR98]; and finally we discuss the symmetries of the system.
- In section I.3, we discuss the non-interacting system. In particular, we derive detailed formulae for the Fermi points and for the asymptotic behavior of the propagator around its singularities.
- In section I.4, we describe the multiscale decomposition used to compute the free energy and Schwinger function.
- In section I.5, we state and prove a *power counting* lemma, which will allow us to compute bounds for the effective potential in each regime. The lemma is based on the Gallavotti-Nicolò tree expansion [GN85, GN85b], and follows [BG90, GM01, Gi10]. We conclude this section by showing how to compute the two-point Schwinger function from the effective potentials.
- In section I.6, we discuss the integration over the *ultraviolet regime*, i.e. scales $h > 0$.

- In sections I.7, I.8 and I.9, we discuss the multiscale integration in the first, second and third regimes, and complete the proofs of theorem I.1.1, as well as of theorems I.1.2, I.1.3, I.1.4.

I.2. The model

From this point on, we set $\gamma_4 = \Delta = 0$.

In this section, we define the model in precise terms, re-express the free energy and two-point Schwinger function in terms of Grassmann integrals and truncated expectations, which we will subsequently explain how to compute, and discuss the symmetries of the model and their representation in this formalism.

I.2.1. Precise definition of the model

In the following, some of the formulae are repetitions of earlier ones, which are recalled for ease of reference. This section complements section I.1.1, where the same definitions were anticipated in a less verbose form. The main novelty lies in the momentum-real space correspondence, which is made explicit.

1 - Lattice. As mentioned in section I.1, the atomic structure of bilayer graphene consists in two honeycomb lattices in so-called *Bernal* or *AB* stacking, as was shown in figure I.1.1. It can be constructed by copying an elementary cell at every integer combination of

$$l_1 := \left(\frac{3}{2}, \frac{\sqrt{3}}{2}, 0\right), \ l_2 := \left(\frac{3}{2}, -\frac{\sqrt{3}}{2}, 0\right) \tag{I.2.1}$$

where we have chosen the unit length to be equal to the distance between two nearest neighbors in a layer (see figure I.2.1). The elementary cell consists of four atoms at the following coordinates

$$(0,0,0);\ (0,0,c);\ (-1,0,c);\ (1,0,0)$$

given relatively to the center of the cell. c is the spacing between layers; it can be measured experimentally, and has a value of approximately 2.4 [TMe92].

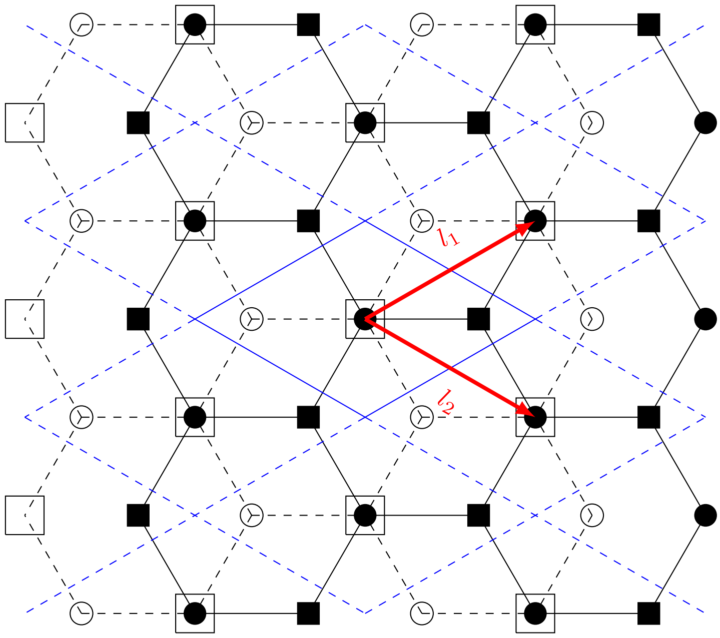

fig I.2.1: decomposition of the crystal into elementary cells, represented by the blue (color online) rhombi. There are four atoms in each elementary cell: ● of type a at $(0,0,0)$, □ of type $\tilde b$ at $(0,0,c)$, ○ of type $\tilde a$ at $(-1,0,c)$ and ■ of type b at $(0,0,c)$.

We define the lattice

$$\Lambda := \left\{n_1 l_1 + n_2 l_2,\ (n_1,n_2)\in\{0,\cdots,L-1\}^2\right\} \tag{I.2.2}$$

where L is a positive integer that determines the size of the crystal, that we will eventually send to infinity, with periodic boundary conditions. We introduce the intra-layer nearest neighbor vectors:

$$\delta_1 := (1,0,0),\quad \delta_2 := \left(-\frac12,\frac{\sqrt3}{2},0\right),\quad \delta_3 := \left(-\frac12,-\frac{\sqrt3}{2},0\right). \tag{I.2.3}$$

The *dual* of Λ is

$$\hat\Lambda := \left\{\frac{m_1}{L}G_1 + \frac{m_2}{L}G_2,\ (m_1,m_2)\in\{0,\cdots,L-1\}^2\right\} \tag{I.2.4}$$

with periodic boundary conditions, where

$$G_1 = \left(\frac{2\pi}{3},\frac{2\pi}{\sqrt3},0\right),\ G_2 = \left(\frac{2\pi}{3},-\frac{2\pi}{\sqrt3},0\right). \tag{I.2.5}$$

It is defined in such a way that $\forall x\in\Lambda$, $\forall k\in\hat\Lambda$,

$$e^{ikxL} = 1.$$

Since the third component of vectors in $\hat\Lambda$ is always 0, we shall drop it and write vectors of $\hat\Lambda$ as elements of $\mathbb R^2$. In the limit $L\to\infty$, the set $\hat\Lambda$ tends to the torus $\hat\Lambda_\infty = \mathbb R^2/(\mathbb Z G_1 + \mathbb Z G_2)$, also called the *Brillouin zone*.

2 - Hamiltonian. Given $x\in\Lambda$, we denote the Fermionic annihilation operators at atoms of type a, $\tilde b$, $\tilde a$ and b within the elementary cell centered at x respectively by a_x, $\tilde b_x$, $\tilde a_{x-\delta_1}$ and $b_{x+\delta_1}$. The corresponding creation operators are their adjoint operators.

We recall the Hamiltonian (I.1.1)

$$\mathcal H = \mathcal H_0 + \mathcal H_I$$

where $\mathcal{H}_0$ is the *free Hamiltonian* and $\mathcal{H}_I$ is the *interaction Hamiltonian.*

2-1 - Free Hamiltonian. As was mentioned in section I.1, the free Hamiltonian describes the *hopping* of electrons from one atom to another. Here, we only consider the hoppings $\gamma_0, \gamma_1, \gamma_3$, see figure I.1.2, so that $\mathcal{H}_0$ has the following expression in x space:

$$\begin{aligned}\mathcal{H}_0 := -\gamma_0 \sum_{\substack{x\in\Lambda\\ j=1,2,3}} \left(a_x^\dagger b_{x+\delta_j} + b_{x+\delta_j}^\dagger a_x + \tilde b_x^\dagger \tilde a_{x-\delta_j} + \tilde a_{x-\delta_j}^\dagger \tilde b_x\right) - \gamma_1 \sum_{x\in\Lambda}\left(a_x^\dagger \tilde b_x + \tilde b_x^\dagger a_x\right)\\ -\gamma_3 \sum_{\substack{x\in\Lambda\\ j=1,2,3}} \left(\tilde a_{x-\delta_1}^\dagger b_{x-\delta_1-\delta_j} + b_{x-\delta_1-\delta_j}^\dagger \tilde a_{x-\delta_1}\right)\end{aligned} \tag{I.2.6}$$

Equation (I.2.6) can be rewritten in Fourier space as follows. We define the Fourier transform of the annihilation operators as

$$\hat a_k := \sum_{x\in\Lambda} e^{ikx} a_x\ , \quad \hat{\tilde b}_k := \sum_{x\in\Lambda} e^{ikx}\tilde b_{x+\delta_1}\ , \quad \hat{\tilde a}_k := \sum_{x\in\Lambda} e^{ikx}\tilde a_{x-\delta_1}\ , \quad \hat b_k := \sum_{x\in\Lambda} e^{ikx} b_{x+\delta_1} \tag{I.2.7}$$

in terms of which

$$\mathcal{H}_0 = -\frac{1}{|\Lambda|}\sum_{k\in\hat\Lambda} \hat A_k^\dagger H_0(k) A_k \tag{I.2.8}$$

where $|\Lambda| = L^2$, $\hat A_k$ is a column vector, whose transpose is $\hat A_k^T = (\hat a_k, \hat{\tilde b}_k, \hat{\tilde a}_k, \hat b_k)$,

$$H_0(k) := \begin{pmatrix} 0 & \gamma_1 & 0 & \gamma_0\Omega^*(k) \\ \gamma_1 & 0 & \gamma_0\Omega(k) & 0 \\ 0 & \gamma_0\Omega^*(k) & 0 & \gamma_3\Omega(k)e^{3ik_x} \\ \gamma_0\Omega(k) & 0 & \gamma_3\Omega^*(k)e^{-3ik_x} & 0 \end{pmatrix} \tag{I.2.9}$$

and

$$\Omega(k) := \sum_{j=1}^{3} e^{ik(\delta_j-\delta_1)} = 1 + 2e^{-i\frac{3}{2}k_x}\cos\left(\frac{\sqrt3}{2}k_y\right).$$

We pick the energy unit in such a way that $\gamma_0 = 1$.

2-2 - Interaction. We now define the interaction Hamiltonian. We first define the number operators n_x^α for $\alpha \in \{a, \tilde b, \tilde a, b\}$ and $x \in \Lambda$ in the following way:

$$n_x^a = a_x^\dagger a_x\ , \quad n_x^{\tilde b} = \tilde b_x^\dagger \tilde b_x\ , \quad n_x^{\tilde a} = \tilde a_{x-\delta_1}^\dagger \tilde a_{x-\delta_1}\ , \quad n_x^b = b_{x+\delta_1}^\dagger b_{x+\delta_1} \tag{I.2.10}$$

and postulate the form of the interaction to be of an extended *Hubbard* form:

$$\mathcal{H}_I := U \sum_{(x,y)\in\Lambda^2}\ \sum_{(\alpha,\alpha')\in\{a,\tilde b,\tilde a,b\}^2} v(x+d_\alpha-y-d_{\alpha'})\left(n_x^\alpha - \frac12\right)\left(n_y^{\alpha'}-\frac12\right) \tag{I.2.11}$$

where the d_α are the vectors that give the position of each atom type with respect to the centers of the lattice Λ: $d_a := 0$, $d_{\tilde b} := (0,0,c)$, $d_{\tilde a} := (0,0,c)-\delta_1$, $d_b := \delta_1$ and v is a bounded, rotationally invariant function, which decays exponentially fast to zero at infinity. In our spin-less case, we can assume without loss of generality that $v(0) = 0$.

I.2.2. Schwinger function as Grassmann integrals and expectations

The aim of the present work is to compute the *specific free energy* and the *two-point Schwinger function*. These quantities are defined for finite lattices by

$$f_\Lambda := -\frac{1}{\beta|\Lambda|}\log\left(\mathrm{Tr}\left(e^{-\beta\mathcal{H}}\right)\right) \tag{I.2.12}$$

where β is inverse temperature and

$$\check{s}_{\alpha',\alpha}(\mathbf{x}_1-\mathbf{x}_2) := \left\langle \mathbf{T}(\alpha'_{\mathbf{x}_1}\alpha^\dagger_{\mathbf{x}_2})\right\rangle := \frac{\mathrm{Tr}(e^{-\beta\mathcal{H}}\mathbf{T}(\alpha'_{\mathbf{x}_1}\alpha^\dagger_{\mathbf{x}_2}))}{\mathrm{Tr}(e^{-\beta\mathcal{H}})} \tag{I.2.13}$$

in which $(\alpha,\alpha')\in\mathcal{A}^2 := \{a,\tilde{b},\tilde{a},b\}^2$; $\mathbf{x}_{1,2}=(t_{1,2},x_{1,2})$ with $t_{1,2}\in[0,\beta)$; $\alpha_{\mathbf{x}} = e^{\mathcal{H}t}\alpha_x e^{-\mathcal{H}t}$; and $\mathbf{T}$ is the *Fermionic time ordering operator* defined in (I.1.8). Our strategy essentially consists in deriving convergent expansions for f_Λ and $\check{s}$, uniformly in $|\Lambda|$ and β, and then to take $\beta,|\Lambda|\to\infty$.

However, the quantities on the right side of (I.2.12) and (I.2.13) are somewhat difficult to manipulate. In this section, we will re-express f_Λ and $\check{s}$ in terms of *Grassmann integrals* and *expectations*, and show how such quantities can be computed using a *determinant expansion*. This formalism will lay the groundwork for the procedure which will be used in the following to express f_Λ and $\check{s}$ as series, and subsequently prove their convergence.

1 - Grassmann integral formulation. We first describe how to express (I.2.12) and (I.2.13) as Grassmann integrals. The procedure is well known and details can be found in many references, see e.g. [GM10, appendix B] and [Gi10] for a discussion adapted to the case of graphene, or [GM01] for a discussion adapted to general low-dimensional Fermi systems, or [BG95] and [Sa13] and references therein for an even more general picture.

1-1 - Definition. We first define a Grassmann algebra and an integration procedure on it. We move to Fourier space: for every $\alpha\in\mathcal{A} := \{a,\tilde{b},\tilde{a},b\}$, the operator $\alpha_{(t,x)}$ is associated

$$\hat{\alpha}_{\mathbf{k}=(k_0,k)} := \frac{1}{\beta}\int_0^\beta dt\ e^{itk_0}e^{\mathcal{H}_0 t}\hat{\alpha}_k e^{-\mathcal{H}_0 t}$$

with $k_0\in 2\pi\beta^{-1}(\mathbb{Z}+1/2)$ (notice that because of the $1/2$ term, $k_0\neq 0$ for finite β). We notice that $\mathbf{k}\in\mathcal{B}_{\beta,L} := (2\pi\beta^{-1}(\mathbb{Z}+1/2))\times\hat{\Lambda}$ varies in an infinite set. Since this will cause trouble when defining Grassmann integrals, we shall impose a cutoff $M\in\mathbb{N}$: let $\chi_0(\rho)$ be a smooth compact support function that returns 1 if $\rho\leqslant 1/3$ and 0 if $\rho\geqslant 2/3$, and let

$$\mathcal{B}^*_{\beta,L} := \mathcal{B}_{\beta,L}\cap\{(k_0,k),\ \chi_0(2^{-M}|k_0|\neq 0)\}.$$

To every $(\hat{\alpha}_{\mathbf{k}},\hat{\alpha}^\dagger_{\mathbf{k}})$ for $\alpha\in\mathcal{A}$ and $\mathbf{k}\in\mathcal{B}^*_{\beta,L}$, we associate a pair of *Grassmann variables* $(\hat{\psi}^-_{\mathbf{k},\alpha},\hat{\psi}^+_{\mathbf{k},\alpha})$, and we consider the finite Grassmann algebra (i.e. an algebra in which the $\hat{\psi}$ anti-commute with each other) generated by the collection $\{\hat{\psi}^\pm_{\mathbf{k},\alpha}\}^{\alpha\in\mathcal{A}}_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}$. We define the Grassmann integral

$$\int\prod_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}^{\alpha\in\mathcal{A}} d\hat{\psi}^+_{\mathbf{k},\alpha}d\hat{\psi}^-_{\mathbf{k},\alpha}$$

as the linear operator on the Grassmann algebra whose action on a monomial in the variables $\hat{\psi}^\pm_{\mathbf{k},\alpha}$ is 0 except if said monomial is $\prod^{\alpha\in\mathcal{A}}_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}\hat{\psi}^-_{\mathbf{k},\alpha}\hat{\psi}^+_{\mathbf{k},\alpha}$ up to a permutation of the variables, in which case the value of the integral is determined using

$$\int\prod_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}^{\alpha\in\mathcal{A}} d\hat{\psi}^+_{\mathbf{k},\alpha}d\hat{\psi}^-_{\mathbf{k},\alpha}\left(\prod_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}^{\alpha\in\mathcal{A}}\hat{\psi}^-_{\mathbf{k},\alpha}\hat{\psi}^+_{\mathbf{k},\alpha}\right)=1 \tag{I.2.14}$$

along with the anti-commutation of the $\hat{\psi}$.

In the following, we will express the free energy and Schwinger function as *Grassmann integrals*, specified by a *propagator* and a *potential*. The propagator is a 4×4 complex matrix $\hat{g}(\mathbf{k})$, supported on some set $\mathcal{B}\subset\mathcal{B}^*_{\beta,L}$, and is associated with the *Gaussian Grassmann integration measure*

$$P_{\hat{g}}(d\psi) := \left(\prod_{\mathbf{k}\in\mathcal{B}}(\beta|\Lambda|)^4 \det\hat{g}(\mathbf{k})\left(\prod_{\alpha\in\mathcal{A}} d\hat{\psi}^+_{\mathbf{k},\alpha}d\hat{\psi}^-_{\mathbf{k},\alpha}\right)\right)\exp\left(-\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}}\hat{\psi}^+_{\mathbf{k}}\hat{g}^{-1}(\mathbf{k})\hat{\psi}^-_{\mathbf{k}}\right). \tag{I.2.15}$$

Gaussian Grassmann integrals satisfy the following *addition principle*: given two propagators $\hat{g}_1$ and $\hat{g}_2$, and any polynomial $\mathfrak{P}(\psi)$ in the Grassmann variables,

$$\int P_{\hat{g}_1+\hat{g}_2}(d\psi)\ \mathfrak{P}(\psi) = \int P_{\hat{g}_1}(d\psi_1)\int P_{\hat{g}_2}(d\psi_2)\ \mathfrak{P}(\psi_1+\psi_2). \tag{I.2.16}$$

1-2 - Free energy. We now express the free energy as a Grassmann integral. We define the *free propagator*

$$\hat{g}_{\leqslant M}(\mathbf{k}) := \chi_0(2^{-M}|k_0|)(-ik_0\mathbb{1}-H_0(k))^{-1} \tag{I.2.17}$$

and the Gaussian integration measure $P_{\leqslant M}(d\psi)\equiv P_{\hat{g}_{\leqslant M}}(d\psi)$. One can prove (see e.g. [GM10, appendix B]) that if

$$\frac{1}{\beta|\Lambda|}\log\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{V}(\psi)} \tag{I.2.18}$$

is analytic in U, uniformly as $M\to\infty$, a fact we will check a posteriori, then the finite volume free energy can be written as

$$f_\Lambda = f_{0,\Lambda} - \lim_{M\to\infty}\frac{1}{\beta|\Lambda|}\log\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{V}(\psi)} \tag{I.2.19}$$

where $f_{0,\Lambda}$ is the free energy in the $U=0$ case and, using the symbol $\int d\mathbf{x}$ as a shorthand for $\int_0^\beta dt\sum_{x\in\Lambda}$,

$$\mathcal{V}(\psi) = U\sum_{(\alpha,\alpha')\in\mathcal{A}^2}\int d\mathbf{x}d\mathbf{y}\ w_{\alpha,\alpha'}(\mathbf{x}-\mathbf{y})\psi^+_{\mathbf{x},\alpha}\psi^-_{\mathbf{x},\alpha}\psi^+_{\mathbf{y},\alpha'}\psi^-_{\mathbf{y},\alpha'} \tag{I.2.20}$$

in which $w_{\alpha,\alpha'}(\mathbf{x}) := \delta(x_0)v(x+d_\alpha-d_{\alpha'})$, where $\delta(x_0)$ denotes the β-periodic Dirac delta function, and

$$\psi^\pm_{\mathbf{x},\alpha} := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}\hat{\psi}^\pm_{\mathbf{k},\alpha}e^{\pm i\mathbf{k}\mathbf{x}}\ . \tag{I.2.21}$$

Notice that the expression of $\mathcal{V}(\psi)$ in (I.2.20) is very similar to that of $\mathcal{H}_I$, with an added imaginary time (x_0,y_0) and the $\alpha_{\mathbf{x}}$ replaced by $\psi_{\mathbf{x},\alpha}$, except that $(\alpha^\dagger_{\mathbf{x}}\alpha_{\mathbf{x}}-1/2)$ becomes $\psi^+_{\mathbf{x},\alpha}\psi^-_{\mathbf{x},\alpha}$. Roughly, the reason why we "drop the 1/2" is because of the difference between the anti-commutation rules of $\alpha_{\mathbf{x}}$ and $\psi_{\mathbf{x},\alpha}$ (i.e., $\{\alpha_{\mathbf{x}},\alpha^\dagger_{\mathbf{x}}\}=1$, vs. $\{\psi^+_{\mathbf{x},\alpha},\psi^-_{\mathbf{x},\alpha}\}=0$). More precisely, taking $\mathbf{x}=(x_0,x)$ with $x_0\in(-\beta,\beta)$, it is easy to check that the limit as $M\to\infty$ of $g_{\leqslant M}(\mathbf{x}) := \int P_{\leqslant M}(d\psi)\psi^-_{\mathbf{x}}\psi^+_{\mathbf{0}}$ is equal to $\check{s}(\mathbf{x})$, if $\mathbf{x}\neq\mathbf{0}$, and equal to $\check{s}(\mathbf{0})+1/2$, otherwise. This extra $+1/2$ accounts for the "dropping of the 1/2" mentioned above.

1-3 - Two-point Schwinger function. The two-point Schwinger function can be expressed as a Grassmann integral as well: under the condition that

$$\frac{\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{V}(\psi)}\hat{\psi}^-_{\mathbf{k},\alpha_1}\hat{\psi}^+_{\mathbf{k},\alpha_2}}{\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{V}(\psi)}} \tag{I.2.22}$$

is analytic in U uniformly in M, a fact we will also check a posteriori, then one can prove (see e.g. [GM10, appendix B]) that the two-point Schwinger function can be written as

$$s_{\alpha_1,\alpha_2}(\mathbf{k}) = \lim_{M\to\infty} \frac{\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{V}(\psi)}\hat\psi^-_{\mathbf{k},\alpha_1}\hat\psi^+_{\mathbf{k},\alpha_2}}{\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{V}(\psi)}}. \tag{I.2.23}$$

In order to facilitate the computation of the right side of (I.2.23), we will first rewrite it as

$$s_{\alpha_1,\alpha_2}(\mathbf{k}) = \lim_{M\to\infty} \int d\hat J^-_{\mathbf{k},\alpha_1} d\hat J^+_{\mathbf{k},\alpha_2} \log \int P_{\leqslant M}(d\psi) e^{-\mathcal{V}(\psi)+\hat J^+_{\mathbf{k},\alpha_1}\hat\psi^-_{\mathbf{k},\alpha_1}+\hat\psi^+_{\mathbf{k},\alpha_2}\hat J^-_{\mathbf{k},\alpha_2}} \tag{I.2.24}$$

where $\hat J^-_{\mathbf{k},\alpha}$ and $\hat J^+_{\mathbf{k},\alpha'}$ are extra Grassmann variables introduced for the purpose of the computation (note here that the Grassmann integral over the variables $\hat J^-_{\mathbf{k},\alpha_1}, \hat J^+_{\mathbf{k},\alpha_2}$ acts as a functional derivative with respect to the same variables, due to the Grassmann integration/derivation rules). We define the *generating functional*

$$\mathcal{W}(\psi, \hat J_{\mathbf{k},\underline{\alpha}}) := \mathcal{V}(\psi) - \hat J^+_{\mathbf{k},\alpha_1}\hat\psi^-_{\mathbf{k},\alpha_1} - \hat\psi^+_{\mathbf{k},\alpha_2}\hat J^-_{\mathbf{k},\alpha_2}. \tag{I.2.25}$$

2 - Expectations. We have seen that the free energy and Schwinger function can be computed as Grassmann integrals, it remains to see how one computes such integrals. We can write (I.2.18) as

$$\log \int P_{\leqslant M}(d\psi) e^{-\mathcal{V}(\psi)} = \sum_{N=1}^\infty \frac{(-1)^N}{N!} \mathcal{E}^T_{\leqslant M}(\underbrace{\mathcal{V},\cdots,\mathcal{V}}_{N\ \text{times}}) =: \sum_{N=1}^\infty \frac{(-1)^N}{N!}\mathcal{E}^T_{\leqslant M}(\mathcal{V};N). \tag{I.2.26}$$

where the *truncated expectation* is defined as

$$\mathcal{E}^T_{\leqslant M}(\mathcal{V}_1,\cdots,\mathcal{V}_N) := \frac{\partial^N}{\partial\lambda_1\cdots\partial\lambda_N} \log\int P_{\leqslant M}(d\psi)\ e^{\lambda_1\mathcal{V}_1+\cdots+\lambda_N\mathcal{V}_N}\Big|_{\lambda_1=\cdots=\lambda_N=0}. \tag{I.2.27}$$

in which $(\mathcal{V}_1,\cdots,\mathcal{V}_N)$ is a collection of commuting polynomials and the index $_{\leqslant M}$ refers to the propagator of $P_{\leqslant M}(d\psi)$. A similar formula holds for (I.2.22).

The purpose of this rewriting is that we can compute truncated expectations in terms of a *determinant expansion*, also known as the Battle-Brydges-Federbush formula [BF78, BF84], which expresses it as the determinant of a Gram matrix. The advantage of this writing is that, provided we first re-express the propagator $\hat g_{\leqslant M}(\mathbf{k})$ in $\mathbf{x}$-space, the afore-mentioned Gram matrix can be bounded effectively (see section I.5.2). We therefore first define an $\mathbf{x}$-space representation for $\hat g(\mathbf{k})$:

$$g_{\leqslant M}(\mathbf{x}) := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}} e^{i\mathbf{k}\cdot\mathbf{x}}\hat g_{\leqslant M}(\mathbf{k}). \tag{I.2.28}$$

The determinant expansion is given in the following lemma, the proof of which can be found in [GM01, appendix A.3.2], [Gi10, appendix B].

Lemma I.2.1

Consider a family of sets $\mathbf{P} = (P_1,\cdots,P_s)$ where every P_j is an ordered collection of Grassmann variables, we denote the product of the elements in P_j by $\Psi_{P_j} := \prod_{\psi\in P_j}\psi$.

We call a pair $(\psi^-_{\mathbf{x},\alpha},\psi^+_{\mathbf{x}',\alpha'})\in\mathbf{P}^2$ a *line*, and define the set of *spanning trees* $\mathbf{T}(\mathbf{P})$ as the set of collections T of lines that are such that upon drawing a vertex for each P_i in $\mathbf{P}$ and a line between the vertices corresponding to P_i and to P_j for each line $(\psi^-_{\mathbf{x},\alpha},\psi^+_{\mathbf{x}',\alpha'})\in T$ that is such that $\psi^-_{\mathbf{x},\alpha}\in P_i$ and $\psi^+_{\mathbf{x}',\alpha'}\in P_j$, the resulting graph is a tree that connects all of the vertices.

For every spanning tree $T \in \mathbf{T}(\mathbf{P})$, to each line $l = (\psi^-_{\mathbf{x},\alpha}, \psi^+_{\mathbf{x}',\alpha'}) \in T$ we assign a *propagator* $g_l := g_{\alpha,\alpha'}(\mathbf{x} - \mathbf{x}')$.

If $\mathbf{P}$ contains $2(n+s-1)$ Grassmann variables, with $n \in \mathbb{N}$, then there exists a probability measure $dP_T(\mathbf{t})$ on the set of $n \times n$ matrices of the form $\mathbf{t} = M^T M$ with M being a matrix whose columns are unit vectors of $\mathbb{R}^n$, such that

$$\mathcal{E}^T_{\leqslant M}(\Psi_{P_1}, \cdots, \Psi_{P_s}) = \sum_{T \in \mathbf{T}(\mathbf{P})} \sigma_T \prod_{l \in T} g_l \int dP_T(\mathbf{t}) \ \det G^{(T)}(\mathbf{t}) \tag{I.2.29}$$

where $\sigma_T \in \{-1, 1\}$ and $G^{(T)}(\mathbf{t})$ is an $n \times n$ complex matrix each of whose components is indexed by a line $l \notin T$ and is given by

$$G^{(T)}_l(\mathbf{t}) = \mathbf{t}_l g_l$$

(if $s = 1$, then $\mathbf{T}(\mathbf{P})$ is empty and both the sum over T and the factor $\sigma_T \prod_{l \in T} g_l$ should be dropped from the right side of (I.2.29)).

Lemma I.2.1 gives us a formal way of computing the right side of (I.2.26). However, proving that this formal expression is correct, in the sense that it is not divergent, will require a control over the quantities involved in the right side of (I.2.29), namely the propagator $g_{\leqslant M}$. Since, as was discussed in the introduction, $g_{\leqslant M}$ is singular, controlling the right side of (I.2.26) is a non-trivial task that will require a multiscale analysis described in section I.4.

I.2.3. Symmetries of the system

In the following, we will rely heavily on the symmetries of the system, whose representation in terms of Grassmann variables is now discussed.

A *symmetry* of the system is a map that leaves *both*

$$h_0 := \sum_{\mathbf{x},\mathbf{y}} \psi^+_{\mathbf{x}} g^{-1}(\mathbf{x} - \mathbf{y}) \psi^-_{\mathbf{y}} \tag{I.2.30}$$

and $\mathcal{V}(\psi)$ invariant ($\mathcal{V}(\psi)$ was defined in (I.2.20)). We define

$$\hat\xi^+_{\mathbf{k}} := \begin{pmatrix} \hat\psi^+_{\mathbf{k},a} & \hat\psi^+_{\mathbf{k},\tilde b} \end{pmatrix}, \quad \hat\xi^-_{\mathbf{k}} := \begin{pmatrix} \hat\psi^-_{\mathbf{k},a} \\ \hat\psi^-_{\mathbf{k},\tilde b} \end{pmatrix}, \quad \hat\phi^+_{\mathbf{k}} := \begin{pmatrix} \hat\psi^+_{\mathbf{k},\tilde a} & \hat\psi^+_{\mathbf{k},b} \end{pmatrix} \hat\phi^-_{\mathbf{k}} := \begin{pmatrix} \hat\psi^-_{\mathbf{k},\tilde a} \\ \hat\psi^-_{\mathbf{k},b} \end{pmatrix} \tag{I.2.31}$$

as well as the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We now enumerate the symmetries of the system, and postpone their proofs to appendix I.A5.

1 - Global $U(1)$. For $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$, the map

$$\left\{ \begin{array}{l} \hat\xi^\pm_{\mathbf{k}} \longmapsto e^{\pm i\theta} \hat\xi^\pm_{\mathbf{k}} \\ \hat\phi^\pm_{\mathbf{k}} \longmapsto e^{\pm i\theta} \hat\phi^\pm_{\mathbf{k}} \end{array} \right. \tag{I.2.32}$$

is a symmetry.

2 - $2\pi/3$ rotation. Let $T\mathbf{k} := (k_0, e^{-i\frac{2\pi}{3}\sigma_2} k)$, $l_2 := (3/2, -\sqrt{3}/2)$ and $\mathcal{T}_{\mathbf{k}} := e^{-i(l_2 \cdot k)\sigma_3}$, the mapping

$$\left\{ \begin{array}{l} \hat\xi^\pm_{\mathbf{k}} \longmapsto \hat\xi^\pm_{T\mathbf{k}} \\ \hat\phi^-_{\mathbf{k}} \longmapsto \mathcal{T}_{T\mathbf{k}} \hat\phi^-_{T\mathbf{k}}, \ \hat\phi^+_{\mathbf{k}} \longmapsto \hat\phi^+_{T\mathbf{k}} \mathcal{T}^\dagger_{T\mathbf{k}} \end{array} \right. \tag{I.2.33}$$

is a symmetry.

3 - Complex conjugation. The map in which

$$\left\{\begin{array}{l}\hat\xi^\pm_{\mathbf k}\longmapsto\hat\xi^\pm_{-\mathbf k}\\ \hat\phi^\pm_{\mathbf k}\longmapsto\hat\phi^\pm_{-\mathbf k}.\end{array}\right.\tag{I.2.34}$$

and every complex coefficient of h_0 and $\mathcal V$ is mapped to its complex conjugate is a symmetry.

4 - Vertical reflection. Let $R_v\mathbf k=(k_0,k_x,-k_y)$,

$$\left\{\begin{array}{l}\hat\xi^\pm_{\mathbf k}\longmapsto\hat\xi^\pm_{R_v\mathbf k}\\ \hat\phi^\pm_{\mathbf k}\longmapsto\hat\phi^\pm_{R_v\mathbf k}\end{array}\right.\tag{I.2.35}$$

is a symmetry.

5 - Horizontal reflection. Let $R_h\mathbf k=(k_0,-k_x,k_y)$,

$$\left\{\begin{array}{l}\hat\xi^-_{\mathbf k}\longmapsto\sigma_1\hat\xi^-_{R_h\mathbf k},\ \hat\xi^+_{\mathbf k}\longmapsto\hat\xi^+_{R_h\mathbf k}\sigma_1\\ \hat\phi^-_{\mathbf k}\longmapsto\sigma_1\hat\phi^-_{R_h\mathbf k},\ \hat\phi^+_{\mathbf k}\longmapsto\hat\phi^+_{R_h\mathbf k}\sigma_1\end{array}\right.\tag{I.2.36}$$

is a symmetry.

6 - Parity. Let $P\mathbf k=(k_0,-k_x,-k_y)$,

$$\left\{\begin{array}{l}\hat\xi^\pm_{\mathbf k}\longmapsto i(\hat\xi^\mp_{P\mathbf k})^T\\ \hat\phi^\pm_{\mathbf k}\longmapsto i(\hat\phi^\mp_{P\mathbf k})^T\end{array}\right.\tag{I.2.37}$$

is a symmetry.

7 - Time inversion. Let $I\mathbf k=(-k_0,k_x,k_y)$, the mapping

$$\left\{\begin{array}{l}\hat\xi^-_{\mathbf k}\longmapsto-\sigma_3\hat\xi^-_{I\mathbf k},\ \hat\xi^+_{\mathbf k}\longmapsto\hat\xi^+_{I\mathbf k}\sigma_3\\ \hat\phi^-_{\mathbf k}\longmapsto-\sigma_3\hat\phi^-_{I\mathbf k},\ \hat\phi^+_{\mathbf k}\longmapsto\hat\phi^+_{I\mathbf k}\sigma_3\end{array}\right.\tag{I.2.38}$$

is a symmetry.

I.3. Free propagator

In section I.2.2, we showed how to express the free energy and the two-point Schwinger function as a formal series of truncated expectations (I.2.26). Controlling the convergence of this series is made difficult by the fact that the propagator $\hat g_{\leqslant M}$ is singular, and will require a finer analysis. In this section, we discuss which are the singularities of $\hat g_{\leqslant M}$ and how it behaves close to them, and identify three regimes in which the propagator behaves differently.

I.3.1. Fermi points

The free propagator is singular if $k_0 = 0$ and k is such that $H_0(k)$ is not invertible. The set of such k's is called the *Fermi surface*. In this subsection, we study the properties of this set. We recall the definition of H_0 in (I.2.9),

$$H_0(k) := -\begin{pmatrix} 0 & \gamma_1 & 0 & \Omega^*(k) \\ \gamma_1 & 0 & \Omega(k) & 0 \\ 0 & \Omega^*(k) & 0 & \gamma_3\Omega(k)e^{3ik_x} \\ \Omega(k) & 0 & \gamma_3\Omega^*(k)e^{-3ik_x} & 0 \end{pmatrix}$$

so that, using corollary I.A2.2 (see appendix I.A2),

$$\det H_0(k) = \left|\Omega^2(k) - \gamma_1\gamma_3\Omega^*(k)e^{-3ik_x}\right|^2. \tag{I.3.1}$$

It is then straightforward to compute the solutions of $\det H_0(k) = 0$ (see appendix I.A1 for details): we find that as long as $0 < \gamma_1\gamma_3 < 2$, there are 8 Fermi points:

$$\left\{\begin{array}{l} p_{F,0}^\omega := \left(\frac{2\pi}{3}, \omega\frac{2\pi}{3\sqrt{3}}\right) \\ p_{F,1}^\omega := \left(\frac{2\pi}{3}, \omega\frac{2}{\sqrt{3}}\arccos\left(\frac{1-\gamma_1\gamma_3}{2}\right)\right) \\ p_{F,2}^\omega := \left(\frac{2\pi}{3} + \frac{2}{3}\arccos\left(\frac{\sqrt{1+\gamma_1\gamma_3(2-\gamma_1\gamma_3)}}{2}\right), \omega\frac{2}{\sqrt{3}}\arccos\left(\frac{1+\gamma_1\gamma_3}{2}\right)\right) \\ p_{F,3}^\omega := \left(\frac{2\pi}{3} - \frac{2}{3}\arccos\left(\frac{\sqrt{1+\gamma_1\gamma_3(2-\gamma_1\gamma_3)}}{2}\right), \omega\frac{2}{\sqrt{3}}\arccos\left(\frac{1+\gamma_1\gamma_3}{2}\right)\right). \end{array}\right. \tag{I.3.2}$$

for $\omega \in \{-,+\}$. Note that

$$\begin{gathered} p_{F,1}^\omega = p_{F,0}^\omega + \left(0, \omega\tfrac{2}{3}\gamma_1\gamma_3\right) + O(\epsilon^4), \quad p_{F,2}^\omega = p_{F,0}^\omega + \left(\tfrac{1}{\sqrt{3}}\gamma_1\gamma_3, -\omega\tfrac{1}{3}\gamma_1\gamma_3\right) + O(\epsilon^4), \\ p_{F,3}^\omega = p_{F,0}^\omega + \left(-\tfrac{1}{\sqrt{3}}\gamma_1\gamma_3, -\omega\tfrac{1}{3}\gamma_1\gamma_3\right) + O(\epsilon^4). \end{gathered} \tag{I.3.3}$$

The points $p_{F,j}^\omega$ for $j = 1, 2, 3$ are labeled as per figure I.1.3.

I.3.2. Behavior around the Fermi points

In this section, we compute the dominating behavior of $\hat{g}(\mathbf{k})$ close to its singularities, that is close to $\mathbf{p}_{F,j}^\omega := (0, p_{F,j}^\omega)$. We recall that $\hat{A}(\mathbf{k}) := (-ik_0\mathbb{1} + H_0(k))$ and $\hat{g}(\mathbf{k}) = \chi_0(2^{-M}|k_0|)\hat{A}^{-1}(\mathbf{k})$.

1 - First regime. We define $k' := k - p_{F,0}^\omega = (k'_x, k'_y)$, $\mathbf{k}' := (k_0, k')$. We have

$$\Omega(p_{F,0}^\omega + k') = \frac{3}{2}(ik'_x + \omega k'_y) + O(|k'|^2) =: \xi + O(|k'|^2) \tag{I.3.4}$$

so that, by using (I.A2.2) with $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{x}, \mathfrak{z}) = -(\gamma_1, \Omega(k), \gamma_3\Omega(k)e^{3ik_x}, k_0, k_0)$,

$$\det \hat{A}(\mathbf{p}_{F,0}^\omega + \mathbf{k}') = (k_0^2 + |\xi|^2)^2 + O(\|\mathbf{k}'\|_{\mathrm{I}}^5, \epsilon^2\|\mathbf{k}'\|_{\mathrm{I}}^2) \tag{I.3.5}$$

where

$$\|\mathbf{k}'\|_{\mathrm{I}} := \sqrt{k_0^2 + |\xi|^2} \tag{I.3.6}$$

in which the label $_{\mathrm{I}}$ stands for "first regime". If

$$\kappa_1\epsilon < \|\mathbf{k}'\|_{\mathrm{I}} < \bar{\kappa}_0 \tag{I.3.7}$$

for suitable constants $\kappa_1, \bar\kappa_0 > 0$, then the remainder term in (I.3.5) is smaller than the explicit term, so that (I.3.5) is adequate in this regime, which we call the "first regime".

We now compute the dominating part of $\hat A^{-1}$ in this regime. The computation is carried out in the following way: we neglect terms of order γ_1, γ_3 and $|k'|^2$ in $\hat A$, invert the resulting matrix using (I.A2.3), prove that this inverse is bounded by (const.) $\|\mathbf{k}'\|_{\mathrm{I}}^{-1}$, and deduce a bound on the error terms. We thus find

$$\hat A^{-1}(\mathbf{p}^\omega_{F,0}+\mathbf{k}') = -\frac{1}{k_0^2+|\xi|^2}\left(\begin{array}{cccc} -ik_0 & 0 & 0 & \xi^* \\ 0 & -ik_0 & \xi & 0 \\ 0 & \xi^* & -ik_0 & 0 \\ \xi & 0 & 0 & -ik_0 \end{array}\right)\left(\mathbb{1}+O(\|\mathbf{k}'\|_{\mathrm{I}}, \epsilon\|\mathbf{k}'\|_{\mathrm{I}}^{-1})\right) \tag{I.3.8}$$

and

$$|\hat A^{-1}(\mathbf{p}^\omega_{F,0}+\mathbf{k}')| \leqslant (\text{const.})\ \|\mathbf{k}'\|_{\mathrm{I}}^{-1}. \tag{I.3.9}$$

Note that, recalling that the basis in which we wrote A^{-1} is $\{a, \tilde b, \tilde a, b\}$, each graphene layer is decoupled from the other in the dominating part of (I.3.8).

2 - Ultraviolet regime. The regime in which $\|\mathbf{k}'\|_{\mathrm{I}} \geqslant \bar\kappa_0$ for both $\omega = \pm$, and is called the *ultraviolet* regime. For such $\mathbf{k}' =: \mathbf{k} - \mathbf{p}^\omega_{F,0}$, one easily checks that

$$|\hat A^{-1}(\mathbf{k})| \leqslant (\text{const.})\ |\mathbf{k}|^{-1}. \tag{I.3.10}$$

3 - Second regime. We now go beyond the first regime: we assume that $\|\mathbf{k}'\|_{\mathrm{I}} \leqslant \kappa_1\epsilon$ and, using again (I.3.4) and (I.A2.2), we write

$$\det \hat A(\mathbf{p}^\omega_{F,0}+\mathbf{k}') = \gamma_1^2 k_0^2 + |\xi|^4 + O(\epsilon^{7/2}\|\mathbf{k}'\|_{\mathrm{II}}^{3/2}, \epsilon^5\|\mathbf{k}'\|_{\mathrm{II}}, \epsilon\|\mathbf{k}'\|_{\mathrm{II}}^3) \tag{I.3.11}$$

where

$$\|\mathbf{k}'\|_{\mathrm{II}} := \sqrt{k_0^2 + \frac{|\xi|^4}{\gamma_1^2}}. \tag{I.3.12}$$

If

$$\kappa_2\ \epsilon^3 < \|\mathbf{k}'\|_{\mathrm{II}} < \bar\kappa_1\ \epsilon \tag{I.3.13}$$

for suitable constants $\kappa_2, \bar\kappa_1 > 0$, then the remainder in (I.3.11) is smaller than the explicit term, and we thus define the "second regime", for which (I.3.11) is appropriate.

We now compute the dominating part of $\hat A^{-1}$ in this regime. To that end, we define the dominating part $\mathfrak{L}_{\mathrm{II}}\hat A$ of $\hat A$ by neglecting the terms of order γ_3 and $|k'|^2$ in $\hat A$ as well as the elements $\hat A_{aa}$ and $\hat A_{\tilde b\tilde b}$ (which are both equal to $-ik_0$), block-diagonalize it using proposition I.A3.1 (see appendix I.A3) and invert it:

$$\left(\mathfrak{L}_{\mathrm{II}}\hat A(\mathbf{k})\right)^{-1} = \left(\begin{array}{cc} \mathbb{1} & M_{\mathrm{II}}(\mathbf{k})^\dagger \\ 0 & \mathbb{1} \end{array}\right)\left(\begin{array}{cc} a^{(M)}_{\mathrm{II}} & 0 \\ 0 & a^{(m)}_{\mathrm{II}}(\mathbf{k}) \end{array}\right)\left(\begin{array}{cc} \mathbb{1} & 0 \\ M_{\mathrm{II}}(\mathbf{k}) & \mathbb{1} \end{array}\right) \tag{I.3.14}$$

where

$$a^{(M)}_{\mathrm{II}} := -\left(\begin{array}{cc} 0 & \gamma_1^{-1} \\ \gamma_1^{-1} & 0 \end{array}\right),\quad a^{(m)}_{\mathrm{II}}(\mathbf{p}^\omega_{F,0}+\mathbf{k}') := \frac{\gamma_1}{\gamma_1^2k_0^2+|\xi|^4}\left(\begin{array}{cc} i\gamma_1 k_0 & (\xi^*)^2 \\ \xi^2 & i\gamma_1 k_0 \end{array}\right) \tag{I.3.15}$$

and

$$M_{\mathrm{II}}(\mathbf{p}^\omega_{F,0}+\mathbf{k}') := -\frac{1}{\gamma_1}\left(\begin{array}{cc} \xi^* & 0 \\ 0 & \xi \end{array}\right). \tag{I.3.16}$$

We then bound the right side of (I.3.14), and find

$$\left|\left(\mathfrak{L}_{\rm II}\hat A(\mathbf{p}^\omega_{F,0}+\mathbf{k}')\right)^{-1}\right|\leqslant(\mathrm{const.})\left(\begin{array}{cc}\epsilon^{-1}&\epsilon^{-1/2}\|\mathbf{k}'\|_{\rm II}^{-1/2}\\ \epsilon^{-1/2}\|\mathbf{k}'\|_{\rm II}^{-1/2}&\|\mathbf{k}'\|_{\rm II}^{-1}\end{array}\right),\tag{I.3.17}$$

in which the bound should be understood as follows: the upper-left element in (I.3.17) is the bound on the upper-left 2×2 block of $(\mathfrak{L}_{\rm II}\hat A)^{-1}$, and similarly for the upper-right, lower-left and lower-right. Using this bound in

$$\hat A^{-1}(\mathbf{k})=\left(\mathfrak{L}_{\rm II}\hat A(\mathbf{k})\right)^{-1}\left[\mathbb{1}+(\hat A(\mathbf{k})-\mathfrak{L}_{\rm II}\hat A(\mathbf{k}))\left(\mathfrak{L}_{\rm II}\hat A(\mathbf{k})\right)^{-1}\right]^{-1}$$

we deduce a bound on the error term in square brackets and find

$$\begin{array}{r}\hat A^{-1}(\mathbf{p}^\omega_{F,0}+\mathbf{k}')=\left(\begin{array}{cc}\mathbb{1}&M_{\rm II}(\mathbf{k})^\dagger\\ 0&\mathbb{1}\end{array}\right)\left(\begin{array}{cc}a^{(M)}_{\rm II}&0\\ 0&a^{(m)}_{\rm II}(\mathbf{k})\end{array}\right)\left(\begin{array}{cc}\mathbb{1}&0\\ M_{\rm II}(\mathbf{k})&\mathbb{1}\end{array}\right)\cdot\\ \cdot(\mathbb{1}+O(\epsilon^{-1/2}\|\mathbf{k}'\|_{\rm II}^{1/2},\epsilon^{3/2}\|\mathbf{k}'\|_{\rm II}^{-1/2}))\end{array}\tag{I.3.18}$$

which implies the analogue of (I.3.17) for $\hat A^{-1}$,

$$|\hat A^{-1}(\mathbf{p}^\omega_{F,0}+\mathbf{k}')|\leqslant(\mathrm{const.})\left(\begin{array}{cc}\epsilon^{-1}&\epsilon^{-1/2}\|\mathbf{k}'\|_{\rm II}^{-1/2}\\ \epsilon^{-1/2}\|\mathbf{k}'\|_{\rm II}^{-1/2}&\|\mathbf{k}'\|_{\rm II}^{-1}\end{array}\right).\tag{I.3.19}$$

Remark: Using the explicit expression for $\hat A^{-1}(\mathbf{p}^\omega_{F,0}+\mathbf{k}')$ obtained by applying proposition I.A2.1 (see appendix I.A2), one can show that the error term on the right side of (I.3.18) can be improved to $O(\epsilon^{-1}\|\mathbf{k}'\|_{\rm II},\epsilon^{3/2}\|\mathbf{k}'\|_{\rm II}^{-1/2}))$. Since we will not need this improved bound in the following, we do not belabor further details.

4 - Intermediate regime. In order to derive (I.3.18), we assumed that $\|\mathbf{k}'\|_{\rm II}<\bar\kappa_1\epsilon$ with $\bar\kappa_1$ small enough. In the intermediate regime defined by $\bar\kappa_1\epsilon<\|\mathbf{k}'\|_{\rm II}$ and $\|\mathbf{k}'\|_{\rm I}<\kappa_1\epsilon$, we have that $\|\mathbf{k}'\|_{\rm I}\sim\|\mathbf{k}'\|_{\rm II}\sim\epsilon$ (given two positive functions $a(\epsilon)$ and $b(\epsilon)$, the symbol $a\sim b$ stands for $cb\leqslant a\leqslant Cb$ for some universal constants $C>c>0$). Moreover,

$$\det\hat A(\mathbf{p}^\omega_{F,0}+\mathbf{k}')=(k_0^2+|\xi|^2)^2+\gamma_1^2k_0^2+O(\epsilon^5)\tag{I.3.20}$$

therefore $|\det\hat A|>(\mathrm{const.})\ \epsilon^4$ and

$$|\hat A^{-1}(\mathbf{p}^\omega_{F,0},\mathbf{k}')|\leqslant(\mathrm{const.})\ \epsilon^{-1}\tag{I.3.21}$$

which is identical to the bound at the end of the first regime and at the beginning of the second.

5 - Third regime. We now probe deeper, beyond the second regime, and assume that $\|\mathbf{k}'\|_{\rm II}\leqslant\kappa_2\epsilon^3$. Since we will now investigate the regime in which $|k'|<(\mathrm{const.})\ \epsilon^2$, we will need to consider all the Fermi points $p^\omega_{F,j}$ with $j\in\{0,1,2,3\}$.

5-1 - Around $\mathbf{p}^\omega_{F,0}$ We start with the neighborhood of $\mathbf{p}^\omega_{F,0}$:

$$\det\hat A(\mathbf{p}^\omega_{F,0}+\mathbf{k}')=\gamma_1^2(k_0^2+\gamma_3^2|\xi|^2)+O(\epsilon^{-1}\|\mathbf{k}'\|_{\rm III}^3)\tag{I.3.22}$$

where

$$\|\mathbf{k}'\|_{\rm III}:=\sqrt{k_0^2+\gamma_3^2|\xi|^2}.\tag{I.3.23}$$

The third regime around $\mathbf{p}^\omega_{F,0}$ is defined by

$$\|\mathbf{k}'\|_{\rm III}<\bar\kappa_2\epsilon^3\tag{I.3.24}$$

for some $\bar\kappa_2<\kappa_2$. The computation of the dominating part of $\hat A^{-1}$ in this regime around $\mathbf{p}_{F,0}^\omega$ is similar to that in the second regime, but for the fact that we only neglect the terms of order $|k'|^2$ in $\hat A$ as well as the elements $\hat A_{aa}$ and $\hat A_{\tilde b\tilde b}$. In addition, the terms that are of order $\epsilon^{-3}\|\mathbf{k}'\|_{\mathrm{III}}^2$ that come out of the computation of the dominating part of $\hat A$ in block-diagonal form are also put into the error term. We thus find

$$\hat A^{-1}(\mathbf{k})=\left(\begin{array}{cc}\mathbb{1}&M_{\mathrm{III},0}(\mathbf{k})^\dagger\\0&\mathbb{1}\end{array}\right)\left(\begin{array}{cc}a_{\mathrm{III},0}^{(M)}&0\\0&a_{\mathrm{III},0}^{(m)}(\mathbf{k})\end{array}\right)\left(\begin{array}{cc}\mathbb{1}&0\\M_{\mathrm{III},0}(\mathbf{k})&\mathbb{1}\end{array}\right)(\mathbb{1}+O(\epsilon^{-3}\|\mathbf{k}'\|_{\mathrm{III}}))\tag{I.3.25}$$

where

$$a_{\mathrm{III},0}^{(M)}:=-\left(\begin{array}{cc}0&\gamma_1^{-1}\\\gamma_1^{-1}&0\end{array}\right),\quad a_{\mathrm{III},0}^{(m)}(\mathbf{p}_{F,0}^\omega+\mathbf{k}'):=-\frac{1}{k_0^2+\gamma_3^2|\xi|^2}\left(\begin{array}{cc}-ik_0&\gamma_3\xi\\\gamma_3\xi^*&-ik_0\end{array}\right)\tag{I.3.26}$$

and

$$M_{\mathrm{III},0}(\mathbf{p}_{F,0}^\omega+\mathbf{k}'):=-\frac{1}{\gamma_1}\left(\begin{array}{cc}\xi^*&0\\0&\xi\end{array}\right)\tag{I.3.27}$$

and

$$|\hat A^{-1}(\mathbf{p}_{F,0}^\omega+\mathbf{k}')|\leqslant(\text{const.})\left(\begin{array}{cc}\epsilon^{-1}&\epsilon^{-2}\\\epsilon^{-2}&\|\mathbf{k}'\|_{\mathrm{III}}^{-1}\end{array}\right).\tag{I.3.28}$$

5-2 - Around $\mathbf{p}_{F,1}^\omega$ We now discuss the neighborhood of $\mathbf{p}_{F,1}^\omega$. We define $k_1':=k-p_{F,1}^\omega=(k_{1,x}',k_{1,y}')$ and $\mathbf{k}_1':=(k_0,k_1')$. We have

$$\Omega(p_{F,1}^\omega+k_1')=\gamma_1\gamma_3+\xi_1+O(\epsilon^2|k_1'|)\tag{I.3.29}$$

where

$$\xi_1:=\frac{3}{2}(ik_{1,x}'+\omega k_{1,y}').$$

Using (I.A2.2) and (I.A2.4), we obtain

$$\det\hat A(\mathbf{p}_{F,1}^\omega+\mathbf{k}_1')=\gamma_1^2k_0^2+|\Omega^2-\gamma_1\gamma_3\Omega^*e^{-3ik_{1,x}'}|^2+O(\epsilon^4|k_0|^2)\tag{I.3.30}$$

where Ω is evaluated at $p_{F,1}^\omega+k_1'$. Injecting (I.3.29) into this equation, we find

$$\det\hat A(\mathbf{p}_{F,1}^\omega+\mathbf{k}_1')=\gamma_1^2(k_0^2+\gamma_3^2|x_1|^2)+O(\epsilon^4\|\mathbf{k}_1'\|_{\mathrm{III}}^2,\epsilon^{-1}\|\mathbf{k}_1'\|_{\mathrm{III}}^3)\tag{I.3.31}$$

where

$$x_1:=\frac{3}{2}(3ik_{1,x}'+\omega k_{1,y}').$$

The third regime around $p_{F,1}^\omega$ is therefore defined by

$$\|\mathbf{k}_1'\|_{\mathrm{III}}<\bar\kappa_2\ \epsilon^3$$

where $\bar\kappa_2$ can be assumed to be the same as in (I.3.24) without loss of generality. The dominating part of $\hat A^{-1}$ in this regime around $\mathbf{p}_{F,1}^\omega$ is similar to that around $\mathbf{p}_{F,0}^\omega$, except that we neglect the terms of order ϵ^2k_1' in $\hat A$ as well as the elements $\hat A_{aa}$ and $\hat A_{\tilde b\tilde b}$. As around $\mathbf{p}_{F,0}^\omega$, the terms of order $\epsilon^{-3}\|\mathbf{k}_1'\|_{\mathrm{III}}^2$ are put into the error term. We thus find

$$\hat A^{-1}(\mathbf{k})=\left(\begin{array}{cc}\mathbb{1}&M_{\mathrm{III},1}(\mathbf{k})^\dagger\\0&\mathbb{1}\end{array}\right)\left(\begin{array}{cc}a_{\mathrm{III},1}^{(M)}&0\\0&a_{\mathrm{III},1}^{(m)}(\mathbf{k})\end{array}\right)\left(\begin{array}{cc}\mathbb{1}&0\\M_{\mathrm{III},1}(\mathbf{k})&\mathbb{1}\end{array}\right)\cdot\\ \cdot(\mathbb{1}+O(\epsilon,\epsilon^{-3}\|\mathbf{k}'\|_{\mathrm{III}}))\tag{I.3.32}$$

where

$$a^{(M)}_{\mathrm{III},1} := -\left(\begin{array}{cc} 0 & \gamma_1^{-1} \\ \gamma_1^{-1} & 0 \end{array}\right), \quad a^{(m)}_{\mathrm{III},1}(\mathbf{p}^{\omega}_{F,1}+\mathbf{k}') := \frac{1}{k_0^2+\gamma_3^2|x_1|^2}\left(\begin{array}{cc} ik_0 & \gamma_3 x_1^* \\ \gamma_3 x_1 & ik_0 \end{array}\right) \tag{I.3.33}$$

and

$$M_{\mathrm{III},1}(\mathbf{p}^{\omega}_{F,1}+\mathbf{k}'_1) := -\gamma_3\mathbb{1} - \frac{1}{\gamma_1}\left(\begin{array}{cc} \xi_1^* & 0 \\ 0 & \xi_1 \end{array}\right) \tag{I.3.34}$$

and

$$|\hat{A}^{-1}(\mathbf{p}^{\omega}_{F,1}+\mathbf{k}'_1)| \leqslant (\mathrm{const.})\left(\begin{array}{cc} \epsilon^2\|\mathbf{k}'_1\|_{\mathrm{III}}^{-1} & \epsilon\|\mathbf{k}'_1\|_{\mathrm{III}}^{-1} \\ \epsilon\|\mathbf{k}'_1\|_{\mathrm{III}}^{-1} & \|\mathbf{k}'_1\|_{\mathrm{III}}^{-1} \end{array}\right). \tag{I.3.35}$$

5-3 - Around $\mathbf{p}^{\omega}_{F,j}$ The behavior of $\hat{g}(\mathbf{k})$ around $p^{\omega}_{F,j}$ for $j\in\{2,3\}$ can be deduced from (I.3.32) by using the symmetry (I.2.33) under $2\pi/3$ rotations: if we define $k'_j := k - p^{\omega}_{F,j} = (k'_{j,x}, k'_{j,y})$, $\mathbf{k}'_j := (k_0, k'_j)$ then, for $j = 2,3$ and $\omega\pm$,

$$\hat{A}^{-1}(\mathbf{k}'_j+\mathbf{p}^{\omega}_{F,j}) = \left(\begin{array}{cc} \mathbb{1} & 0 \\ 0 & \mathcal{T}_{T\mathbf{k}'_j+\mathbf{p}^{\omega}_{F,j-\omega}} \end{array}\right)\hat{A}^{-1}(T\mathbf{k}'_j+\mathbf{p}^{\omega}_{F,j-\omega})\left(\begin{array}{cc} \mathbb{1} & 0 \\ 0 & \mathcal{T}^{\dagger}_{T\mathbf{k}'_j+\mathbf{p}^{\omega}_{F,j-\omega}} \end{array}\right) \tag{I.3.36}$$

where T and $\mathcal{T}_{\mathbf{k}}$ were defined above (I.2.33), and $\mathbf{p}^{-}_{F,4}\equiv\mathbf{p}^{-}_{F,1}$. In addition, if $\mathbf{k}'_2$ and $\mathbf{k}'_3$ are in the third regime, then $\mathcal{T}_{T\mathbf{k}'_j+\mathbf{p}^{\omega}_{F,j}} = e^{-i\omega\frac{2\pi}{3}\sigma_3}+O(\epsilon^2)$.

6 - Intermediate regime. We are left with an intermediate regime between the second and third regimes, defined by

$$\bar{\kappa}_2\epsilon^3 < \|\mathbf{k}'\|_{\mathrm{III}}\ , \quad \|\mathbf{k}'\|_{\mathrm{II}} < \kappa_2\epsilon^3 \quad \text{and} \quad \bar{\kappa}_2\epsilon^3 < \|\mathbf{k}'_j\|_{\mathrm{III}},\ \forall j\in\{1,2,3\}, \tag{I.3.37}$$

which implies

$$\|\mathbf{k}'\|_{\mathrm{III}} \sim \|\mathbf{k}'\|_{\mathrm{II}} \sim \|\mathbf{k}'_j\|_{\mathrm{III}} \sim \epsilon^3$$

and

$$\det\hat{A}(\mathbf{p}^{\omega}_{F,0}+\mathbf{k}') = \gamma_1^2k_0^2 + \left|\xi^2-\gamma_1\gamma_3\xi^*\right|^2 + O(\epsilon^{10}). \tag{I.3.38}$$

One can prove (see appendix I.A4) that injecting (I.3.37) into (I.3.38) implies that $|\det\hat{A}| \geqslant (\mathrm{const.})\ \epsilon^8$, which in turn implies that

$$|\hat{A}^{-1}(\mathbf{p}^{\omega}_{F,0}+\mathbf{k}')| \leqslant (\mathrm{const.})\left(\begin{array}{cc} \epsilon^{-1} & \epsilon^{-2} \\ \epsilon^{-2} & \epsilon^{-3} \end{array}\right) \tag{I.3.39}$$

which is identical to the bound at the end of the second regime and at the beginning of the third.

7 - Summary. Let us briefly summarize this sub-section: we defined the norms

$$\|\mathbf{k}'\|_{\mathrm{I}} := \sqrt{k_0^2+|\xi|^2}, \quad \|\mathbf{k}'\|_{\mathrm{II}} := \sqrt{k_0^2+\frac{|\xi^4|}{\gamma_1^2}}, \quad \|\mathbf{k}'\|_{\mathrm{III}} := \sqrt{k_0^2+\gamma_3^2|\xi|^2}, \tag{I.3.40}$$

and identified an *ultraviolet* regime and three *infrared* regimes in which the free propagator $\hat{g}(\mathbf{k})$ behaves differently:

- for $\|\mathbf{k}'\|_{\mathrm{I}} > \bar{\kappa}_0$, (I.3.10) holds.
- for $\kappa_1\epsilon < \|\mathbf{k}'\|_{\mathrm{I}} < \bar{\kappa}_0$, (I.3.8) holds.
- for $\kappa_2\epsilon^3 < \|\mathbf{k}'\|_{\mathrm{II}} < \bar{\kappa}_1\epsilon$, (I.3.18) holds.
- for $\|\mathbf{k}'\|_{\mathrm{III}} < \bar{\kappa}_2\epsilon^3$, (I.3.25) holds, for $\|\mathbf{k}'_1\|_{\mathrm{III}} < \bar{\kappa}_2\epsilon^3$, (I.3.32) holds, and similarly for the $j = 2,3$ cases.

I.4. Multiscale integration scheme

In this section, we describe the scheme that will be followed in order to compute the right side of (I.2.26). We will first define a *multiscale decomposition* in each regime which will play an essential role in showing that the formal series in (I.2.26) converges. In doing so, we will define *effective* interactions and propagators, which will be defined in **k**-space, but since we wish to use the determinant expansion in lemma I.2.1 to compute and bound the effective truncated expectations, we will have to define the effective quantities in **x**-space as well. Once this is done, we will write bounds for the propagator in terms of scales.

I.4.1. Multiscale decomposition

We will now discuss the scheme we will follow to compute the Gaussian Grassmann integrals in terms of which the free energy and two-point Schwinger function were expressed in (I.2.19) and (I.2.24). The main idea is to decompose them into scales, and compute them one scale at a time. The result of the integration over one scale will then be considered as an *effective theory* for the remaining ones.

Throughout this section, we will use a smooth cutoff function $\chi_0(\rho)$, which returns 1 for $\rho \leqslant 1/3$ and 0 for $\rho \geqslant 2/3$.

1 - Ultraviolet regime. Let $\bar{\mathfrak{h}}_0 := \lfloor \log_2(\bar\kappa_0) \rfloor$ (in which $\bar\kappa_0$ is the constant that appeared after (I.3.40) which defines the inferior bound of the ultraviolet regime). For $h \in \{\bar{\mathfrak{h}}_0, \cdots, M\}$ and $h' \in \{\bar{\mathfrak{h}}_0 + 1, \cdots, M\}$, we define

$$\begin{gathered} f_{\leqslant h'}(\mathbf{k}) := \chi_0(2^{-h'}|k_0|), \quad f_{\leqslant \bar{\mathfrak{h}}_0}(\mathbf{k}) := \textstyle\sum_{\omega=\pm} \chi_0(2^{-\bar{\mathfrak{h}}_0}\|\mathbf{k} - \mathbf{p}_{F,0}^{\omega}\|_{\mathrm{I}}), \\ f_{h'}(\mathbf{k}) := f_{\leqslant h'}(\mathbf{k}) - f_{\leqslant h'-1}(\mathbf{k}) \\ \mathcal{B}_{\beta,L}^{(\leqslant h)} := \mathcal{B}_{\beta,L} \cap \mathrm{supp} f_{\leqslant h}, \quad \mathcal{B}_{\beta,L}^{(h')} := \mathcal{B}_{\beta,L} \cap \mathrm{supp} f_{h'}, \end{gathered} \tag{I.4.1}$$

in which $\|\cdot\|_{\mathrm{I}}$ is the norm defined in (I.3.40). In addition, we define

$$\hat{g}_{h'}(\mathbf{k}) := f_{h'}(\mathbf{k})\hat{A}^{-1}(\mathbf{k}), \quad \hat{g}_{\leqslant h}(\mathbf{k}) := f_{\leqslant h}(\mathbf{k})\hat{A}^{-1}(\mathbf{k}) \tag{I.4.2}$$

so that, in particular,

$$\hat{g}_{\leqslant M}(\mathbf{k}) = \hat{g}_{\leqslant M-1}(\mathbf{k}) + \hat{g}_M(\mathbf{k}).$$

Furthermore, it follows from the addition property (I.2.16) that for all $h \in \{\bar{\mathfrak{h}}_0, \cdots, M-1\}$,

$$\left\{ \begin{aligned} &\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{V}(\psi)} = e^{-\beta|\Lambda|F_h} \int P_{\leqslant h}(d\psi^{(\leqslant h)})\ e^{-\mathcal{V}^{(h)}(\psi^{(\leqslant h)})} \\ &\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{W}(\psi,\hat{J}_{\mathbf{k},\underline{\alpha}})} = e^{-\beta|\Lambda|F_h} \int P_{\leqslant h}(d\psi^{(\leqslant h)})\ e^{-\mathcal{W}^{(h)}(\psi^{(\leqslant h)},\hat{J}_{\mathbf{k},\underline{\alpha}})} \end{aligned} \right. \tag{I.4.3}$$

where $P_{\leqslant h}(d\psi^{(\leqslant h)}) \equiv P_{\hat{g}_{\leqslant h}}(d\psi^{(\leqslant h)})$,

$$\begin{aligned} -\beta|\Lambda|F_h - \mathcal{V}^{(h)}(\psi^{(\leqslant h)}) &:= -\beta|\Lambda|F_{h+1} + \log \int P_{h+1}(d\psi^{(h+1)})\ e^{-\mathcal{V}^{(h+1)}(\psi^{(h+1)}+\psi^{(\leqslant h)})} \\ &= -\beta|\Lambda|F_{h+1} + \sum_{N=1}^{\infty} \frac{(-1)^N}{N!} \mathcal{E}_{h+1}^T(\mathcal{V}^{(h+1)}(\psi^{(h+1)} + \psi^{(\leqslant h)}); N) \end{aligned} \tag{I.4.4}$$

and

$$\begin{aligned} &-\beta|\Lambda|(F_h - F_{h+1}) - \mathcal{W}^{(h)}(\psi^{(\leqslant h)}, \hat{J}_{\mathbf{k},\underline{\alpha}}) \\ &\qquad := \sum_{N=1}^{\infty} \frac{(-1)^N}{N!} \mathcal{E}_{(h+1)}^T(\mathcal{W}^{(h+1)}(\psi^{(h+1)} + \psi^{(\leqslant h)}, \hat{J}_{\mathbf{k},\underline{\alpha}}); N) \end{aligned} \tag{I.4.5}$$

in which the induction is initialized by

$$\mathcal{V}^{(M)} := \mathcal{V}, \quad \mathcal{W}^{(M)} := \mathcal{W}, \quad F_M := 0.$$

2 - First regime. We now decompose the first regime into scales. The main difference with the ultraviolet regime is that we incorporate the quadratic part of the effective potential into the propagator at each step of the multiscale integration. This is necessary to get satisfactory bounds later on. The propagator will therefore be changed, or *dressed*, inductively at every scale, as discussed below.

Let $\mathfrak{h}_1 := \lceil \log_2(\kappa_1\epsilon)\rceil$ (in which κ_1 is the constant that appears after (I.3.40) which defines the inferior bound of the first regime), and $\|\cdot\|_{\mathrm{I}}$ be the norm defined in (I.3.40). We define for $h \in \{\mathfrak{h}_1, \cdots, \bar{\mathfrak{h}}_0\}$ and $h' \in \{\mathfrak{h}_1+1, \cdots, \bar{\mathfrak{h}}_0\}$,

$$\begin{array}{c} f_{\leqslant h,\omega}(\mathbf{k}) := \chi_0(2^{-h}\|\mathbf{k}-\mathbf{p}_{F,0}^{\omega}\|_{\mathrm{I}}), \quad f_{h',\omega}(\mathbf{k}) := f_{\leqslant h',\omega}(\mathbf{k}) - f_{\leqslant h'-1,\omega}(\mathbf{k}) \\ \mathcal{B}_{\beta,L}^{(\leqslant h,\omega)} := \mathcal{B}_{\beta,L} \cap \mathrm{supp} f_{\leqslant h,\omega}, \quad \mathcal{B}_{\beta,L}^{(h',\omega)} := \mathcal{B}_{\beta,L} \cap \mathrm{supp} f_{\leqslant h',\omega} \end{array} \tag{I.4.6}$$

and

$$\hat{g}_{h',\omega}(\mathbf{k}) := f_{h',\omega}(\mathbf{k})\hat{A}^{-1}(\mathbf{k}), \quad \hat{g}_{\leqslant h,\omega}(\mathbf{k}) := f_{\leqslant h,\omega}(\mathbf{k})\hat{A}^{-1}(\mathbf{k}). \tag{I.4.7}$$

For $h \in \{\mathfrak{h}_1, \cdots, \bar{\mathfrak{h}}_0-1\}$, we define

$$\begin{array}{l} -\beta|\Lambda|(F_h - F_{h+1}) - \mathcal{Q}^{(h)}(\psi^{(\leqslant h)}) - \bar{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)}) \\ \qquad\qquad := \displaystyle\sum_{N=1}^{\infty} \frac{(-1)^N}{N!} \bar{\mathcal{E}}_{h+1}^T(\bar{\mathcal{V}}^{(h+1)}(\psi^{(h+1)} + \psi^{(\leqslant h)}); N) \\ \mathcal{Q}^{(\bar{\mathfrak{h}}_0)}(\psi^{(\leqslant \bar{\mathfrak{h}}_0)}) + \bar{\mathcal{V}}^{(\bar{\mathfrak{h}}_0)}(\psi^{(\leqslant \bar{\mathfrak{h}}_0)}) := \mathcal{V}^{(\bar{\mathfrak{h}}_0)}(\psi^{(\leqslant \bar{\mathfrak{h}}_0)}) \end{array} \tag{I.4.8}$$

and

$$\begin{array}{l} -\beta|\Lambda|(F_h - F_{h+1}) - \mathcal{Q}^{(h)}(\psi^{(\leqslant h)}) - \bar{\mathcal{W}}^{(h)}(\psi^{(\leqslant h)}, \hat{J}_{\mathbf{k},\underline{\alpha}}) \\ \qquad\qquad := \displaystyle\sum_{N=1}^{\infty} \frac{(-1)^N}{N!} \bar{\mathcal{E}}_{h+1}^T(\bar{\mathcal{W}}^{(h+1)}(\psi^{(h+1)} + \psi^{(\leqslant h)}, \hat{J}_{\mathbf{k},\underline{\alpha}}); N) \\ \mathcal{Q}^{(\bar{\mathfrak{h}}_0)}(\psi^{(\leqslant \bar{\mathfrak{h}}_0)}) + \bar{\mathcal{W}}^{(\bar{\mathfrak{h}}_0)}(\psi^{(\leqslant \bar{\mathfrak{h}}_0)}, \hat{J}_{\mathbf{k},\underline{\alpha}}) := \mathcal{W}^{(\bar{\mathfrak{h}}_0)}(\psi^{(\leqslant \bar{\mathfrak{h}}_0)}, \hat{J}_{\mathbf{k},\underline{\alpha}}) \end{array} \tag{I.4.9}$$

in which $\mathcal{Q}^{(h)}$ is quadratic in the ψ, $\bar{\mathcal{V}}^{(h)}$ is at least quartic and $\bar{\mathcal{W}}^{(h)}$ has no terms that are both quadratic in ψ and constant in $\hat{J}_{\mathbf{k},\underline{\alpha}}$; and $\bar{\mathcal{E}}_{h+1}^T$ is the truncated expectation defined from the Gaussian measure $P_{\hat{\bar{g}}_{h+1,+}}(d\psi_+^{(h+1)})P_{\hat{\bar{g}}_{h+1,-}}(d\psi_-^{(h+1)})$; in which $\hat{\bar{g}}_{h+1,\omega}$ is the *dressed propagator* and is defined as follows. Let $\hat{W}_2^{(h)}(\mathbf{k})$ denote the *kernel* of $\mathcal{Q}^{(h)}$ i.e.

$$\mathcal{Q}^{(h)}(\psi^{(\leqslant h)}) =: \frac{1}{\beta|\Lambda|} \sum_{\omega,(\alpha,\alpha')} \sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(\leqslant h,\omega)}} \hat{\psi}_{\mathbf{k},\omega,\alpha}^{(\leqslant h)+} \hat{W}_{2,(\alpha,\alpha')}^{(h)}(\mathbf{k}) \hat{\psi}_{\mathbf{k},\omega,\alpha'}^{(\leqslant h)-} \tag{I.4.10}$$

(remark: the ω index in $\hat{\psi}_{\mathbf{k},\omega,\alpha}^{\pm}$ is redundant since given $\mathbf{k}$, it is defined as the unique ω that is such that $\mathbf{k} \in \mathcal{B}_{\beta,L}^{(\leqslant h,\omega)}$; it will however be needed when defining the $\mathbf{x}$-space counterpart of $\hat{\psi}_{\mathbf{k},\omega,\alpha}^{\pm}$ below). We define $\hat{\bar{g}}_{h,\omega}$ and $\hat{\bar{g}}_{\leqslant h,\omega}$ by induction: $\hat{\bar{g}}_{\leqslant \bar{\mathfrak{h}}_0,\omega}(\mathbf{k}) := (\hat{g}_{\leqslant \bar{\mathfrak{h}}_0,\omega}^{-1}(\mathbf{k}) + \hat{W}_2^{(\bar{\mathfrak{h}}_0)}(\mathbf{k}))^{-1}$ and, for $h \in \{\mathfrak{h}_1+1, \ldots, \bar{\mathfrak{h}}_0\}$,

$$\left\{ \begin{array}{l} \hat{\bar{g}}_{h,\omega}(\mathbf{k}) := f_{h,\omega}(\mathbf{k}) f_{\leqslant h,\omega}^{-1}(\mathbf{k}) \hat{\bar{g}}_{\leqslant h,\omega}(\mathbf{k}) \\ \left(\hat{\bar{g}}_{\leqslant h-1,\omega}(\mathbf{k})\right)^{-1} := f_{\leqslant h-1,\omega}^{-1}(\mathbf{k}) \left(\hat{\bar{g}}_{\leqslant h,\omega}(\mathbf{k})\right)^{-1} + \hat{W}_2^{(h-1)}(\mathbf{k}) \end{array} \right. \tag{I.4.11}$$

in which $f^{-1}_{\leqslant h,\omega}(\mathbf{k})$ is equal to $1/f_{\leqslant h,\omega}(\mathbf{k})$ if $f_{\leqslant h,\omega}(\mathbf{k})\neq 0$ and to 0 if not.

The dressed propagator is thus defined so that

$$\left\{\begin{array}{l}\displaystyle\int P_M(d\psi)\ e^{-\mathcal{V}(\psi)}=e^{-\beta|\Lambda|F_h}\int \bar P_{\leqslant h}(\psi^{(\leqslant h)})\ e^{-\bar{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)})}\\ \displaystyle\int P_M(d\psi)\ e^{-\mathcal{W}(\psi,\hat J_{\mathbf{k},\underline{\alpha}})}=e^{-\beta|\Lambda|F_h}\int \bar P_{\leqslant h}(\psi^{(\leqslant h)})\ e^{-\bar{\mathcal{W}}^{(h)}(\psi^{(\leqslant h)},\hat J_{\mathbf{k},\underline{\alpha}})}\end{array}\right. \tag{I.4.12}$$

in which $\bar P_{\leqslant h}\equiv P_{\hat{\bar g}_{\leqslant h,+}}(d\psi_+^{(\leqslant h)})P_{\hat{\bar g}_{\leqslant \underline{h},-}}(d\psi_-^{(\leqslant h)})$. Equation (I.4.11) can be expanded into a more explicit form: for $h'\in\{\mathfrak{h}_1+1,\ldots,\bar{\mathfrak{h}}_0\}$ and $h\in\{\mathfrak{h}_1,\cdots,\bar{\mathfrak{h}}_0\}$,

$$\hat{\bar g}_{h',\omega}(\mathbf{k})=f_{h',\omega}(\mathbf{k})\left(\hat{\bar A}_{h',\omega}(\mathbf{k})\right)^{-1},\quad \hat{\bar g}_{\leqslant h,\omega}(\mathbf{k})=f_{\leqslant h,\omega}\left(\hat{\bar A}_{h,\omega}(\mathbf{k})\right)^{-1} \tag{I.4.13}$$

where

$$\hat{\bar A}_{h,\omega}(\mathbf{k}):=\hat A(\mathbf{k})+f_{\leqslant h,\omega}(\mathbf{k})\hat W_2^{(h)}(\mathbf{k})+\sum_{h'=h+1}^{\bar{\mathfrak{h}}_0}\hat W_2^{(h')}(\mathbf{k}) \tag{I.4.14}$$

(in which the sum should be interpreted as zero if $h=\bar{\mathfrak{h}}_0$).

3 - Intermediate regime. We briefly discuss the intermediate region between regimes 1 and 2. We define

$$f_{\mathfrak{h}_1,\omega}(\mathbf{k}):=\chi_0(2^{-\mathfrak{h}_1}\|\mathbf{k}-\mathbf{p}_{F,0}^{\omega}\|_{\mathrm{I}})-\chi_0(2^{-\bar{\mathfrak{h}}_1}\|\mathbf{k}-\mathbf{p}_{F,0}^{\omega}\|_{\mathrm{II}})=:f_{\leqslant\mathfrak{h}_1,\omega}(\mathbf{k})-f_{\leqslant\bar{\mathfrak{h}}_1,\omega}(\mathbf{k}) \tag{I.4.15}$$

where $\bar{\mathfrak{h}}_1:=\lfloor\log_2(\bar\kappa_1\epsilon)\rfloor$, from which we define $\hat{\bar g}_{\mathfrak{h}_1,\omega}(\mathbf{k})$ and $\hat{\bar g}_{\leqslant\bar{\mathfrak{h}}_1,\omega}(\mathbf{k})$ in the same way as in (I.4.13) with

$$\hat{\bar A}_{\bar{\mathfrak{h}}_1,\omega}(\mathbf{k}):=\hat A(\mathbf{k})+f_{\leqslant\bar{\mathfrak{h}}_1,\omega}(\mathbf{k})\hat W_2^{(\bar{\mathfrak{h}}_1)}(\mathbf{k})+\sum_{h'=\mathfrak{h}_1}^{\bar{\mathfrak{h}}_0}\hat W_2^{(h')}(\mathbf{k}). \tag{I.4.16}$$

The analogue of (I.4.12) holds here as well.

4 - Second regime. We now define a multiscale decomposition for the integration in the second regime. Proceeding as we did in the first regime, we define $\mathfrak{h}_2:=\lceil\log_2(\kappa_2\epsilon^3)\rceil$, for $h\in\{\mathfrak{h}_2,\cdots,\bar{\mathfrak{h}}_1\}$ and $h'\in\{\mathfrak{h}_2+1,\cdots,\bar{\mathfrak{h}}_1\}$, we define

$$\begin{array}{c}f_{\leqslant h,\omega}(\mathbf{k}):=\chi_0(2^{-h}\|\mathbf{k}-\mathbf{p}_{F,0}^{\omega}\|_{\mathrm{II}}),\quad f_{h',\omega}(\mathbf{k}):=f_{\leqslant h',\omega}(\mathbf{k})-f_{\leqslant h'-1,\omega}(\mathbf{k})\\ \mathcal{B}_{\beta,L}^{(\leqslant h,\omega)}:=\mathcal{B}_{\beta,L}\cap\mathrm{supp}f_{\leqslant h,\omega},\quad \mathcal{B}_{\beta,L}^{(h',\omega)}:=\mathcal{B}_{\beta,L}\cap\mathrm{supp}f_{\leqslant h',\omega}.\end{array} \tag{I.4.17}$$

The analogues of (I.4.12), and (I.4.13) hold with

$$\hat{\bar A}_{h-1,\omega}(\mathbf{k}):=\hat A(\mathbf{k})+f_{\leqslant h-1,\omega}(\mathbf{k})\hat W_2^{(h-1)}(\mathbf{k})+\sum_{h'=h}^{\bar{\mathfrak{h}}_1}\hat W_2^{(h')}(\mathbf{k})+\sum_{h'=\mathfrak{h}_1}^{\bar{\mathfrak{h}}_0}\hat W_2^{(h')}(\mathbf{k}). \tag{I.4.18}$$

5 - Intermediate regime. The intermediate region between regimes 2 and 3 is defined in analogy with that between regimes 1 and 2: we let

$$\begin{array}{c}f_{\mathfrak{h}_2,\omega}(\mathbf{k}):=\chi_0(2^{-\mathfrak{h}_2}\|\mathbf{k}-\mathbf{p}_{F,0}^{\omega}\|_{\mathrm{II}})-\displaystyle\sum_{j\in\{0,1,2,3\}}\chi_0(2^{-\bar{\mathfrak{h}}_2}\|\mathbf{k}-\mathbf{p}_{F,j}^{\omega}\|_{\mathrm{III}})\\ f_{\leqslant\bar{\mathfrak{h}}_2,\omega,j}(\mathbf{k}):=\chi_0(2^{-\bar{\mathfrak{h}}_2}\|\mathbf{k}'_{\omega,j}\|_{\mathrm{III}})\end{array} \tag{I.4.19}$$

where $\bar{\mathfrak{h}}_2 := \lfloor \log_2(\bar\kappa_2\epsilon^3)\rfloor$ from which we define $\hat{\bar g}_{\mathfrak{h}_2,\omega}(\mathbf{k})$ and $\hat{\bar g}_{\leqslant\bar{\mathfrak{h}}_2,\omega}(\mathbf{k})$ in the same way as in (I.4.13) with

$$\hat{\bar A}_{\mathfrak{h}_2,\omega}(\mathbf{k}) := \hat A(\mathbf{k}) + f_{\leqslant\bar{\mathfrak{h}}_2,\omega}(\mathbf{k})\hat W_2^{(\bar{\mathfrak{h}}_2)}(\mathbf{k}) + \sum_{h'=\mathfrak{h}_2}^{\bar{\mathfrak{h}}_1}\hat W_2^{(h')}(\mathbf{k}) + \sum_{h'=\mathfrak{h}_1}^{\bar{\mathfrak{h}}_0}\hat W_2^{(h')}(\mathbf{k}). \tag{I.4.20}$$

The analogue of (I.4.12) holds here as well.

6 - Third regime. There is an extra subtlety in the third regime: we will see in section I.9 that the singularities of the dressed propagator are slightly different from those of the bare (i.e. non-interacting) propagator: at scale h the effective Fermi points $\mathbf{p}_{F,j}^{\omega}$ with $j=1,2,3$ are moved to $\tilde{\mathbf{p}}_{F,j}^{(\omega,h)}$, with

$$\|\tilde{\mathbf{p}}_{F,j}^{(\omega,h)} - \mathbf{p}_{F,j}^{\omega}\|_{\mathrm{III}} \leqslant (\mathrm{const.})\ |U|\epsilon^3. \tag{I.4.21}$$

The central Fermi points, $j=0$, are left invariant by the interaction. For notational uniformity we set $\tilde{\mathbf{p}}_{F,0}^{(\omega,h)} \equiv \mathbf{p}_{F,0}^{\omega}$. Keeping this in mind, we then proceed in a way reminiscent of the first and second regimes: let $\mathfrak{h}_\beta := \lfloor\log_2(\pi/\beta)\rfloor$, for $h\in\{\mathfrak{h}_\beta,\cdots,\bar{\mathfrak{h}}_2\}$ and $h'\in\{\mathfrak{h}_\beta+1,\cdots,\bar{\mathfrak{h}}_2\}$, we define

$$\begin{array}{c}
f_{\leqslant h,\omega,j}(\mathbf{k}) := \chi_0(2^{-h}\|\mathbf{k}-\tilde{\mathbf{p}}_{F,j}^{(\omega,h+1)}\|_{\mathrm{III}}),\quad f_{h',\omega,j}(\mathbf{k}) := f_{\leqslant h',\omega,j}(\mathbf{k}) - f_{\leqslant h'-1,\omega,j}(\mathbf{k})\\
\mathcal{B}_{\beta,L}^{(\leqslant h,\omega,j)} := \mathcal{B}_{\beta,L}\cap\mathrm{supp} f_{\leqslant h,\omega,j},\quad \mathcal{B}_{\beta,L}^{(h',\omega,j)} := \mathcal{B}_{\beta,L}\cap\mathrm{supp} f_{\leqslant h',\omega,j}
\end{array} \tag{I.4.22}$$

and the analogues of (I.4.12), and (I.4.13) hold with

$$\begin{aligned}\hat{\bar A}_{h-1,\omega,j}(\mathbf{k}) := {}& \hat A(\mathbf{k}) + f_{\leqslant h-1,\omega,j}(\mathbf{k})\hat W_2^{(h-1)}(\mathbf{k})\\
&+\sum_{h'=h}^{\bar{\mathfrak{h}}_2}\hat W_2^{(h')}(\mathbf{k}) + \sum_{h'=\mathfrak{h}_2}^{\bar{\mathfrak{h}}_1}\hat W_2^{(h')}(\mathbf{k}) + \sum_{h'=\mathfrak{h}_1}^{\bar{\mathfrak{h}}_0}\hat W_2^{(h')}(\mathbf{k}).\end{aligned} \tag{I.4.23}$$

7 - Last scale. Recalling that $|k_0|\geqslant\pi/\beta$, the smallest possible scale is $\mathfrak{h}_\beta := \lfloor\log_2(\pi/\beta)\rfloor$. The last integration is therefore that on scale $h=\mathfrak{h}_\beta+1$, after which, we are left with

$$\left\{\begin{array}{l}
\displaystyle\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{V}(\psi)} = e^{-\beta|\Lambda|F_{\mathfrak{h}_\beta}}\\[2ex]
\displaystyle\int P_{\leqslant M}(d\psi)\ e^{-\mathcal{W}(\psi,\hat J_{\mathbf{k},\underline{\alpha}})} = e^{-\beta|\Lambda|F_{\mathfrak{h}_\beta}}e^{-\mathcal{W}^{(\mathfrak{h}_\beta)}(\hat J_{\mathbf{k},\underline{\alpha}})}.
\end{array}\right. \tag{I.4.24}$$

The subsequent sections are dedicated to the proof of the fact that both $F_{\mathfrak{h}_\beta}$ and $\mathcal{W}^{(\mathfrak{h}_\beta)}$ are analytic in U, uniformly in L, β and ϵ. We will do this by studying each regime, one at a time, performing a *tree expansion* in each of them in order to bound the terms of the series (see section I.5 and following).

I.4.2. x-space representation of the effective potentials

We will compute the truncated expectations arising in (I.4.4), (I.4.5), (I.4.8) and (I.4.9) using a determinant expansion (see lemma I.2.1) which, as was mentioned above, is only useful if the propagator and effective potential are expressed in **x**-space. We will discuss their definition in this section. We restrict our attention to the effective potentials $\mathcal{V}^{(h)}$ since, in order to compute the two-point Schwinger function in the regimes we are interested in, we will not need to express the kernels of $\mathcal{W}^{(h)}$ in **x**-space.

1 - Ultraviolet regime. We first discuss the ultraviolet regime, which differs from the others in that the propagator does not depend on the index ω. We write $\mathcal{V}^{(h)}$ in terms of its *kernels* (anti-symmetric in the exchange of their indices), defined as

$$\mathcal{V}^{(h)}(\psi^{(\leqslant h)}) =: \sum_{l=1}^{\infty} \frac{1}{(\beta|\Lambda|)^{2l-1}} \sum_{\underline{\alpha}=(\alpha_1,\cdots,\alpha_{2l})} \sum_{\substack{(\mathbf{k}_1,\cdots,\mathbf{k}_{2l})\in\mathcal{B}_{\beta,L}^{(\leqslant h)2l}\\ \mathbf{k}_1-\mathbf{k}_2+\cdots+\mathbf{k}_{2l-1}-\mathbf{k}_{2l}=0}} \hat{W}_{2l,\underline{\alpha}}^{(h)}(\mathbf{k}_1,\cdots,\mathbf{k}_{2l-1})\cdot \\ \cdot\hat{\psi}_{\mathbf{k}_1,\alpha_1}^{(\leqslant h)+}\hat{\psi}_{\mathbf{k}_2,\alpha_2}^{(\leqslant h)-}\cdots\hat{\psi}_{\mathbf{k}_{2l-1},\alpha_{2l-1}}^{(\leqslant h)+}\hat{\psi}_{\mathbf{k}_{2l},\alpha_{2l}}^{(\leqslant h)-}. \tag{I.4.25}$$

The $\mathbf{x}$-space expression for $\hat{\psi}_{\mathbf{k},\alpha}^{(\leqslant h)\pm}$ is defined as

$$\psi_{\mathbf{x},\alpha}^{(\leqslant h)\pm} := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(\leqslant h)}} e^{\pm i\mathbf{k}\cdot\mathbf{x}}\hat{\psi}_{\mathbf{k},\alpha}^{(\leqslant h)\pm} \tag{I.4.26}$$

so that the propagator's formulation in $\mathbf{x}$-space is

$$g_h(\mathbf{x}-\mathbf{y}) := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(\leqslant h)}} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}\hat{g}_h(\mathbf{k}) \tag{I.4.27}$$

and similarly for $g_{\leqslant h}$, and the effective potential (I.4.25) becomes

$$\mathcal{V}^{(h)}(\psi^{(\leqslant h)}) = \sum_{l=1}^{\infty}\sum_{\underline{\alpha}}\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}\; W_{2l,\underline{\alpha}}^{(h)}(\mathbf{x}_1-\mathbf{x}_{2l},\cdots,\mathbf{x}_{2l-1}-\mathbf{x}_{2l})\cdot \\ \cdot\psi_{\mathbf{x}_1,\alpha_1}^{(\leqslant h)+}\psi_{\mathbf{x}_2,\alpha_2}^{(\leqslant h)-}\cdots\psi_{\mathbf{x}_{2l-1},\alpha_{2l-1}}^{(\leqslant h)+}\psi_{\mathbf{x}_{2l},\alpha_{2l}}^{(\leqslant h)-} \tag{I.4.28}$$

with

$$W_{2l,\underline{\alpha}}^{(h)}(\mathbf{u}_1,\cdots,\mathbf{u}_{2l-1}) := \frac{1}{(\beta|\Lambda|)^{2l-1}}\sum_{(\mathbf{k}_1,\cdots,\mathbf{k}_{2l-1})\in\mathcal{B}_{\beta,L}^{2l-1}} e^{i(\sum_{i=1}^{2l-1}(-1)^i\mathbf{k}_i\cdot\mathbf{u}_i)}\hat{W}_{2l,\underline{\alpha}}^{(h)}(\mathbf{k}_1,\cdots,\mathbf{k}_{2l-1}). \tag{I.4.29}$$

Remark: From (I.4.25), $\hat{W}_{2l,\underline{\alpha}}^{(h)}(\underline{\mathbf{k}})$ is not defined for $\mathbf{k}_i\notin\mathcal{B}_{\beta,L}^{(\leqslant h)}$, however, one can easily check that (I.4.29) holds for any extension of $\hat{W}_{2l,\underline{\alpha}}^{(h)}$ to $\mathcal{B}_{\beta,L}^{2l-1}$, thanks to the compact support properties of $\psi^{(\leqslant h)}$ in momentum space. In order to get satisfactory bounds on $W_{2l,\underline{\alpha}}^{(h)}(\underline{\mathbf{x}})$, that is in order to avoid Gibbs phenomena, we define the extension of $\hat{W}_{2l,\underline{\alpha}}^{(h)}(\underline{\mathbf{k}})$ similarly to (I.4.25) by relaxing the condition that $\psi^{(\leqslant h)}$ is supported on $\mathcal{B}_{\beta,L}^{(\leqslant h)}$ and iterating (I.4.4). In other words, we let $\hat{W}_{2l,\underline{\alpha}}^{(h)}(\underline{\mathbf{k}})$ for $\underline{\mathbf{k}}\in\mathcal{B}_{\beta,L}^{2l-1}$ be the kernels of $\mathcal{V}^{*(h)}$ defined inductively by

$$-\beta|\Lambda|\mathfrak{e}_h - \mathcal{V}^{*(h)}(\Psi) := \sum_{N=1}^{\infty}\frac{(-1)^N}{N!}\mathcal{E}_{h+1}^T(\mathcal{V}^{*(h+1)}(\psi^{(h+1)}+\Psi);N) \tag{I.4.30}$$

in which $\{\hat{\Psi}_{\mathbf{k},\alpha}\}_{\mathbf{k}\in\mathcal{B}_{\beta,L},\alpha\in\mathcal{A}}$ is a collection of *external fields* (in reference to the fact that, contrary to $\psi^{(\leqslant h)}$, they have a non-compact support in momentum space). The use of this specific extension can be justified *ab-initio* by re-defining the cutoff function χ in such a way that its support is $\mathbb{R}$, e.g. using exponential tails that depend on a parameter ϵ_χ in such a way that the support tends to be compact as ϵ_χ goes to 0. Following this logic, we could first define $\hat{W}$ using the non-compactly supported cutoff function and then take the $\epsilon_\chi\to 0$ limit, thus recovering (I.4.30). Such an approach is dicussed in [BM02]. From now on, with some abuse of notation, we shall identify $\mathcal{V}^{*(h)}$ with $\mathcal{V}^{(h)}$ and denote them by the same symbol $\mathcal{V}^{(h)}$, which is justified by the fact that their kernels are (or can be chosen, from what said above, to be) the same.

2 - First and second regimes. We now discuss the first and second regimes (the third regime is very slightly different in that the index ω is complemented by an extra index j and the Fermi points are shifted). Similarly to (I.4.25), we define the *kernels* of $\bar{\mathcal{V}}$:

$$\bar{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)}) =: \sum_{l=2}^{\infty} \frac{1}{(\beta|\Lambda|)^{2l-1}} \sum_{\underline{\omega},\underline{\alpha}} \sum_{\substack{(\mathbf{k}_1,\cdots,\mathbf{k}_{2l})\in\mathcal{B}_{\beta,L}^{(\leqslant h,\underline{\omega})}\\ \mathbf{k}_1-\mathbf{k}_2+\cdots+\mathbf{k}_{2l-1}-\mathbf{k}_{2l}=0}} \hat{W}_{2l,\underline{\alpha}}^{(h)}(\mathbf{k}_1,\cdots,\mathbf{k}_{2l-1})\cdot \cdot\hat{\psi}^{(\leqslant h)+}_{\mathbf{k}_1,\alpha_1,\omega_1}\hat{\psi}^{(\leqslant h)-}_{\mathbf{k}_2,\alpha_2,\omega_2}\cdots\hat{\psi}^{(\leqslant h)+}_{\mathbf{k}_{2l-1},\alpha_{2l-1},\omega_{2l-1}}\hat{\psi}^{(\leqslant h)-}_{\mathbf{k}_{2l},\alpha_{2l},\omega_{2l}}. \tag{I.4.31}$$

where $\mathcal{B}_{\beta,L}^{(\leqslant h,\underline{\omega})} = \mathcal{B}_{\beta,L}^{(\leqslant h,\omega_1)}\times\cdots\times\mathcal{B}_{\beta,L}^{(\leqslant h,\omega_{2l})}$. Note that the kernel $\hat{W}_{2l,\underline{\alpha}}^{(h)}$ is independent of $\underline{\omega}$, which can be easily proved using the symmetry $\omega_i\mapsto-\omega_i$. The $\mathbf{x}$-space expression for $\hat{\psi}^{(\leqslant h)\pm}_{\mathbf{k},\alpha,\omega}$ is

$$\psi^{(\leqslant h)\pm}_{\mathbf{x},\alpha,\omega} := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(\leqslant h,\omega)}} e^{\pm i(\mathbf{k}-\mathbf{p}_{F,0}^{\omega})\cdot\mathbf{x}}\hat{\psi}^{(\leqslant h)\pm}_{\mathbf{k},\alpha,\omega}. \tag{I.4.32}$$

Remark: Unlike $\hat{\psi}_{\mathbf{k},\alpha,\omega}$, the ω index in $\psi^{(\leqslant h)\pm}_{\mathbf{x},\alpha,\omega}$ is *not* redundant. Keeping track of this dependence is required to prove properties of $\hat{W}_{2l}(\mathbf{k})$ and $\hat{\bar{g}}_h(\mathbf{k})$ close to $\mathbf{p}_{F,0}^{\omega}$ while working in $\mathbf{x}$-space. Such considerations were first discussed in [BG90] in which $\psi_{\mathbf{x},\alpha,\omega}$ were called *quasi-particle* fields.

We then define the propagator in $\mathbf{x}$-space:

$$\hat{g}_{h,\omega}(\mathbf{x}-\mathbf{y}) := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(\leqslant h,\omega)}} e^{i(\mathbf{k}-\mathbf{p}_{F,0}^{\omega})\cdot(\mathbf{x}-\mathbf{y})}\hat{\bar{g}}_{h,\omega}(\mathbf{k}) \tag{I.4.33}$$

and similarly for $\bar{g}_{\leqslant h,\omega}$, and the effective potential (I.4.31) becomes

$$\bar{\mathcal{V}}^{(h)}(\psi^{(\leqslant h)}) = \sum_{l=2}^{\infty}\sum_{\underline{\omega},\underline{\alpha}}\frac{1}{\beta|\Lambda|}\int d\mathbf{x}_1\cdots d\mathbf{x}_{2l}\ W_{2l,\underline{\alpha},\underline{\omega}}^{(h)}(\mathbf{x}_1-\mathbf{x}_{2l},\cdots,\mathbf{x}_{2l-1}-\mathbf{x}_{2l})\cdot \cdot\psi^{(\leqslant h)+}_{\mathbf{x}_1,\alpha_1,\omega_1}\psi^{(\leqslant h)-}_{\mathbf{x}_2,\alpha_2,\omega_2}\cdots\psi^{(\leqslant h)+}_{\mathbf{x}_{2l-1},\alpha_{2l-1},\omega_{2l-1}}\psi^{(\leqslant h)-}_{\mathbf{x}_{2l},\alpha_{2l},\omega_{2l}} \tag{I.4.34}$$

and

$$\mathcal{Q}^{(h)}(\psi^{(\leqslant h)}) = \sum_{\omega,(\alpha,\alpha')}\int d\mathbf{x}d\mathbf{y}\ \psi^{(\leqslant h)+}_{\mathbf{x},\omega,\alpha}W^{(h)}_{2,\omega,(\alpha,\alpha')}(\mathbf{x}-\mathbf{y})\psi^{(\leqslant h)-}_{\mathbf{y},\omega,\alpha'} \tag{I.4.35}$$

in which

$$W_{2l,\underline{\alpha},\underline{\omega}}^{(h)}(\mathbf{u}_1,\cdots,\mathbf{u}_{2l-1}) := \frac{\delta_{0,\sum_{j=1}^{2l}(-1)^j\mathbf{p}_{F,0}^{\omega_j}}}{(\beta|\Lambda|)^{2l-1}}\sum_{(\mathbf{k}_1,\cdots,\mathbf{k}_{2l-1})\in\mathcal{B}_{\beta,L}^{2l-1}} e^{i(\sum_{j=1}^{2l-1}(-1)^j(\mathbf{k}_j-\mathbf{p}_{F,0}^{\omega_j})\cdot\mathbf{u}_j)}\hat{W}_{2l,\underline{\alpha}}^{(h)}(\mathbf{k}_1,\cdots,\mathbf{k}_{2l-1}). \tag{I.4.36}$$

As in the ultraviolet, the definition of $\hat{W}_{2l,\underline{\alpha}}^{(h)}(\underline{\mathbf{k}})$ is extended to $\mathcal{B}_{\beta,L}^{2l-1}$ by defining it as the kernel of $\mathcal{V}^{*(h)}$:

$$-\beta|\Lambda|\mathfrak{e}_h-\mathcal{V}^{*(h)}(\Psi) := \sum_{N=1}^{\infty}\frac{(-1)^N}{N!}\bar{\mathcal{E}}^T_{h+1}(\mathcal{V}^{*(h+1)}(\psi^{(h+1)}+\Psi);N) \tag{I.4.37}$$

in which $\{\hat{\Psi}_{\mathbf{k},\alpha}\}_{\mathbf{k}\in\mathcal{B}_{\beta,L},\alpha\in\mathcal{A}}$ is a collection of *external fields*. The definition (I.4.36) suggests a definition for $\bar{A}_{h,\omega}$ (see (I.4.14) and (I.4.18)):

$$\bar{A}_{h,\omega}(\mathbf{x}) := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}} e^{i(\mathbf{k}-\mathbf{p}_{F,0}^{\omega})\cdot\mathbf{x}}\hat{\bar{A}}_{h,\omega}(\mathbf{k}). \tag{I.4.38}$$

3 - Third regime. We now turn our attention to the third regime. As discussed in section I.4.1, in addition to there being an extra index j, the Fermi points are also shifted in the third regime. The kernels of $\bar{\mathcal{V}}$ and $\mathcal{Q}$ are defined as in (I.4.31), but with ω replaced by (ω, j). The $\mathbf{x}$-space representation of $\hat{\psi}^{(\leqslant h)\pm}_{\mathbf{k},\alpha,\omega,j}$ is defined as

$$\psi^{(\leqslant h)\pm}_{\mathbf{x},\alpha,\omega,j} := \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^{(\leqslant h,\omega,j)}_{\beta,L}} e^{\pm i(\mathbf{k}-\tilde{\mathbf{p}}^{(\omega,h)}_{F,j})\cdot\mathbf{x}} \hat{\psi}^{(\leqslant h)\pm}_{\mathbf{k},\alpha,\omega,j} \tag{I.4.39}$$

and the $\mathbf{x}$-space expression of the propagator and the kernels of $\bar{\mathcal{V}}$ and $\mathcal{Q}$ are defined by analogy with the first regime:

$$\hat{g}_{h,\omega,j}(\mathbf{x}-\mathbf{y}) := \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^{(\leqslant h,\omega,j)}_{\beta,L}} e^{i(\mathbf{k}-\tilde{\mathbf{p}}^{(\omega,h)}_{F,j})\cdot(\mathbf{x}-\mathbf{y})} \hat{\bar{g}}_{h,\omega,j}(\mathbf{k}) \tag{I.4.40}$$

and

$$\begin{aligned} W^{(h)}_{2l,\underline{\alpha},\underline{\omega},\underline{j}}(\mathbf{u}_1,\cdots,\mathbf{u}_{2l-1}) :=& \frac{\delta_{0,\sum_{n=1}^{2l}(-1)^n \tilde{\mathbf{p}}^{(h,\omega_n)}_{F,j}}}{(\beta|\Lambda|)^{2l-1}} \cdot \\ &\cdot \sum_{(\mathbf{k}_1,\cdots,\mathbf{k}_{2l-1})\in\mathcal{B}^{2l-1}_{\beta,L}} e^{i(\sum_{n=1}^{2l-1}(-1)^n(\mathbf{k}_n-\tilde{\mathbf{p}}^{(\omega_n,h)}_{F,j})\cdot\mathbf{u}_j)} \hat{W}^{(h)}_{2l,\underline{\alpha}}(\mathbf{k}_1,\cdots,\mathbf{k}_{2l-1}). \end{aligned} \tag{I.4.41}$$

In addition

$$\bar{A}_{h,\omega,j}(\mathbf{x}) := \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}} e^{i(\mathbf{k}-\tilde{\mathbf{p}}^{(\omega,h)}_{F,j})\cdot\mathbf{x}} \hat{\bar{A}}_{h,\omega,j}(\mathbf{k}). \tag{I.4.42}$$

I.4.3. Estimates of the free propagator

Before moving along with the tree expansion, we first compute a bound on $\hat{g}_h$ in the different regimes, which will be used in the following.

1 - Ultraviolet regime. We first study the ultraviolet regime, i.e. $h \in \{1,\cdots,M\}$.

1-1 - Fourier space bounds. We have

$$\hat{A}(\mathbf{k})^{-1} := -(ik_0\mathbb{1} + H_0(k))^{-1} = -\frac{1}{ik_0}\left(\mathbb{1} + \frac{H_0(k)}{ik_0}\right)^{-1}$$

and

$$|\hat{g}_h(\mathbf{k})| = |f_h(\mathbf{k})\hat{A}^{-1}(\mathbf{k})| \leqslant (\text{const.})\ 2^{-h},$$

where $|\cdot|$ is the operator norm. Therefore

$$\frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}} |\hat{g}_h(\mathbf{k})| \leqslant (\text{const.}). \tag{I.4.43}$$

Furthermore, for all $m_0 + m_k \leqslant 7$ (we choose the constant 7 in order to get adequate bounds on the real-space decay of the free propagator, good enough for performing the localization and renormalization procedure described below; any other larger constant would yield identical results),

$$|2^{hm_0}\partial^{m_0}_{k_0}\partial^{m_k}_{k}\hat{g}_h(\mathbf{k})| \leqslant (\text{const.})\ 2^{-h} \tag{I.4.44}$$

in which ∂_{k_0} denotes the discrete derivative with respect to k_0 and, with a slightly abusive notation, ∂_k the discrete derivative with respect to either k_x or k_y. Indeed the derivatives over k land on $ik_0\hat{A}^{-1}$, which does not change the previous estimate, and the derivatives over k_0 either land on f_h, $1/(ik_0)$, or $ik_0\hat{A}^{-1}$, which yields an extra 2^{-h} in the estimate.

Remark: The previous argument implicitly uses the Leibnitz rule, which must be used carefully since the derivatives are discrete. However, since the estimate is purely dimensional, we can replace the discrete discrete derivative with a continuous one without changing the order of magnitude of the resulting bound.

1-2 - Configuration space bounds. We now prove that the inverse Fourier transform of $\hat{g}_h$

$$g_h(\mathbf{x}) := \frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{g}_h(\mathbf{k}) \tag{I.4.45}$$

satisfies the following estimate: for all $m_0 + m_k \leqslant 3$,

$$\int d\mathbf{x}\; x_0^{m_0} x^{m_k} |g_h(\mathbf{x})| \leqslant (\text{const.})\; 2^{-h-m_0h}, \tag{I.4.46}$$

where we recall that $\int d\mathbf{x}$ is a shorthand for $\int_0^\beta dt \sum_{x\in\Lambda}$. Indeed, note that the right side of (I.4.45) can be thought of as the Riemann sum approximation of

$$\int_{\mathbb{R}} \frac{dk_0}{2\pi} \int_{\hat{\Lambda}_\infty} \frac{dk}{|\hat{\Lambda}_\infty|} e^{-i\mathbf{k}\cdot\mathbf{x}} \hat{g}_h(\mathbf{k}) \tag{I.4.47}$$

where $\hat{\Lambda}_\infty = \{t_1G_1 + t_2G_2 : t_i \in [0,1)\}$ is the limit as $L\to\infty$ of $\hat{\Lambda}$, see (I.2.4) and following lines. The dimensional estimates one finds using this continuum approximation are the same as those using (I.4.45) therefore, integrating (I.4.47) 7 times by parts and using (I.4.44) we find

$$|g_h(\mathbf{x})| \leqslant \frac{(\text{const.})}{1 + (2^h|x_0| + |x|)^7}$$

so that by changing variables in the integral over x_0 to $2^h x_0$, and using

$$\int d\mathbf{x}\; \frac{x_0^{m_0} x^{m_k}}{1 + (|x_0| + |x|)^7} < (\text{const.})$$

we find (I.4.46).

2 - First regime. We now consider the first regime, i.e. $h \in \{\mathfrak{h}_1 + 1, \cdots, \bar{\mathfrak{h}}_0\}$.

2-1 - Fourier space bounds. From (I.3.8) we find

$$|\hat{g}_{h,\omega}(\mathbf{k})| \leqslant (\text{const.})\; 2^{-h}$$

therefore

$$\frac{1}{\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}} |\hat{g}_{h,\omega}(\mathbf{k})| \leqslant (\text{const.})\; 2^{2h} \tag{I.4.48}$$

and for $m \leqslant 7$,

$$|2^{mh}\partial_{\mathbf{k}}^m \hat{g}_{h,\omega}(\mathbf{k})| \leqslant (\text{const.})\; 2^{-h} \tag{I.4.49}$$

in which we again used the slightly abusive notation of writing $\partial_{\mathbf{k}}$ to mean any derivative with respect to k_0, k_x or k_y. Equation (I.4.49) then follows from similar considerations as those in the ultraviolet regime.

2-2 - Configuration space bounds. We estimate the real-space counterpart of $\hat{g}_{h,\omega}$,

$$g_{h,\omega}(\mathbf{x}) := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(h,\omega)}_{\beta,L}} e^{-i(\mathbf{k}-\mathbf{p}^{\omega}_{F,0})\cdot\mathbf{x}}\hat{g}_{h,\omega}(\mathbf{k}),$$

and find that for $m\leqslant 3$,

$$\int d\mathbf{x}\ |\mathbf{x}^m g_{h,\omega}(\mathbf{x})|\leqslant (\text{const.})\ 2^{-(1+m)h} \tag{I.4.50}$$

which follows from very similar considerations as the ultraviolet estimate.

3 - Second regime. We treat the second regime, i.e. $h\in\{\mathfrak{h}_2+1,\cdots,\bar{\mathfrak{h}}_1\}$ in a very similar way (we skip the intermediate regime which can be treated in the same way as either the first or second regimes):

$$\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}|\hat{g}_{h,\omega}(\mathbf{k})|\leqslant(\text{const.})\ 2^{h+h_\epsilon} \tag{I.4.51}$$

and for all $m_0+m_k\leqslant 7$,

$$|2^{m_0h}\partial_{k_0}^{m_0}2^{m_k\frac{h+h_\epsilon}{2}}\partial_k^{m_k}\hat{g}_{h,\omega}(\mathbf{k})|\leqslant(\text{const.})\ 2^{-h} \tag{I.4.52}$$

where $h_\epsilon:=\log_2(\epsilon)$. Therefore for all $m_0+m_k\leqslant 3$,

$$\int d\mathbf{x}\ |x_0^{m_0}x^{m_k}g_{h,\omega}(\mathbf{x})|\leqslant(\text{const.})\ 2^{-h-m_0h-m_k\frac{h+h_\epsilon}{2}}. \tag{I.4.53}$$

4 - Third regime. Finally, the third regime, i.e. $h\in\{\mathfrak{h}_3+1,\cdots,\bar{\mathfrak{h}}_2\}$:

$$\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}}|\hat{g}_{h,\omega}(\mathbf{k})|\leqslant(\text{const.})\ 2^{2h-2h_\epsilon} \tag{I.4.54}$$

and for all $m_0+m_k\leqslant 7$,

$$|2^{m_0h}\partial_{k_0}^{m_0}2^{m_k(h-h_\epsilon)}\hat{g}_{h,\omega,j}(\mathbf{k})|\leqslant(\text{const.})\ 2^{-h}. \tag{I.4.55}$$

Therefore for all $m_0+m_k\leqslant 3$,

$$\int d\mathbf{x}\ |x_0^{m_0}x^{m_k}g_{h,\omega,j}(\mathbf{x})|\leqslant(\text{const.})\ 2^{-h-m_0h-m_k(h-h_\epsilon)} \tag{I.4.56}$$

where

$$g_{h,\omega,j}(\mathbf{x}) := \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(h,\omega,j)}_{\beta,L}} e^{-i(\mathbf{k}-\tilde{\mathbf{p}}^{(\omega,h+1)}_{F,j})\cdot\mathbf{x}}\hat{g}_{h,\omega}(\mathbf{k}).$$

I.5. Tree expansion and constructive bounds

In this section, we shall define the Gallavotti-Nicolò tree expansion [GN85, GN85b], and show how it can be used to compute bounds for the $\mathfrak{e}_h$, $\mathcal{V}^{(h)}$, $\mathcal{Q}^{(h)}$ and $\bar{\mathcal{V}}^{(h)}$ defined above in (I.4.4) and (I.4.8), using the estimates (I.4.46), (I.4.50), (I.4.53) and (I.4.56). We follow [BG90, GM01, GM10]. We conclude the section by showing how to compute the terms in $\bar{\mathcal{W}}^{(h)}$ that are quadratic in $\hat{J}_{\mathbf{k},\underline{\alpha}}$ from $\mathcal{V}^{(h)}$ and $\hat{\bar{g}}_h$.

The discussion in this section is meant to be somewhat general, in order to be applied to the ultraviolet, first, second and third regimes (except for lemma I.5.2 which does not apply to the ultraviolet regime).

I.5.1. Gallavotti-Nicolò Tree expansion

In this section, we will define a tree expansion to re-express equations *of the type*

$$-v^{(h)}(\psi^{(\leqslant h)})-\mathcal V^{(h)}(\psi^{(\leqslant h)})=\sum_{N=1}^\infty\frac{(-1)^N}{N!}\mathcal E^T_{h+1}\left(\mathcal V^{(h+1)}(\psi^{(\leqslant h)}+\psi^{(h+1)});N\right)\tag{I.5.1}$$

for $h\in\{h_2^*,\cdots,h_1^*-1\}$ (in the ultraviolet regime $h_2^*=\bar{\mathfrak h}_0$, $h_1^*=M$; in the first $h_2^*=\mathfrak h_1$, $h_1^*=\bar{\mathfrak h}_0$; in the second $h_2^*=\mathfrak h_2$, $h_1^*=\bar{\mathfrak h}_1$; and in the third, $h_2^*=\mathfrak h_\beta$, $h_1^*=\bar{\mathfrak h}_2$), with

$$\left\{\begin{array}{l}\mathcal V^{(h)}(\psi^{(\leqslant h)})=\displaystyle\sum_{l=q}^\infty\sum_{\underline\varpi}\int d\underline{\mathbf x}\ W^{(h)}_{2l,\underline\varpi}(\underline{\mathbf x})\psi^{(\leqslant h)+}_{\mathbf x_1,\varpi_1}\psi^{(\leqslant h)-}_{\mathbf x_2,\varpi_2}\cdots\psi^{(\leqslant h)+}_{\mathbf x_{2l-1},\varpi_{2l-1}}\psi^{(\leqslant h)-}_{\mathbf x_{2l},\varpi_{2l}}\\ v^{(h)}(\psi^{(\leqslant h)})=\displaystyle\sum_{l=0}^{q-1}\sum_{\underline\varpi}\int d\underline{\mathbf x}\ W^{(h)}_{2l,\underline\varpi}(\underline{\mathbf x})\psi^{(\leqslant h)+}_{\mathbf x_1,\varpi_1}\psi^{(\leqslant h)-}_{\mathbf x_2,\varpi_2}\cdots\psi^{(\leqslant h)+}_{\mathbf x_{2l-1},\varpi_{2l-1}}\psi^{(\leqslant h)-}_{\mathbf x_{2l},\varpi_{2l}}\end{array}\right.\tag{I.5.2}$$

($q=1$ in the ultraviolet regime and $q=2$ in the first, second and third) in which $\underline\varpi$ and $\underline{\mathbf x}$ are shorthands for $(\varpi_1,\cdots,\varpi_{2l})$ and $(\mathbf x_1,\cdots,\mathbf x_{2l})$; ϖ denotes a collection of indices: (α,ω) in the first and second regimes, (α,ω,j) in the third, and (α) in the ultraviolet; and $W^{(h)}_{2l,\underline\varpi}(\underline{\mathbf x})$ is a function that only depends on the differences $\mathbf x_i-\mathbf x_j$. The propagator associated with $\mathcal E^T_{h+1}$ will be denoted $g_{(h+1),(\varpi,\varpi')}(\mathbf x-\mathbf x')$ and is to be interpreted as the dressed propagator $\bar g_{(h+1,\omega),(\alpha,\alpha')}$ in the first and second regimes, and as $\bar g_{(h+1,\omega,j),(\alpha,\alpha')}$ in the third. Note in particular that in the first and second regimes the propagator is diagonal in the ω indices, and is diagonal in (ω,j) in the third. In all cases, we write

$$g_{(h+1),(\varpi,\varpi')}(\mathbf x-\mathbf x')=\frac{1}{\beta|\Lambda|}\sum_{\mathbf k\in\mathcal B_{\beta,L}}e^{-i(\mathbf k-\mathbf p^{(h+1)}_\varpi)(\mathbf x-\mathbf x')}\hat g_{(h+1),(\varpi,\varpi')}(\mathbf k)\ ,\tag{I.5.3}$$

where $\mathbf p^{(h+1)}_\varpi$ should be interpreted as $\mathbf 0$ in the ultraviolet regime, as $\mathbf p^\omega_{F,0}$ in the first and second, and as $\tilde{\mathbf p}^{(\omega,h+1)}_{F,j}$ in the third, see (I.4.21).

Remark: The usual way of computing expressions of the form (I.5.1) is to write the right side as a sum over Feynman diagrams. The tree expansion detailed below provides a way of identifying the sub-diagrams that scale in the same way (see the remark at the end of this section). In the proofs below, there will be no mention of Feynman diagrams, since a diagramatic expansion would yield insufficient bounds.

We will now be a bit rough for a few sentences, in order to carry the main idea of the tree expansion across: equation (I.5.1) is an inductive equation for the $\mathcal V^{(h)}$, which we will pictorially think of as the *merging* of a selection of N potentials $\mathcal V^{(h+1)}$ via a truncated expectation. If we iterate (I.5.1) all the way to scale h_2^*, then we get a set of *merges* that *fit* into each other, creating a tree structure. The sum over the choice of N's at every step will be expressed as a sum over Gallavotti-Nicolò trees, which we will now define precisely.
Given a scale $h\in\{h_2^*,\cdots,h_1^*-1\}$ and an integer $N\geqslant 1$, we define the set $\mathcal T_N^{(h)}$ of Gallavotti-Nicolò (GN) trees as a set of *labeled* rooted trees with N leaves in the following way.

- We define the set of unlabeled trees inductively: we start with a *root*, that is connected to a node v_0 that we will call the *first node* of the tree; every node is assigned an ordered set of child nodes. v_0 must have at least one child, while the other nodes may be childless. We denote the parent-child partial ordering by $v'\prec v$ (v' is the parent of v). The nodes that have no children are called *leaves* or *endpoints*. By convention, the root is not considered to be a node, but we will still call it the parent of v_0.

- Each node is assigned a *scale label* $h' \in \{h+1, \cdots, h_1^*+1\}$ and the root is assigned the *scale label* h, in such a way that the children of the root or of a node on scale h' are on scale $h'+1$ (keep in mind that it is possible for a node to have a single child).
- The leaves whose scale is $\leqslant h_1^*$ are called *local*. The leaves on scale h_1^*+1 can either be *local* or *irrelevant* (see figure I.5.1).
- Every local leaf must be preceeded by a *branching node*, i.e. a node with at least two children. In other words, every local leaf must have at least one sibling.
- We denote the set of nodes of a tree τ by $\bar{\mathfrak{V}}(\tau)$, the set of nodes that are not leaves by $\mathfrak{V}(\tau)$ and the set of leaves by $\mathfrak{E}(\tau)$.

Remark: Local leaves are called "local" because those nodes are usually applied a *localization* operation (see e.g. [BG95]). In the present case, such a step is not needed, due to the super-renormalizable nature of the first and third regimes.

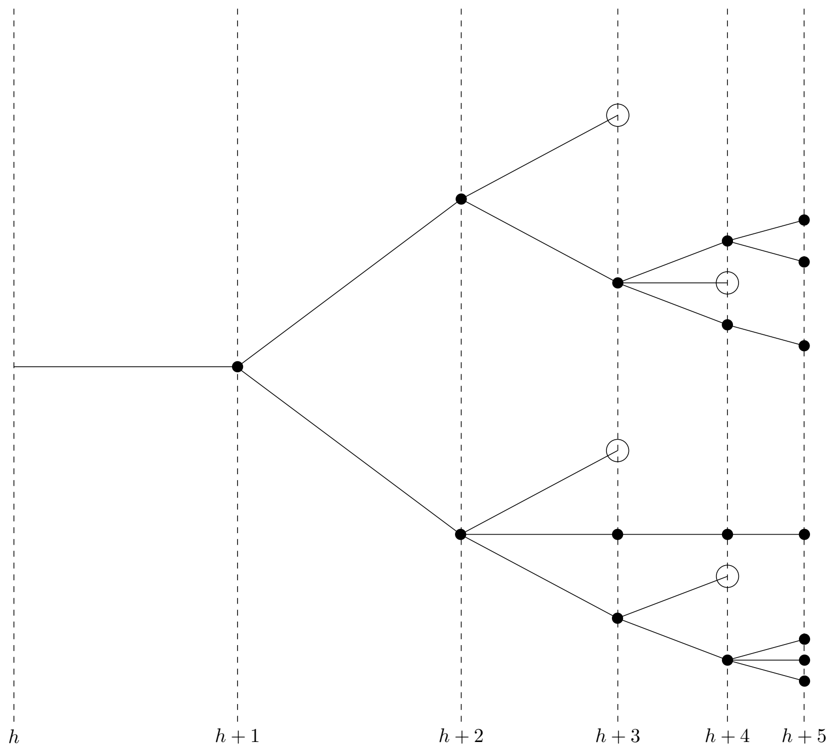

fig I.5.1: example of a tree on scale h up to scale $h_1^*+1 = h+5$ with 11 leaves, 5 of which are local and 6 irrelevant. Local leaves are represented as empty circles, whereas irrelevant leaves are represented as full circles.

Every node of a Gallavotti-Nicolò tree τ corresponds to a truncated expectation of effective potentials of the form (I.5.1). If one expands the product of factors of the form $(\psi_{\mathbf{x},\varpi}^{(\leqslant h)\pm} + \psi_{\mathbf{x},\varpi}^{(h+1)\pm})$ in every term in the right side of (I.5.1), then one finds a sum over *choices* between $\psi^{(\leqslant h)}$ and $\psi^{(h)}$ for every $(\mathbf{x}, \varpi, \pm)$. We will express this sum as a sum over a set of *external field labels* (corresponding to the labels of $\psi^{(\leqslant h)}$ which are called external because they can be factored out of the truncated expectation) defined in the following way. Given an integer $\ell_0 \geqslant q$, whose purpose will become clear in lemma I.5.2 (we will choose ℓ_0 to be $=1$ in the ultraviolet regime, and $=2,3,2$ in the first, second, third infrared regimes, respectively), a tree $\tau \in \mathcal{T}_N^{(h)}$ whose endpoints are denoted by $(v_1, \cdots, v_N)$, as well as a collection of integers $\underline{l}_\tau := (l_{v_1}, \cdots, l_{v_N}) \in \mathbb{N}^N$ such that $l_{v_i} \geqslant q$ and, if v_i is a local leaf, $l_{v_i} < \ell_0$ (in particular, if $\ell_0 = q$ there are no local leaves), we introduce an ordered collection of *fields*, i.e. triplets

$$F = ((\mathbf{x}_1, \varpi_1, +), (\mathbf{x}_2, \varpi_2, -), \cdots, (\mathbf{x}_{2L-1}, \varpi_{2L-1}, +), (\mathbf{x}_{2L}, \varpi_{2L}, -)) \,. \tag{I.5.4}$$

where $L := l_{v_1} + \cdots + l_{v_N}$. We then define the set of *external field labels* of each endpoint v_i as the following ordered collections of integers

$$I_{v_1} := (1, \cdots, 2l_{v_1}), \cdots, I_{v_N} := (2l_{v_{N-1}} + 1, \cdots, 2l_{v_N}).$$

We define the set $\mathcal{P}_{\tau,\underline{l}_\tau,\ell_0}$ of *external field labels* compatible with a tree $\tau \in \mathcal{T}_N^{(h)}$ as the set of all the collections $\mathbf{P} = \{P_v\}_{v\in\mathfrak{V}(\tau)}$ where P_v are themselves collections of integers that satisfy the following constraints:

- For every $v \in \mathfrak{V}(\tau)$ whose children are $(v_1, \cdots, v_s)$, $P_v \subset P_{v_1}\cup\cdots\cup P_{v_s}$ in which, by convention, if v_i is an endpoint then $P_{v_i} = I_{v_i}$; and the order of the elements of P_v is that of P_{v_1} through P_{v_s} (in particular the integers coming from P_{v_1} precede those from P_{v_2} and so forth).
- For all $v \in \mathfrak{V}(\tau)$, P_v must contain as many even integers as odd ones (even integers correspond to fields with a $-$, and odd ones to a $+$).
- If v has more than one child, then $P_v \neq P_{v'}$ for all $v' \succ v$
- For all $v \in \bar{\mathfrak{V}}(\tau)\setminus\{v_0\}$ which is not a local leaf, the cardinality of P_v must satisfy $|P_v| \geqslant 2\ell_0$.

Furthemore, given a node v whose children are $(v_1, \cdots, v_s)$, we define $R_v := \bigcup_{i=1}^{s} P_{v_i} \setminus P_v$.

We associate a *value* to each node v of such a tree in the following way. If v is a leaf, then its value is

$$\rho_v := W^{(h_v-1)}_{|P_v|,\underline{\varpi}_v}(\underline{\mathbf{x}}_v) \tag{I.5.5}$$

where $|P_v|$ denotes the cardinality of P_v, and $\underline{\varpi}_v$ and $\underline{\mathbf{x}}_v$ are the *field labels* (i.e. elements of F) specified by the indices in P_v. If v is not a leaf and $R_v \neq \emptyset$, then its value is

$$\rho_v := \sum_{T_v\in\mathbf{T}(\mathbf{R}_v)} \sigma_{T_v} \prod_{l\in T_v} g_{(h_v),l} \int dP_{T_v}(\mathbf{t})\ \det G^{(T_v,h_v)}(\mathbf{t}) =: \sum_{T_v\in\mathbf{T}(\mathbf{R}_v)} \rho_v^{(T_v)} \tag{I.5.6}$$

where $\mathbf{T}(\mathbf{R}_v)$, $g_{(h_v),l}$, $dP_{T_v}(\mathbf{t})$ and $G^{(T_v,h_v)}$ are defined as in lemma I.2.1 with g replaced by g_{h_v}, and if the children of v are denoted by $(v_1, \cdots, v_s)$, then $\mathbf{R}_v := (P_{v_1} \setminus P_v, \ldots, P_{v_s} \setminus P_v)$. If v is not a leaf and $R_v = \emptyset$, then it has exactly one child and we let its value be $\rho_v = 1$.

Lemma I.5.1

Equation (I.5.1) can be re-written as

$$-v^{(h)}(\psi^{(\leqslant h)}) - \mathcal{V}^{(h)}(\psi^{(\leqslant h)}) = \sum_{N=1}^{\infty} \sum_{\tau\in\mathcal{T}_N^{(h)}} \sum_{\underline{l}_\tau} \sum_{\underline{\varpi}_\tau} \int d\underline{\mathbf{x}}_\tau \sum_{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,\ell_0}} \Psi^{(\leqslant h)}_{P_{v_0}} \prod_{v\in\bar{\mathfrak{V}}(\tau)} \frac{(-1)^{s_v}}{s_v!}\rho_v \tag{I.5.7}$$

where $\underline{l}_\tau := (l_{v_1}, \cdots, l_{v_N})$ (see above), $\underline{\varpi}_\tau$ and $\underline{\mathbf{x}}_\tau$ are the field labels in F, s_v is the number of children of v, ρ_v was defined above in (I.5.5) and (I.5.6), v_0 is the first node of τ and

$$\Psi^{(\leqslant h)}_{P_{v_0}} := \prod_{i\in P_{v_0}} \psi^{(\leqslant h)\epsilon_i}_{\mathbf{x}_i,\varpi_i}$$

where ϵ_i is the third component of the i-th triplet in F.

Remark: The sum over $\mathbf{P} \in \mathcal{P}_{\tau,\underline{l}_\tau,\ell_0}$ is a sum over the assignement of P_v for nodes that are not endpoints. The sets I_v are not summed over, instead they are fixed by $\underline{l}_\tau$. Furthermore, if $\mathcal{P}_{\tau,\underline{l}_\tau,\ell_0} = \emptyset$ (e.g. if $\ell_0 = q$ and τ contains local leaves), then the sum should be interpreted as 0.

By injecting (I.5.6) into (I.5.7), we can re-write

$$-v^{(h)}(\psi^{(\leqslant h)}) - \mathcal{V}^{(h)}(\psi^{(\leqslant h)}) = \sum_{N=1}^{\infty} \sum_{\tau\in\mathcal{T}_N^{(h)}} \sum_{T\in\mathbf{T}(\tau)} \sum_{\underline{l}_\tau} \sum_{\underline{\varpi}_\tau} \int d\underline{\mathbf{x}}_\tau \sum_{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,\ell_0}} \Psi^{(\leqslant h)}_{P_{v_0}} \prod_{v\in\bar{\mathfrak{V}}(\tau)} \frac{(-1)^{s_v}}{s_v!}\rho_v^{(T_v)} \tag{I.5.8}$$

where $\mathbf{T}(\tau)$ is the set of collections of $(T_v \in \mathbf{T}(\mathbf{R}_v))_{v\in\mathfrak{V}(\tau)}$. Moreover, while $\rho_v^{(T_v)}$ was defined in (I.5.6) if $v \in \mathfrak{V}(\tau)$, it stands for ρ_v if $v \in \mathfrak{E}(\tau)$ (note that in this case $T_v = \emptyset$).

Idea of the proof: The proof of this lemma can easily be reconstructed from the schematic description below. We do not present it in full detail here because its proof has already been discussed in several references, among which [BG95, GM01, Gi10].

The lemma follows from an induction on h, in which we write the truncated expectation in the right side of (I.5.1) as

$$\sum_{l_1,\cdots,l_N}\sum_{\underline{\varpi}_1,\cdots,\underline{\varpi}_N}\int d\underline{\mathbf{x}}_1\cdots d\underline{\mathbf{x}}_N\ W^{(h+1)}_{2l_1,\underline{\varpi}_1}(\underline{\mathbf{x}}_1)\cdots W^{(h+1)}_{2l_N,\underline{\varpi}_N}(\underline{\mathbf{x}}_N)\cdot$$
$$\cdot\mathcal{E}^T_{h+1}\left(\prod_{j=1}^{l_1}(\psi^{(\leqslant h)+}_{x_{1,2j-1},\varpi_{1,2j-1}}+\psi^{(h+1)+}_{x_{1,2j-1},\varpi_{1,2j-1}})(\psi^{(\leqslant h)-}_{x_{1,2j},\varpi_{1,2j}}+\psi^{(h+1)-}_{x_{1,2j},\varpi_{1,2j}}),\cdots\right.$$
$$\left.\cdots,\prod_{j=1}^{l_N}(\psi^{(\leqslant h)+}_{x_{N,2j-1},\varpi_{N,2j-1}}+\psi^{(h+1)+}_{x_{N,2j-1},\varpi_{N,2j-1}})(\psi^{(\leqslant h)-}_{x_{N,2j},\varpi_{N,2j}}+\psi^{(h+1)-}_{x_{N,2j},\varpi_{N,2j}})\right)$$

which yields a sum over the choices between $\psi^{(\leqslant h)}$ and $\psi^{(h+1)}$, with each choice corresponding to an instance of P_v: each $\psi^{(\leqslant h)\epsilon}_{\mathbf{x},\varpi}$ "creates" the element $(\mathbf{x},\varpi,\epsilon)$ in P_v. The remaining truncated expectation is then computed by applying lemma I.2.1. Finally, the $W^{(h+1)}_{2l_j,\underline{\varpi}_j}$ with $l_j < \ell_0$ are left as such, and yield a *local leaf* in the tree expansion, the others are expanded using the inductive hypothesis.

Remark: For readers who are familiar with Feynman diagram expansions, it may be worth pointing out that a Gallavotti-Nicolò tree paired up with a set of external field labels $\mathbf{P}$ represents a class of labeled Feynman diagrams (the labels being the scales attached to the lines, or equivalently to the propagators) with similar scaling properties. In fact, given a labeled Feynman diagram, one defines a tree and a set of external field labels by the following procedure. For every h, we define the *clusters on scale* h as the connected components of the diagram one obtains by removing the lines with a scale label that is $< h$. We assign a node with scale label h to every cluster on scale h. The set P_v contains the indices of the legs of the Feynman diagram that exit the corresponding cluster. If a cluster on scale h contains a cluster on scale $h+1$, then we draw a branch between the two corresponding nodes. See figure I.5.2 for an example.

Local leaves correspond to clusters that have *few* external legs. They are considered as "black boxes": the clusters on larger scales contained inside them are discarded.

A more detailed discussion of this correspondence can be found in [GM01, section 5.2] among other references.

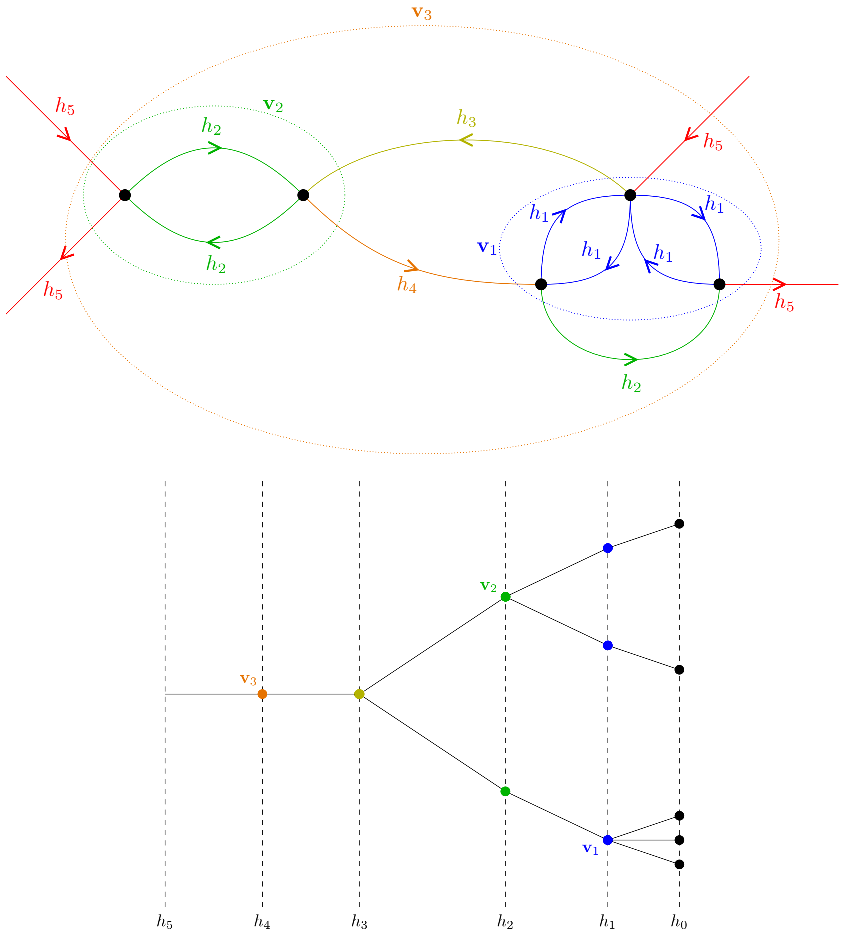

fig I.5.2: Example of a labeled Feynman diagram and its corresponding tree. Three clusters, denoted by $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, on scale h_1, h_2 and h_4 respectively, are explicitely drawn as dotted ellipses. There are 4 more clusters (2 on scale h_1, 1 on scale h_2 and 1 on scale h_3) which are not represented. The scales are drawn in different colors (color online): red for h_5, orange for h_4, yellow for h_3, green for h_2 and blue for h_1.

I.5.2. Power counting lemma

We will now state and prove the *power counting lemma*, which is an important step in bounding the elements in the tree expansion (I.5.8) in the first, second and third regimes.

In the following, we will use a slightly abusive notation: given $\underline{\mathbf{x}} = (\mathbf{x}_1, \dots, \mathbf{x}_n)$, we will write $\underline{\mathbf{x}}^m$ to mean "any of the products of the following form"

$$x_{j_1,i_1} \cdots x_{j_m,i_m}$$

where $i_\nu \in \{0,1,2\}$ indexes the components of $\mathbf{x}$ and $j_\nu \in \{1,\cdots,n\}$ indexes the components of $\underline{\mathbf{x}}$. We will also denote the translate of $\underline{\mathbf{x}}$ by $\mathbf{y}$ by $\underline{\mathbf{x}} - \mathbf{y} \equiv (\mathbf{x}_1 - \mathbf{y}, \dots, \mathbf{x}_n - \mathbf{y})$. Furthermore, given $\underline{\mathbf{x}}^m$, we define the vector $\underline{m}$ whose i-th component is the number of occurrences of $x_{\cdot,i}$ in the product $\underline{\mathbf{x}}^m$ (note that $m_0 + m_1 + m_2 = m$).

The power counting lemma will be stated as an inequality on the so-called *beta function* of the renormalization group flow, defined as

$$B^{(h)}_{2l,\underline{\varpi}}(\underline{\mathbf{x}}) := \begin{cases} W^{(h)}_{2l,\underline{\varpi}}(\underline{\mathbf{x}}) - W^{(h+1)}_{2l,\underline{\varpi}}(\underline{\mathbf{x}}) & \text{if } l \geqslant q \\ \\ W^{(h)}_{2l,\underline{\varpi}}(\underline{\mathbf{x}}) & \text{if } l < q. \end{cases} \tag{I.5.9}$$

In terms of the tree expansion (I.5.7), $B_{2l}^{(h)}$ is the sum of the contributions to $W_{2l}^{(h)}$ whose field label assignment $\mathbf{P}$ is such that every node $v\in\mathfrak{V}(\tau)\setminus\{v_0\}$ that is connected to the root by a chain of nodes with only one child satisfies $|P_v|>2l$. We denote the set of such field label assignments by $\tilde{\mathcal{P}}_{\tau,\underline{l}_\tau,\ell_0}$ for any given τ, $\underline{l}_\tau$ and ℓ_0. In other words, $B_{2l}^{(h)}$ contains all the contributions that have at least one propagator on scale $h+1$. If $l<q$, then all the contributions have a propagator on scale $h+1$, so $B_{2l}=W_{2l}$.

Lemma I.5.2

Assume that the propagator $g_{(h),(\varpi,\varpi')}(\mathbf{x}-\mathbf{x}')$ can be written as in (I.5.3). Given $h\in\{h_2^*,\cdots,h_1^*-1\}$, if $\forall m\in\{0,1,2,3\}$, and

$$\left\{\begin{array}{l}\displaystyle\int d\mathbf{x}\ |\mathbf{x}^m g_{h'}(\mathbf{x})|\leqslant C_g2^{-c_gh'}\mathfrak{F}_{h'}(\underline{m})\\ \displaystyle\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}}|\hat g_{h'}(\mathbf{k})|\leqslant C_G2^{(c_k-c_g)h'}\end{array}\right.,\qquad\forall h'\in\{h+1,\cdots,h_1^*\},\tag{I.5.10}$$

where c_g, c_k, C_g and C_G are constants, independent of h, and $\mathfrak{F}_{h'}(\underline{m})$ is a shorthand for

$$A_0^{m_0}A_1^{m_1}A_2^{m_2}2^{-h'(d_0m_0+d_1m_1+d_2m_2)}$$

in which $A_0,A_1,A_2>0$, $d_0,d_1,d_2\geqslant0$, and m_i is the number of times any of the $x_{j,i}$ appears in $\mathbf{x}^m$; if

$$\ell_0>\frac{c_k}{c_k-c_g}\tag{I.5.11}$$

and

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^mW_{2l,\underline{\varpi}}^{(h')}(\underline{\mathbf{x}})\right|\leqslant\mathfrak{C}_{2l}|U|^{\max(1,l-1)}2^{h'(c_k-(c_k-c_g)l)}\mathfrak{F}_{h'}(\underline{m}),\qquad\forall h'\in\{h+1,\cdots,h_1^*\}\tag{I.5.12}$$

where $q\leqslant l<\ell_0$ for $h'<h_1^*$, $l\geqslant q$ for $h'=h_1^*$ (in particular, if $q\geqslant\ell_0$, then $h'=h_1^*$), and $\mathfrak{C}_{2l}$ are constants, then

$$\begin{array}{r}\displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^mB_{2l,\underline{\varpi}}^{(h)}(\underline{\mathbf{x}})\right|\leqslant2^{h(c_k-(c_k-c_g)l)}\mathfrak{F}_h(\underline{m})(C_3C_G^{-1})^l\sum_{N=1}^\infty\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\underline{l}_\tau}\\ \displaystyle\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}_{\tau,\underline{l}_\tau,\ell_0}\\|P_{v_0}|=2l}}C_1^N(C_gC_G^{-1})^{N-1}\left(\prod_{v\in\mathfrak{V}(\tau)}2^{(c_k-(c_k-c_g)\frac{|P_v|}{2})}\right)\left(\prod_{v\in\mathfrak{E}(\tau)}(C_2C_G)^{l_v}\mathfrak{C}_{2l_v}|U|^{\max(1,l_v-1)}\right)\end{array}\tag{I.5.13}$$

where C_1, C_2 and C_3 are constants, independent of c_g, c_k, C_g, C_G and h.

Remarks: Here are a few comments about this lemma.

- Combining this lemma with (I.5.9) yields a bound on $W_{2l,\underline{\varpi}}^{(h)}(\underline{\mathbf{x}})$. In particular, if $l\geqslant\ell_0$ and $h<h_1^*$, then

$$\begin{array}{r}\displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^mW_{2l,\underline{\varpi}}^{(h)}(\underline{\mathbf{x}})\right|\leqslant2^{h(c_k-(c_k-c_g)l)}\mathfrak{F}_h(\underline{m})(C_3C_G^{-1})^l\sum_{N=1}^\infty\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\underline{l}_\tau}\\ \displaystyle\sum_{\substack{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,\ell_0}\\|P_{v_0}|=2l}}C_1^N(C_gC_G^{-1})^{N-1}\left(\prod_{v\in\mathfrak{V}(\tau)}2^{(c_k-(c_k-c_g)\frac{|P_v|}{2})}\right)\left(\prod_{v\in\mathfrak{E}(\tau)}(C_2C_G)^{l_v}\mathfrak{C}_{2l_v}|U|^{\max(1,l_v-1)}\right).\end{array}\tag{I.5.14}$$

- The lemma cannot be used in this form in the ultraviolet regime, since in that case the right side of (I.5.11) is infinite, because $c_k = c_g = 1$. In the ultraviolet we will need to re-organize the tree expansion, in order to derive convergent bounds on the series, as discussed in section I.6 below.

- The lemma gives a bound on the m-th derivative of $\hat W^{(h)}_{2l,\underline{\varpi}}(\underline{\mathbf{k}})$, which we will need in order to write the dominating behavior of the two-point Schwinger function as stated in theorems I.1.2, I.1.3, I.1.4; however, we will never need to take m larger than 3, which is important because the bound (I.5.13), if generalized to larger values of m, would diverge faster than $m!$ as $m \to \infty$.

- Recall that the propagator g_h appearing in the statement should be interpreted as the dressed propagator $\bar g_h$ in the first, second and third regimes. Since $\bar g_h$ depends on $W^{(h')}_{2l,\underline{\varpi}}$ for $h' \geqslant h$, we will have to apply the lemma inductively, proving at each step that the dressed propagator satisfies the bounds (I.5.10).

- Similarly, the bounds (I.5.12) will have to be proved inductively.

- In this lemma, the purpose of ℓ_0, which up until now may have seemed like an arbitrary definition, is made clear. In fact, the condition that $\ell_0 > c_k/(c_k - c_g)$ implies that $c_k - (c_k - c_g)|P_v|/2 < 0$, $\forall v \in \mathfrak{V}(\tau) \setminus \{v_0\}$. If this were not the case, then the weight of each tree τ could increase with the size of the tree, making the right side of (I.5.13) divergent.

- The combination $c_k - (c_k - c_g)|P_v|/2$ is called the *scaling dimension* of the cluster v. Under the assumptions of the lemma, the scaling dimension is negative, $\forall v \in \mathfrak{V}(\tau) \setminus \{v_0\}$. The clusters with non-negative scaling dimensions are necessarily leaves, and condition (I.5.12) corresponds to the requirement that we can control the size of these dangerous clusters. Essentially, what this lemma shows is that the only terms that are potentially problematic are those with non-negative scaling dimension. This prompts the following definitions: a node with negative scaling dimension will be called *irrelevant*, one with vanishing scaling dimension *marginal* and one with positive scaling dimension *relevant*.

- We will show that in the first and third regimes $c_k = 3$ and $c_g = 1$, so that the scaling dimension is $3 - |P_v|$. Therefore, the nodes with $|P_v| = 2$ are relevant whereas all the others are irrelevant. In the second regime, $c_k = 2$ and $c_g = 1$, so that the scaling dimension is $2 - |P_v|/2$. Therefore, the nodes with $|P_v| = 2$ are relevant, those with $|P_v| = 4$ are marginal, and all other nodes are irrelevant.

- The purpose of the factor $\mathfrak{F}_h(\underline{m})$ is to take into account the dependence of the order of magnitude of the different components k_0, k_x and k_y in the different regimes. In other words, as was shown in (I.4.46), (I.4.50), (I.4.53) and (I.4.56), the effect of multiplying g by $x_{j,i}$ depends on i, which is a fact the lemma must take into account.

- The reason why we have stated this bound in $\mathbf{x}$-space is because of the estimate of $\det(G^{(h_v,T_v)})$ detailed below, which is very inefficient in $\mathbf{k}$-space.

Proof: The proof proceeds in five steps: first we estimate the determinant appearing in (I.5.6) using the Gram-Hadamard inequality; then we perform a change of variables in the integral over $\underline{\mathbf{x}}_\tau$ in the right side of (I.5.8) in order to re-express it as an integration on differences $\mathbf{x}_i - \mathbf{x}_j$; we then decompose $(\underline{\mathbf{x}} - \mathbf{x}_{2l})^m$; and then compute a bound, which we re-arrange; and finally we use a bound on the number of spanning trees $\mathbf{T}(\tau)$ to conclude the proof.

1 - Gram bound. We first estimate $|\det G^{(T_v,h_v)}|$.

1-1 - Gram-Hadamard inequality. We shall make use of the Gram-Hadamard inequality, which states that the determinant of a matrix M whose components are given by $M_{i,j} = \mathbf{A}_i \odot \mathbf{B}_j$ where $(\mathbf{A}_i)$ and $(\mathbf{B}_i)$ are vectors in some Hilbert space with scalar product $\odot$ (writing M as a scalar product is called writing it in *Gram form*) can be bounded by

$$|\det(M)| \leqslant \prod_i \sqrt{\mathbf{A}_i \odot \mathbf{A}_i}\sqrt{\mathbf{B}_i \odot \mathbf{B}_i}. \tag{I.5.15}$$

The proof of this inequality is based on applying a Gram-Schmidt process to turn $(\mathbf{A}_i)$ and $(\mathbf{B}_i)$ into orthonormal families, at which point the inequality follows trivially. We recall that $G^{(T_v,h_v)}$ is an $(n_v-(s_v-1))\times(n_v-(s_v-1))$ matrix in which s_v denotes the number of children of v and if we denote the children of v by $(v_1,\cdots,v_{s_v})$, then $n_v = |R_v|/2 = (\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|)/2$. Its components are of the form $\mathbf{t}_\ell g_{(h_v),\ell}$ (see lemma I.2.1), with $\mathbf{t}_{(i,j)} = u_i\cdot u_j$ in which the u_i are unit vectors.

1-2 - Gram form. We now put $(g_{(h),(\alpha,\alpha')}(\mathbf{x}-\mathbf{x}'))_{(\mathbf{x},\alpha),(\mathbf{x}',\alpha')}$ in Gram form by using the $\mathbf{k}$-space representation of g_h in (I.5.3). Let $\mathcal{H} = \ell_2(\mathcal{B}_{\beta,L}\times\{a,\tilde b,\tilde a,b\})$ denote the Hilbert space of square summable sequences indexed by $(\mathbf{k},\alpha)\in\mathcal{B}_{\beta,L}\times\{a,\tilde b,\tilde a,b\}$. For every $h\in\{h_2^*,\cdots,h_1^*-1\}$ and $(\mathbf{x},\alpha)\in([0,\beta)\times\Lambda)\times\{a,\tilde b,\tilde a,b\}$, we define a pair of vectors $(\mathbf{A}_\alpha^{(h)}(\mathbf{x}),\mathbf{B}_\alpha^{(h)}(\mathbf{x}))\in\mathcal{H}^2$ by

$$\left\{\begin{array}{l}(\mathbf{A}_\alpha^{(h)}(\mathbf{x}))_{\mathbf{k},\alpha'} := \frac{1}{\sqrt{\beta|\Lambda|}}e^{-i\mathbf{k}\cdot\mathbf{x}}\hat V^{(h)}_{\alpha',\alpha}(\mathbf{k})\sqrt{\hat\lambda^{(h)}_{\alpha'}(\mathbf{k})}\\ (\mathbf{B}_\alpha^{(h)}(\mathbf{x}))_{\mathbf{k},\alpha'} := \frac{1}{\sqrt{\beta|\Lambda|}}e^{-i\mathbf{k}\cdot\mathbf{x}}\hat U^{(h)}_{\alpha,\alpha'}(\mathbf{k})\ \sqrt{\hat\lambda^{(h)}_{\alpha'}(\mathbf{k})}\end{array}\right. \tag{I.5.16}$$

where $\hat\lambda^{(h)}_{\alpha'}(\mathbf{k})$ denotes the α-th eigenvalue of $\sqrt{\hat g_h^\dagger(\mathbf{k})\hat g_h(\mathbf{k})}$ (i.e. the *singular values* of $\hat g_h(\mathbf{k})$) and $\hat V^{(h)}(\mathbf{k})$ and $\hat U^{(h)}(\mathbf{k})$ are unitary matrices that are such that

$$\hat g_h(\mathbf{k}) = \hat V^{(h)\dagger}(\mathbf{k})\hat D^{(h)}(\mathbf{k})\hat U^{(h)}(\mathbf{k}),$$

where $\hat D^{(h)}(\mathbf{k})$ is the diagonal matrix with entries $\hat\lambda^{(h)}_\alpha(\mathbf{k})$. We can now write g_h as

$$g_{(h),(\alpha,\alpha')}(\mathbf{x}-\mathbf{x}') = \mathbf{A}_\alpha^{(h)}(\mathbf{x})\odot\mathbf{B}_{\alpha'}^{(h)}(\mathbf{x}') \tag{I.5.17}$$

where $\odot$ denotes the scalar product on $\mathcal{H}$. Furthermore, recalling that $|\hat g_h(\mathbf{k})|$ is the operator norm of $\hat g_h(\mathbf{k})$, so that $|\hat g_h(\mathbf{k})| = \max\operatorname{spec}\sqrt{\hat g_h^\dagger(\mathbf{k})\hat g_h(\mathbf{k})}$, we have

$$\mathbf{A}_\alpha^{(h)}(\mathbf{x})\odot\mathbf{A}_\alpha^{(h)}(\mathbf{x}) = \mathbf{B}_\alpha^{(h)}(\mathbf{x})\odot\mathbf{B}_\alpha^{(h)}(\mathbf{x}) \leqslant \frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}}|\hat g_h(\mathbf{k})| \leqslant C_G 2^{(c_k-c_g)h} \tag{I.5.18}$$

The Gram form for $G^{(T_v,h_v)}$ is then

$$t_{(i,j)}g_{(h),(\varpi_i,\varpi_j)}(\mathbf{x}_i-\mathbf{x}_j) = (u_i\cdot u_j)(\mathbf{A}_{\varpi_i}(\mathbf{x}_i)\odot\mathbf{B}_{\varpi_j}(\mathbf{x}_j)) \tag{I.5.19}$$

so that, using (I.5.15) and (I.5.18),

$$|\det G^{(T_v,h_v)}| \leqslant (C_G 2^{(c_k-c_g)h_v})^{n_v-(s_v-1)}. \tag{I.5.20}$$

2 - Change of variables. We change variables in the integration over $\underline{\mathbf{x}}_\tau$. For every $v\in\bar{\mathfrak{V}}(\tau)$, let $P_v =: (j_1^{(v)},\cdots,j_{2l_v}^{(v)})$. We recall that a spanning tree $T\in\mathbf{T}(\tau)$ is a diagram connecting the fields specified by the I_v's for $v\in\mathfrak{E}(\tau)$: more precisely, if we draw a vertex for each $v\in\mathfrak{E}(\tau)$ with $|I_v|$ half-lines attached to it that are labeled by the elements of I_v, then $T\in\mathbf{T}(\tau)$ is a pairing of some of the half-lines that results in a tree called a *spanning tree* (not to be confused with a Gallavotti-Nicolò tree) (for an example, see figure I.5.3). The vertex v_r of a spanning tree that contains the *last external field*, i.e. that is such that $j^{(v_0)}_{2l_{v_0}} \in I_{v_r}$, is defined as its root, which allows us to unambiguously define a parent-child partial order, so that we can dress each branch with an arrow that is directed away from the root. For every $v \in \mathfrak{E}(\tau)$ that is not the root of T, we define $J^{(v)} \in I_v$ as the index of the field in which T *enters*, i.e. the index of the half-line of T with an arrow pointing towards v. We also define $J^{(v_r)} := j^{(v_0)}_{2l_{v_0}}$. Now, for every $v \in \mathfrak{E}(\tau)$, we define

$$\mathbf{z}_{j^{(v)}} := \mathbf{x}_{j^{(v)}} - \mathbf{x}_{J^{(v)}}$$

for all $j^{(v)} \in I_v \setminus \{J^{(v)}\}$, and given a line of T connecting $j^{(v)}$ to $J^{(v')}$, we define

$$\mathbf{z}_{J^{(v')}} := \mathbf{x}_{J^{(v')}} - \mathbf{x}_{j^{(v)}}.$$

We have thus defined $(\sum_{v\in\mathfrak{E}(\tau)} |I_v|) - 1$ variables $\mathbf{z}$, so that we are left with $\mathbf{x}_{J^{(v_r)}}$, which we call $\mathbf{x}_0$. It follows directly from the definitions that the change of variables from $\underline{\mathbf{x}}_\tau$ to $\{\mathbf{x}_0, \{\mathbf{z}_j\}_{j\in\mathcal{I}_\tau\setminus\{J^{(v_r)}\}}\}$, where $\mathcal{I}_\tau = \bigcup_{v\in\mathfrak{E}(\tau)} I_v$, has Jacobian equal to 1.

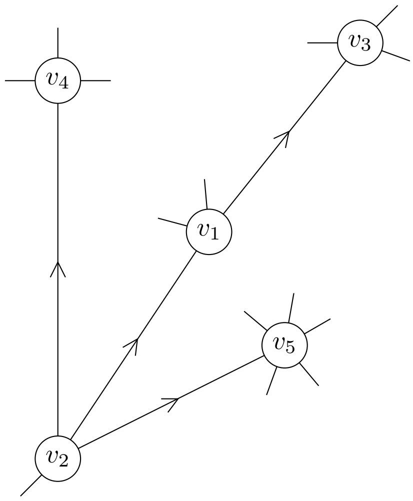

fig I.5.3: example of a spanning tree with $s_v = 5$ and $|P_{v_1}| = |P_{v_2}| = |P_{v_3}| = |P_{v_4}| = 4$, $|P_{v_5}| = 6$; whose root is v_2.

3 - Decomposing $(\underline{\mathbf{x}} - \mathbf{x}_{2l})^m$ We now decompose the $(\underline{\mathbf{x}} - \mathbf{x}_{2l})^m$ factor in (I.5.13) in the following way (note that in terms of the indices in P_{v_0}, $\mathbf{x}_{2l} \equiv \mathbf{x}_{J^{(v_r)}}$): $(\underline{\mathbf{x}} - \mathbf{x}_{2l})^m$ is a product of terms of the form $(x_{j,i} - x_{J^{(v_r)},i})$ which we rewrite as a sum of $z_{j',i}$'s for $v \in \mathfrak{E}(\tau)$ *on the path from* $J^{(v_r)}$ *to* j, a concept we will now make more precise. j and $J^{(v_r)}$ are in $I_{v(j)}$ and I_{v_r} respectively, where $v(j)$ is the unique node in $\mathfrak{E}(\tau)$ such that $j \in I_{v(j)}$. There exists a unique sequence of lines of T that links v_r to $v(j)$, which we denote by $((j_1, j'_1), \cdots, (j_\rho, j'_\rho))$, the convention being that the line (j, j') is oriented from j to j'. The *path from* $J^{(v_r)}$ *to* j is the sequence $\mathbf{z}_{j_1}, \mathbf{z}_{j'_1}, \mathbf{z}_{j_2}, \cdots$ and so forth, until j is reached. We can therefore write

$$x_{j,i} - x_{J^{(v_r)},i} = \sum_{p=1}^{\rho} (z_{j_p,i} + z_{j'_p,i}).$$

4 - Bound in terms of number of spanning trees. Let us now turn to the object of interest, namely the left side of (I.5.13). It follows from (I.5.2) and (I.5.8) that

$$B^{(h)}_{2l,\underline{\varpi}}(\underline{\mathbf{x}}) = \sum_{N=1}^{\infty} \sum_{\tau\in\mathcal{T}^{(h)}_N} \sum_{T\in\mathbf{T}(\tau)} \sum_{\underline{l}_\tau} \sum_{\underline{\varpi}_\tau} \int d\underline{\mathbf{x}}_\tau \sum_{\mathbf{P}\in\tilde{\mathcal{P}}_{\tau,\underline{l}_\tau,\ell_0 \, |P_{v_0}|=2l}} \prod_{v\in\tilde{\mathfrak{V}}(\tau)} \frac{(-1)^{s_v}}{s_v!} \rho^{(T_v)}_v. \tag{I.5.21}$$

Therefore, using the bound (I.5.20), the change of variables defined above and the decomposition of $(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m$ described above, we find

$$\frac{1}{\beta|\Lambda|}\sum_{\underline{\varpi}}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\varpi}}(\underline{\mathbf{x}})\right| \leqslant \frac{1}{\beta|\Lambda|}\sum_{N=1}^{\infty}\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{T\in\mathbf{T}(\tau)}\sum_{\underline{l}_\tau}\sum_{\underline{\varpi}_\tau}\int d\mathbf{x}_0 \sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}_{\tau,\underline{l}_\tau,\ell_0}\\ |P_{v_0}|=2l}}$$
$$\sum_{\substack{(m_\ell)_{\ell\in T},(m_v)_{v\in\mathfrak{E}(\tau)}\\ \sum(m_\ell+m_v)=m}}\prod_{v\in\mathfrak{V}(\tau)}\left(\frac{1}{s_v!}\left(C_G 2^{(c_k-c_g)h_v}\right)^{n_v-(s_v-1)}\prod_{\ell\in T_v}\left(\int d\mathbf{z}_\ell\ \left|\mathbf{z}_\ell^{m_\ell}g_{(h_v),\ell}(\mathbf{z}_\ell)\right|\right)\right)\cdot \tag{I.5.22}$$
$$\cdot\prod_{v\in\mathfrak{E}(\tau)}\int d\underline{\mathbf{z}}^{(v)}\ \left|(\underline{\mathbf{z}}^{(v)})^{m_v}W^{(h_v-1)}_{2l_v,\underline{\varpi}_v}(\underline{\mathbf{z}}^{(v)})\right|$$

(we recall that by definition, if $v\in\mathfrak{E}(\tau)$, $I_v=P_v$ and $|I_v|=2l_v$) in which we inject (I.5.10) and (I.5.12) to find

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\varpi}}(\underline{\mathbf{x}})\right| \leqslant \sum_{N=1}^{\infty}\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{T\in\mathbf{T}(\tau)}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}_{\tau,\underline{l}_\tau,\ell_0}\\ |P_{v_0}|=2l}} C_1^N\mathfrak{F}_h(\underline{m})\cdot$$
$$\cdot\prod_{v\in\mathfrak{V}(\tau)}\frac{1}{s_v!}C_G^{n_v-s_v+1}C_g^{s_v-1}2^{h_v((c_k-c_g)n_v-c_k(s_v-1))}\cdot \tag{I.5.23}$$
$$\cdot\prod_{v\in\mathfrak{E}(\tau)}c_2^{2l_v}\mathfrak{C}_{2l_v}|U|^{\max(1,l_v-1)}2^{(h_v-1)(c_k-(c_k-c_g)l_v)}$$

in which C_1^N is an upper bound on the number of terms in the sum over (m_l) and (m_v) in the previous equation, and c_2 denotes the number of elements in the sum over ϖ_v. Recalling that $n_v=|R_v|/2=(\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|)/2$, we re-arrange (I.5.23) by using

$$\left\{\begin{array}{l}\displaystyle\sum_{v\in\mathfrak{V}(\tau)}h_v|R_v| = -h|P_{v_0}|-\sum_{v\in\mathfrak{V}(\tau)}|P_v|+\sum_{v\in\mathfrak{E}(\tau)}(h_v-1)|I_v|\\ \displaystyle\sum_{v\in\mathfrak{V}(\tau)}h_v(s_v-1)=-h-\sum_{v\in\mathfrak{V}(\tau)}1+\sum_{v\in\mathfrak{E}(\tau)}(h_v-1)\end{array}\right.$$

and

$$\left\{\begin{array}{l}\displaystyle\sum_{v\in\mathfrak{V}(\tau)}|R_v|=|I_{v_0}|-|P_{v_0}|\\ \displaystyle\sum_{v\in\mathfrak{V}(\tau)}(s_v-1)=N-1\end{array}\right.$$

to find

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\varpi}}(\underline{\mathbf{x}})\right| \leqslant C_G^{-l}\sum_{N=1}^{\infty}\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{T\in\mathbf{T}(\tau)}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}_{\tau,\underline{l}_\tau,\ell_0}\\ |P_{v_0}|=2l}} C_1^N(C_gC_G^{-1})^{N-1}\cdot$$
$$\cdot 2^{h(c_k-(c_k-c_g)l)}\mathfrak{F}_h(\underline{m})\prod_{v\in\mathfrak{V}(\tau)}\frac{1}{s_v!}2^{c_k-(c_k-c_g)\frac{|P_v|}{2}}\prod_{v\in\mathfrak{E}(\tau)}(c_2^2C_G)^{l_v}\mathfrak{C}_{2l_v}|U|^{\max(1,l_v-1)}. \tag{I.5.24}$$

5 - Bound on the number of spanning trees. Finally, the number of choices for T can be bounded (see [GM01, lemma A.5])

$$\sum_{T\in\mathbf{T}(\tau)}1\leqslant\prod_{v\in\mathfrak{V}(\tau)}c_3^{\frac{|R_v|}{2}}s_v! \tag{I.5.25}$$

so that by injecting (I.5.25) into (I.5.24), we find (I.5.13), with $C_2 = c_2^2 c_3$ and $C_3 = c_3^{-1}$. $\square$

I.5.3. Schwinger function from the effective potential

In this section we show how to compute $\bar{\mathcal{W}}^{(h)}$ in a similarly general setting as above: consider

$$-\beta|\Lambda|\mathfrak{e}_h - \mathcal{Q}^{(h)}(\psi^{(\leqslant h)}) - \bar{\mathcal{W}}^{(h)}(\psi^{(\leqslant h)}, \hat{J}_{\mathbf{k},\underline{\alpha}}) = \sum_{N=1}^{\infty} \frac{(-1)^N}{N!} \mathcal{E}^T_{h+1}\left(\bar{\mathcal{W}}^{(h+1)}(\psi^{(\leqslant h)} + \psi^{(h+1)}, \hat{J}_{\mathbf{k},\underline{\alpha}}); N\right) \tag{I.5.26}$$

for $h \in \{h_2^*, \cdots, h_1^*-1\}$. This discussion will not be used in the ultraviolet regime, so we can safely discard the cases in which the propagator is not renormalized. Unlike (I.5.1), it is necessary to separate the α indices from the (ω, j) indices, so we write the propagator of $\mathcal{E}^T_{h+1}$ as $g_{(h+1,\varpi),(\alpha,\alpha')}$ where ϖ stands for ω in the first and second regimes, and (ω, j) in the third.

We now rewrite the terms in the right side of (I.5.26) in terms of the effective potential $\mathcal{V}^{(h)}$. Let

$$\mathcal{X}^{(h)}(\psi, \hat{J}_{\mathbf{k},\underline{\alpha}}) := \mathcal{V}^{(h)}(\psi) - \bar{\mathcal{W}}^{(h)}(\psi, \hat{J}_{\mathbf{k},\underline{\alpha}}). \tag{I.5.27}$$

Note that the terms in $\mathcal{X}^{(h)}$ are either linear or quadratic in $\hat{J}_{\mathbf{k},\underline{\alpha}}$, simply because the two J variables we have at our disposal, $\hat{J}^+_{\mathbf{k},\alpha_1}, \hat{J}^-_{\mathbf{k},\alpha_2}$, are Grassmann variables and square to zero. We define the functional derivative of $\mathcal{V}^{(h)}$ with respect to $\hat{\psi}^\pm_{\mathbf{k},\alpha}$:

$$\partial^\pm_{\mathbf{k},\alpha}\mathcal{V}^{(h)}(\psi) := \int d\hat{\psi}^\pm_{\mathbf{k},\alpha}\ \mathcal{V}^{(h)}(\psi).$$

Lemma I.5.3

Assume that, for $h = h_1^*$,

$$\begin{aligned}\mathcal{X}^{(h)}(\psi, \hat{J}_{\mathbf{k},\underline{\alpha}}) = {} & \hat{J}^+_{\mathbf{k},\alpha_1} s^{(h)}_{\alpha_1,\alpha_2}(\mathbf{k})\hat{J}^-_{\mathbf{k},\alpha_2} + \sum_{\alpha'}(\hat{J}^+_{\mathbf{k},\alpha_1} q^{+(h)}_{\alpha_1,\alpha'}(\mathbf{k})\hat{\psi}^-_{\mathbf{k},\alpha'} + \hat{\psi}^+_{\mathbf{k},\alpha'} q^{-(h)}_{\alpha',\alpha_2}(\mathbf{k})\hat{J}^-_{\mathbf{k},\alpha_2}) \\ & + \sum_{\alpha'}\left(\partial^-_{\mathbf{k},\alpha'}\mathcal{V}^{(h)}(\psi)\bar{G}^{-(h)}_{\alpha',\alpha_2}(\mathbf{k})\hat{J}^-_{\mathbf{k},\alpha_2} - \hat{J}^+_{\mathbf{k},\alpha_1}\bar{G}^{+(h)}_{\alpha_1,\alpha'}(\mathbf{k})\partial^+_{\mathbf{k},\alpha'}\mathcal{V}^{(h)}(\psi)\right) \\ & + \sum_{\alpha',\alpha''}\left(\hat{J}^+_{\mathbf{k},\alpha_1}\bar{G}^{+(h)}_{\alpha_1,\alpha'}(\mathbf{k})\partial^+_{\mathbf{k},\alpha'}\partial^-_{\mathbf{k},\alpha''}\mathcal{V}^{(h)}(\psi)\bar{G}^{-(h)}_{\alpha'',\alpha_2}(\mathbf{k})\hat{J}^-_{\mathbf{k},\alpha_2}\right)\end{aligned} \tag{I.5.28}$$

for some $s^{(h_1^*)}_{\alpha_1,\alpha_2}(\mathbf{k})$, $q^{\pm(h_1^*)}_{\alpha,\alpha'}(\mathbf{k})$, $\bar{G}^{(h_1^*)}_{\alpha,\alpha'}(\mathbf{k})$. Then (I.5.28) holds for $h \in \{h_2^*, \ldots, h_1^* - 1\}$ as well, with

$$\left\{\begin{aligned} \bar{G}^{+(h)}_{\alpha,\alpha'}(\mathbf{k}) &:= \bar{G}^{+(h+1)}_{\alpha,\alpha'}(\mathbf{k}) + \sum_{\alpha'',\varpi} q^{+(h+1)}_{\alpha,\alpha''}(\mathbf{k})\hat{g}_{(h+1,\varpi),(\alpha'',\alpha')}(\mathbf{k}) \\ \bar{G}^{-(h)}_{\alpha,\alpha'}(\mathbf{k}) &:= \bar{G}^{-(h+1)}_{\alpha,\alpha'}(\mathbf{k}) + \sum_{\alpha'',\varpi} \hat{g}_{(h+1,\varpi),(\alpha,\alpha'')}(\mathbf{k}) q^{-(h+1)}_{\alpha'',\alpha'}(\mathbf{k}) \end{aligned}\right. \tag{I.5.29}$$

$$\left\{\begin{aligned} q^{+(h)}_{\alpha,\alpha'}(\mathbf{k}) &:= q^{+(h+1)}_{\alpha,\alpha'}(\mathbf{k}) - \sum_{\alpha''}\bar{G}^{+(h)}_{\alpha,\alpha''}(\mathbf{k})\hat{W}^{(h)}_{2,(\alpha'',\alpha')}(\mathbf{k}) \\ q^{-(h)}_{\alpha,\alpha'}(\mathbf{k}) &:= q^{-(h+1)}_{\alpha,\alpha'}(\mathbf{k}) - \sum_{\alpha''}\hat{W}^{(h)}_{2,(\alpha,\alpha'')}(\mathbf{k})\bar{G}^{-(h)}_{\alpha'',\alpha'}(\mathbf{k}) \end{aligned}\right. \tag{I.5.30}$$

and

$$\begin{aligned} s^{(h)}_{\alpha_1,\alpha_2}(\mathbf{k}) := {} & s^{(h+1)}_{\alpha_1,\alpha_2}(\mathbf{k}) + \sum_{\alpha',\alpha'',\omega} q^{+(h+1)}_{\alpha_1,\alpha'}(\mathbf{k})\hat{g}_{(h+1,\varpi),(\alpha',\alpha'')}(\mathbf{k}) q^{-(h+1)}_{\alpha'',\alpha_2}(\mathbf{k}) \\ & - \sum_{\alpha',\alpha''}\bar{G}^{+(h)}_{\alpha_1,\alpha'}(\mathbf{k})\hat{W}^{(h)}_{2,(\alpha',\alpha'')}(\mathbf{k})\bar{G}^{-(h)}_{\alpha'',\alpha_2}(\mathbf{k}) \end{aligned} \tag{I.5.31}$$

in which the sums over α are sums over the indices of g.

The (inductive) proof of lemma I.5.3 is straightforward, although it requires some bookkeeping, and is left to the reader.

Remark: It follows from (I.4.24) and (I.2.24) that the two-point Schwinger function $s_2(\mathbf{k})$ is given by $s_2(\mathbf{k}) = s^{(\mathfrak{h}_\beta)}(\mathbf{k})$ (indeed, once all of the fields have been integrated, $\mathcal{X}^{(\mathfrak{h}_\beta)} = \hat J^+_{\mathbf{k}} s^{(h)}(\mathbf{k})\hat J^-_{\mathbf{k}}$). Therefore (I.5.31) is an inductive formula for the two-point Schwinger function.

I.6. Ultraviolet integration

We now detail the integration over the ultraviolet regime. We start from the tree expansion in the general form discussed in section I.5, with $q = \ell_0 = 1$ and $h_1^* = M$; note that by construction these trees have no local leaves. As mentioned in the first remark after lemma I.5.2, we cannot apply that lemma to prove convergence of the tree expansion: however, as we shall see in a moment, a simple re-organization of it will allow to derive uniformly convergent bounds. We recall the estimates (I.4.46) and (I.4.43) of $\hat g_h$ in the ultraviolet regime: for $m_0 + m_k \leqslant 3$

$$\left\{\begin{array}{l} \displaystyle\int d\mathbf{x}\ x_0^{m_0} x^{m_k} |g_h(\mathbf{x})| \leqslant (\text{const.})\ 2^{-h-m_0h} \\ \displaystyle\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}} |\hat g_h(\mathbf{k})| \leqslant (\text{const.}). \end{array}\right. \tag{I.6.1}$$

Equation (I.6.1) has the same form as (I.5.10), with

$$c_g = c_k = 1, \quad \mathfrak{F}_h(\underline{m}) = 2^{-m_0 h}.$$

We now move on to the power counting estimate. The first remark to be made is that the values of the leaves have a much better dimensional estimate than the one assumed in lemma I.5.2. In fact, the value of any leaf, called $W^{(M)}_{4,\underline{\alpha}}(\underline{\mathbf{x}})$, is the antisymmetric part of

$$\delta_{\alpha_1,\alpha_2}\delta_{\alpha_3,\alpha_4}\delta(\mathbf{x}_1-\mathbf{x}_2)\delta(\mathbf{x}_3-\mathbf{x}_4)Uw_{\alpha_1,\alpha_3}(\mathbf{x}_1-\mathbf{x}_3) \tag{I.6.2}$$

so that

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ |(\underline{\mathbf{x}}-\mathbf{x}_4)^m W^{(M)}_{4,\underline{\alpha}}(\underline{\mathbf{x}})| \leqslant \mathfrak{C}'_4|U|. \tag{I.6.3}$$

1 - Resumming trivial branches. Next, we re-sum the branches of Gallavotti-Nicolò trees that are only followed by a single endpoint: the naive dimensional bound on the value of these branches tends to diverge logarithmically as $M\to\infty$, but one can easily exhibit a cancellation that improves their estimate, as explained below. Consider a tree τ made of a single branch, with a root on scale h and a single leaf on scale $M+1$ with value $W_4^{(M)}$. The 4-field kernel associated with such a tree is $K^{(h)}_{4,\underline{\alpha}}(\underline{\mathbf{x}}) := W^{(M)}_{4,\underline{\alpha}}(\underline{\mathbf{x}})$. The 2-field kernel associated with τ, once summed over the choices of P_v and over the field labels it indexes for $h+1 < h_v \leqslant M$, keeping P_{v_0} and its field labels fixed, can be computed explicitly:

$$K^{(h)}_{2,(\alpha,\alpha')}(\mathbf{x}) = 2U\sum_{h'=h+1}^{M}\left(w_{\alpha,\alpha'}(\mathbf{x})g^{(h')}_{\alpha,\alpha'}(\mathbf{x}) - \delta_{\alpha,\alpha'}\delta(\mathbf{x})\sum_{\alpha_2}\int d\mathbf{y}\ w_{\alpha,\alpha_2}(\mathbf{y})g^{(h')}_{\alpha_2,\alpha_2}(\mathbf{0})\right). \tag{I.6.4}$$

If one were to bound the right side of (I.6.4) term by term in the sum over h' using the dimensional estimates on the propagator (see (I.4.46) and following), one would find a *logarithmic* divergence for $\int d\mathbf{x}|K^{(h)}_{2,(\alpha,\alpha')}(\mathbf{x})|$, i.e. a bound proportional to $M-h$. However, the right side of (I.6.4) depends on propagators evaluated at $x_0=0$ (because $w(\mathbf{x})$ is proportional to $\delta(x_0)$), so we can use an improved bound on the propagator $g_{h'}$: the dominant terms in $\hat g_h(\mathbf{k})$ are odd in k_0, so they cancel when considering

$$\sum_{k_0\in\frac{2\pi}{\beta}(\mathbb{Z}+\frac12)}\hat g_h(\mathbf{k}).$$

From this idea, we compute an improved bound for $|g_h(\mathbf{x})|$ with $x_0=0$:

$$|g_h(0,x,y)|\leqslant\sum_{k_x,k_y}\left|\sum_{k_0}\hat g_h(\mathbf{k})\right|\leqslant(\mathrm{const.})\ 2^{-h}.$$

All in all, we find

$$\int d\mathbf{x}\ |\mathbf{x}^mK^{(h)}_{2,(\alpha,\alpha')}(\mathbf{x})|\leqslant\mathfrak{C}_4|U|,\quad\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ |(\underline{\mathbf{x}}-\mathbf{x}_4)^mK^{(h)}_{4,\underline{\alpha}}(\underline{\mathbf{x}})|\leqslant\mathfrak{C}_4|U| \tag{I.6.5}$$

for some constant $\mathfrak{C}_4$. We then re-organize the right side of (I.5.7) by:

1. summing over the set of *contracted* trees $\tilde{\mathcal{T}}^{(h)}_N$, which is defined like $\mathcal{T}^{(h)}_N$ but for the fact that every node $v\succ v_0$ that is not an endpoint must have at least two endpoints following it, and the endpoints can be on any scale in $[h+2,M+1]$;
2. re-defining the value of the endpoints to be $\tilde\rho_v=K^{(h_v-1)}_{2l_v}$, with $l_v=1,2$.

2 - Contracted tree expansion. We can now estimate the "contracted tree" expansion, by repeating the steps of the proof of lemma I.5.2, thus finding

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^mW^{(h)}_{2l,\underline{\alpha}}(\underline{\mathbf{x}})\right|\leqslant\sum_{N=1}^{\infty}\sum_{\tau\in\tilde{\mathcal{T}}^{(h)}_N}\sum_{T\in\mathbf{T}(\tau)}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\mathcal{P}^{(h)}_{\tau,\underline{l}_\tau,1}\\|P_{v_0}|=2l}}c_1^N\cdot\\ \cdot\prod_{v\in\mathfrak{V}(\tau)}\frac{1}{s_v!}2^{-h_v(s_v-1)}\prod_{v\in\mathfrak{E}(\tau)}c_2^4\mathfrak{C}_4|U| \tag{I.6.6}$$

for two constants c_1 and c_2 in which the sum over $\underline{l}_\tau$ is a sum over the $l_v\in\{1,2\}$. It then follows from the following equation

$$\sum_{v\in\mathfrak{V}(\tau)}h_v(s_v-1)=h(N-1)+\sum_{v\in\mathfrak{V}(\tau)}(N_v-1)$$

in which N_v denotes the number of endpoints following $v\in\tau$, which can be proved by induction, that

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^mW^{(h)}_{2l,\underline{\alpha}}(\underline{\mathbf{x}})\right|\\ \leqslant\sum_{N=1}^{\infty}(|U|c_3)^N2^{-h(N-1)}\sum_{\tau\in\tilde{\mathcal{T}}^{(h)}_N}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\mathcal{P}^{(h)}_{\tau,\underline{l}_\tau,1}\\|P_{v_0}|=2l}}\prod_{v\in\mathfrak{V}(\tau)}2^{-(N_v-1)}. \tag{I.6.7}$$

Furthermore, we notice that by the definition of $\mathcal{P}^{(h)}_{\tau,\underline{l}_\tau,1}$, $|P_v|\leqslant 2N_v+2$. In particular, for $v=v_0$, $2l\leqslant 2N+2$, so the sum over N actually starts at $\max\{1,l-1\}$:

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m W^{(h)}_{2l,\underline{\alpha}}(\underline{\mathbf{x}})\right|$$

$$\leqslant\sum_{N=\max\{1,l-1\}}^{\infty}(|U|c_3)^N 2^{-h(N-1)}\sum_{\tau\in\tilde{\mathcal{T}}^{(h)}_N}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\mathcal{P}^{(h)}_{\tau,\underline{l}_\tau,1}\\ |P_{v_0}|=2l}}\prod_{v\in\mathfrak{V}(\tau)}2^{-(N_v-1)}. \tag{I.6.8}$$

3 - Bound on the contribution at fixed N. We temporarily restrict to the case $N>1$. We bound

$$\mathfrak{T}_N:=\sum_{\tau\in\tilde{\mathcal{T}}^{(h)}_N}\sum_{\underline{l}_\tau}\sum_{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,1}}\prod_{v\in\mathfrak{V}(\tau)}2^{-(N_v-1)}.$$

Since $N_v\geqslant 2$ and $|P_v|\leqslant 2N_v+2$, $\forall\mu\in(0,1)$,

$$-(N_v-1)\leqslant\min\{2-\frac{|P_v|}{2},-1\}\leqslant(1-\mu)\min\{2-\frac{|P_v|}{2},-1\}-\mu\leqslant-(1-\mu)\frac{|P_v|}{6}-\mu$$

so that

$$\mathfrak{T}_N\leqslant\sum_{\tau\in\tilde{\mathcal{T}}^{(h)}_N}\sum_{\underline{l}_\tau}\sum_{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,1}}\prod_{v\in\mathfrak{V}(\tau)}2^{-(1-\mu)\frac{|P_v|}{6}}2^{-\mu}.$$

3-1 - Bound on the field label assignments. We bound

$$\sum_{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,1}}\prod_{v\in\mathfrak{V}(\tau)}2^{-(1-\mu)\frac{|P_v|}{6}}.$$

We proceed by induction: if v_0 denotes the first node of τ (i.e. the node immediately following the root), $(v_1,\cdots,v_s)$ its children, and $(\tau_1,\cdots,\tau_s)$ the sub-trees with first node $(v_1,\cdots,v_s)$, then

$$\begin{aligned}\sum_{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,1}}\prod_{v\in\mathfrak{V}(\tau)}2^{-(1-\mu)\frac{|P_v|}{6}}&\leqslant\sum_{\mathbf{P}_1\in\mathcal{P}(\tau_1)}\cdots\sum_{\mathbf{P}_s\in\mathcal{P}(\tau_s)}\sum_{p_{v_0}=0}^{|P_{v_1}|+\cdots+|P_{v_s}|}\binom{|P_{v_1}|+\cdots+|P_{v_s}|}{p_{v_0}}\cdot\\&\qquad\cdot 2^{-\frac{1-\mu}{6}p_{v_0}}\prod_{i=1}^{s}\prod_{v\in\mathfrak{V}(\tau_i)}2^{-(1-\mu)\frac{|P_v|}{6}}\\&=\prod_{i=1}^{s}\left(\sum_{\mathbf{P}_i\in\mathcal{P}(\tau_i)}(1+2^{-\frac{1-\mu}{6}})^{|P_{v_i}|}\prod_{v\in\mathfrak{V}(\tau_i)}2^{-(1-\mu)\frac{|P_v|}{6}}\right)\end{aligned}$$

so that by iterating this step down to the leaves, we find

$$\sum_{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,1}}\prod_{v\in\mathfrak{V}(\tau)}2^{-(1-\mu)\frac{|P_v|}{6}}\leqslant\left(\sum_{p=0}^{M-h}2^{-\frac{1-\mu}{6}p}\right)^{4N}\leqslant C_P^N \tag{I.6.9}$$

for some constant C_P.

3-2 - Bound on trees. Finally, we bound

$$\sum_{\tau\in\mathcal{T}^{(h)}_N}\prod_{v\in\mathfrak{V}(\tau)}2^{-\mu}.$$

We can re-express the sum over τ as a sum over trees with no scale labels that are such that each node that is not a leaf has at least two children, and a sum over scale labels:

$$\sum_{\tau\in\mathcal{T}_N^{(h)}} = \sum_{\tau^*\in\mathcal{T}_N^*}\sum_{\mathbf{h}\in\mathbf{H}_h(\tau^*)}$$

in which $\mathcal{T}_N^*$ denotes the set of unlabeled rooted trees with N endpoints and $\mathbf{H}_h(\tau^*)$ denotes the set of scale labels *compatible* with τ^*. Therefore

$$\sum_{\tau\in\mathcal{T}_N^{(h)}}\prod_{v\in\mathfrak{V}(\tau)}2^{-\mu} = \sum_{\tau^*\in\mathcal{T}_N^*}\sum_{\mathbf{h}\in\mathbf{H}_h(\tau^*)}\prod_{v\in\mathfrak{V}(\tau^*)}2^{-\mu(h_v-h_{p(v)})}$$

in which $p(v)$ denotes the parent of v, so that

$$\sum_{\tau\in\mathcal{T}_N^{(h)}}\prod_{v\in\mathfrak{V}(\tau)}2^{-\mu} \leqslant \sum_{\tau^*\in\mathcal{T}_N^*}\prod_{v\in\mathfrak{V}(\tau^*)}\sum_{q=1}^{\infty}2^{-\mu q} \leqslant \sum_{\tau^*\in\mathcal{T}_N^*}C_{T,1}^N$$

for some constant $C_{T,1}$, in which we used the fact that $|\mathfrak{V}(\tau^*)|\leqslant N$. Furthermore, it is a well known fact that $\sum_{\tau^*}1\leqslant 4^N$ (see e.g. [GM01, lemma A.1], the proof is based on constructing an injective map to the set of random walks with $2N$ steps: given a tree, consider a walker that starts at the root, and then travels over branches towards the right until it reaches a leaf, and then travels left until it can go right again on a different branch). Therefore

$$\sum_{\tau\in\mathcal{T}_N^{(h)}}\prod_{v\in\mathfrak{V}(\tau)}2^{-\mu}\leqslant C_T^N \tag{I.6.10}$$

for some constant C_T.

3-3 - Conclusion of the proof. Therefore, by combining (I.6.9) and (I.6.10) with the trivial estimate $\sum_{\underline{l}_\tau}1\leqslant 2^N$, we find

$$\mathfrak{T}_N\leqslant(\text{const.})^N. \tag{I.6.11}$$

Equation (I.6.11) trivially holds for $N=1$ as well. If we inject (I.6.11) into (I.6.8) we get:

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m W_{2l,\underline{\alpha}}^{(h)}(\underline{\mathbf{x}})\right| \leqslant \sum_{N=\max\{1,l-1\}}^{\infty}(|U|C')^N 2^{-h(N-1)} \tag{I.6.12}$$

for some constant C' and $h\geqslant 0$. In conclusion, if $|U|$ is small enough (uniformly in h and l),

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m W_{2l,\underline{\alpha}}^{(h)}(\underline{\mathbf{x}})\right| \leqslant (|U|C_0)^{\max\{1,l-1\}}2^{-h(\max\{1,l-1\}-1)} \tag{I.6.13}$$

for some constant $C_0>0$.

I.7. First regime

We now study the first regime. We consider the tree expansion in the general form discussed in section I.5, with $h_1^*=\mathfrak{h}_0$ and $q=\ell_0=2$, so that there are no local leaves, i.e., all leaves are irrelevant, on scale $\bar{\mathfrak{h}}_0+1$. Recall that the truncated expectation $\mathcal{E}_{h+1}^T$ in the right side of (I.5.1) is with respect to the dressed propagator $\bar{g}_{h+1}$ in (I.4.13), so that (I.5.1) is to be interpreted as (I.4.8). A non trivial aspect of the analysis is that we do not have a priori bounds on the dressed propagator, but just on the "bare" one $g_{h,\omega}$, see (I.4.50), (I.4.48). The goal is to show inductively on h that the same qualitative bounds are valid for $\bar g_{h,\omega}$, namely

$$\left\{\begin{array}{l}\displaystyle\int d\mathbf{x}\ |\mathbf{x}^m\bar g_{h,\omega}(\mathbf{x})|\leqslant C_g2^{-h}2^{-mh}\\[2ex] \displaystyle\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}}|\hat{\bar g}_{h,\omega}(\mathbf{k})|\leqslant C_G2^{2h}\end{array}\right. \tag{I.7.1}$$

which in terms of the hypotheses of lemma I.5.2 means

$$c_k=3,\ c_g=1,\quad \mathfrak{F}_h(\underline{m})=2^{-mh}.$$

Note that $\ell_0=\lceil c_k/(c_k-c_g)\rceil>c_k/(c_k-c_g)$, as desired.

I.7.1. Power counting in the first regime

It follows from lemma I.5.2 and (I.6.13) that

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right| \leqslant 2^{h(3-2l)}2^{-mh}\sum_{N=1}^{\infty}\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\|P_{v_0}|=2l}}C_1'^N\prod_{v\in\mathfrak{V}(\tau)}2^{(3-|P_v|)}\prod_{v\in\mathfrak{E}(\tau)}C_1''^{l_v}|U|^{\max(1,l_v-1)} \tag{I.7.2}$$

for two constants C_1' and C_1''.

1 - Bounding the sum on trees. First, we notice that the sum over $\underline{l}_\tau$ can be written as a sum over $l_1,\cdots,l_N$, so that it can be moved before $\sum_\tau$. We focus on the sum

$$\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\|P_{v_0}|=2l}}\prod_{v\in\mathfrak{V}(\tau)}2^{(3-|P_v|)}. \tag{I.7.3}$$

We first consider the case $l\geqslant 2$. For all $\theta\in(0,1)$,

$$\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\|P_{v_0}|=2l}}\prod_{v\in\mathfrak{V}(\tau)}2^{(3-|P_v|)}=\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\|P_{v_0}|=2l}}\prod_{v\in\mathfrak{V}(\tau)}2^{(\theta+(1-\theta))(3-|P_v|)}$$

and since $\ell_0=2$, $|P_v|\geqslant 4$ for every node v that is not the first node or a leaf, so that $3-|P_v|\leqslant -|P_v|/4$. Now, if $N\geqslant 2$, then given τ, let v_τ^* be the node with at least two children that is closest to the root, and h_τ^* its scale. Using the fact that $|P_v|\geqslant 2l+2$ for all $v\prec v_\tau^*$ and the fact that τ has at least two branches on scales $\geqslant h_\tau^*$, we have

$$\prod_{v\in\mathfrak{V}(\tau)}2^{\theta(3-|P_v|)}\leqslant 2^{\theta(2l-1)(h-h_\tau^*)}2^{2\theta h_\tau^*}.$$

If $N=1$, we let $h_\tau^*:=0$, and note that the same estimate holds. Therefore

$$\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\|P_{v_0}|=2l}}\prod_{v\in\mathfrak{V}(\tau)\setminus\{v_0\}}2^{(\theta+(1-\theta))(3-|P_v|)} \leqslant\sum_{h_\tau^*=h+1}^{0}2^{\theta(2l-1)(h-h_\tau^*)+2\theta h_\tau^*}\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,2}}\prod_{v\in\mathfrak{V}(\tau)\setminus\{v_0\}}2^{-(1-\theta)\frac{|P_v|}{4}}$$

which we bound in the same way as in the proof of (I.6.11), i.e. splitting

$$(1-\theta)\frac{|P_v|}{2} = (1-\theta)(1-\mu)\frac{|P_v|}{4} + (1-\theta)\mu\frac{|P_v|}{4} \geqslant (1-\theta)(1-\mu)\frac{|P_v|}{4} + (1-\theta)\mu$$

for all $\mu \in (0,1)$ and bounding

$$\sum_{\mathbf{P}\in\mathcal{P}_{\tau,\underline{l}_\tau,2}} \prod_{v\in\mathfrak{V}(\tau)\setminus\{v_0\}} 2^{-(1-\theta)(1-\mu)\frac{|P_v|}{4}} \leqslant C_P^{\sum_{i=1}^N l_i}$$

and

$$\sum_{\tau\in\mathcal{T}_N^{(h)}} \prod_{v\in\mathfrak{V}(\tau)\setminus\{v_0\}} 2^{-(1-\theta)\mu} \leqslant C_T^N.$$

Therefore if $l \geqslant 2$, then

$$\sum_{\tau\in\mathcal{T}_N^{(h)}} \sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\ |P_{v_0}|=2l}} \prod_{v\in\mathfrak{V}(\tau)\setminus\{v_0\}} 2^{(\theta+(1-\theta))(3-|P_v|)} \leqslant 2^{2\theta h} C_T^N \prod_{i=1}^N C_P^{l_i}. \tag{I.7.4}$$

Consider now the case with $l=1$. If $N=1$ then the sum over τ is trivial, i.e., $\mathcal{T}_1^{(h)}$ consists of a single element, and the sum over $\mathbf{P}$ can be bounded as

$$\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,l_1,2}\\ |P_{v_0}|=2}} \prod_{v\in\mathfrak{V}(\tau)} 2^{(3-|P_v|)} \leqslant 2^h \sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,l_1,2}\\ |P_{v_0}|=2}} \prod_{\substack{v\in\mathfrak{V}(\tau)\\ v\succeq v'}} 2^{4-|P_v|}, \tag{I.7.5}$$

where v' is, if it exists, the leftmost node such that $|P_v|>4$, in which case $4-|P_v| \leqslant -|P_v|/3$; otherwise, we interpret the product over v as 1. Proceeding as in the case $l \geqslant 2$, we bound the right side of (I.7.5) by

$$2^h C^{l_1} \sum_{h_{v'}=h+2}^{0} 2^{2\theta h_{v'}} \leqslant 2^h C' C^{l_1}. \tag{I.7.6}$$

If $N \geqslant 2$, then we denote by τ^* the subtree with $v_\tau^* : v^*$ as first node, and τ' the linear tree with root on scale h and the endpoint on scale h^*, so that $\tau\tau' \cup \tau^*$. We split (I.7.3) as

$$\sum_{l^*=2}^{\sum_{i=1}^N l_i - N+1} \sum_{h^*=h}^{-2} \sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}_{\tau',l^*,2}\\ |P_{v_0}|=2}} \Big(\prod_{v\in\mathfrak{V}(\tau')} 2^{3-|P_v|}\Big)\Big(\sum_{\tau^*\in\mathcal{T}_N^{(h^*)}} \sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}_{\tau^*,\underline{l}_{\tau^*},2}\\ |P_{v^*}|=2l^*}} \prod_{v\in\mathfrak{V}(\tau^*)} 2^{3-|P_v|}\Big). \tag{I.7.7}$$

The sum in the last parentheses can be bounded as in the case $l \geqslant 2$, yielding $C^{\sum_i l_i} 2^{2\theta h^*}$. The remaining sum can be bounded as in (I.7.5)-(I.7.6) so that, in conclusion,

$$\begin{aligned}\sum_{\tau\in\mathcal{T}_N^{(h)}} \sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\ |P_{v_0}|=2}} \prod_{v\in\mathfrak{V}(\tau)} 2^{(3-|P_v|)} &\leqslant (C')^{\sum_{i=1}^N l_i} \sum_{h^*=h}^{-2} 2^{h-h^*} \sum_{h'=h+2}^{h^*} 2^{2\theta(h'-h^*)} 2^{2\theta h^*}\\ &\leqslant (C'')^{\sum_{i=1}^N l_i} 2^h.\end{aligned} \tag{I.7.8}$$

2-1 - $l=1$. Therefore, if $l=1$, (I.7.2) becomes (we recall that $q=2>1$ so that $B_2 = W_2$, see (I.5.9))

$$\int d\mathbf{x}\ \left|\mathbf{x}^m W^{(h)}_{2,\omega,\underline{\alpha}}(\mathbf{x})\right| \leqslant 2^{2h} 2^{-mh} \sum_{N=1}^{\infty} \sum_{l_1,\cdots,l_N \geqslant 2}^{\infty} (C_1'''|U|)^{\sum_{i=1}^N \max(1,l_i-1)} \tag{I.7.9}$$

Assuming $|U|$ is small enough and using the subadditivity of the max function, we rewrite (I.7.9) as

$$\int d\mathbf{x}\ \left|\mathbf{x}^m W^{(h)}_{2,\omega,\underline{\alpha}}(\mathbf{x})\right| \leqslant 2^{2h}2^{-mh}C_1|U| \tag{I.7.10}$$

which we recall holds for $m \leqslant 3$.

2-2 - $l \geqslant 2$**.** Similarly, if $l \geqslant 2$,

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right| \leqslant 2^{h(3-2l+2\theta)}2^{-mh}\sum_{N=1}^{\infty}\sum_{\substack{l_1,\cdots,l_N\geqslant 2\\ (l_1-1)+\cdots+(l_N-1)\geqslant l-1+\delta_{N,1}}}^{\infty}(C_1'''|U|)^{\sum_{i=1}^N \max(1,l_i-1)} \tag{I.7.11}$$

in which the constraint on $l_1,\cdots,l_N$ arises from the fact that, if $N>1$,

$$|P_{v_0}| \leqslant |I_{v_0}| - 2(N-1),$$

while, if $N=1$, $|P_{v_0}|<|I_{v_0}|$. Therefore, assuming that $|U|$ is small enough and summing (I.7.11) over h, we find

$$\left\{\begin{array}{l} \frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_4)^m W^{(h)}_{4,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right| \leqslant 2^{-mh}C_1|U| \\ \frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m W^{(h)}_{2l,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right| \leqslant 2^{h(3-2l+2\theta)}2^{-mh}(C_1|U|)^{l-1} \end{array}\right. \tag{I.7.12}$$

for $l \geqslant 3$ and $m \leqslant 3$.

Remark: The estimates (I.7.2) and (I.7.8) imply the convergence of the tree expansion (I.5.8), thus providing a convergent expansion of $W^{(h)}_{2l,\underline{\omega},\underline{\alpha}}$ in U.

I.7.2. The dressed propagator

We now prove the estimate (I.7.1) on the dressed propagator by induction. We recall (I.4.13)

$$\left(\hat{\bar{g}}_{h,\omega}(\mathbf{k})\right)^{-1} = f^{-1}_{h,\omega}(\mathbf{k})\hat{\bar{A}}^{(h,\omega)}(\mathbf{k}) \tag{I.7.13}$$

with

$$\hat{\bar{A}}^{(h,\omega)}(\mathbf{k}) := \hat{A}(\mathbf{k}) + f_{\leqslant h,\omega}(\mathbf{k})\hat{W}^{(h)}_2(\mathbf{k}) + \sum_{h'=h+1}^{\bar{\mathfrak{h}}_0}\hat{W}^{(h')}_2(\mathbf{k})$$

whose inverse Fourier transform is denoted by $\bar{A}^{(h,\omega)}$. Note that (I.7.10) on its own does not suffice to prove (I.7.1) because the bound on

$$f_{\leqslant h,\omega}(\mathbf{k})\hat{W}^{(h)}_2(\mathbf{k}) + \sum_{h'=h+1}^{\bar{\mathfrak{h}}_0}\hat{W}^{(h')}_2(\mathbf{k}) \tag{I.7.14}$$

that it would yield is (const.) $|U|$ whereas on the support of $f_{h,\omega}$, $\hat{g}^{-1}\sim 2^h$, which we cannot compare with $|U|$ unless we impose an ϵ-dependent smallness condition on U, which we do not want. In addition, even if (I.7.14) were bounded by (const.) $|U|2^h$, we would have to face an extra difficulty to bound $\bar{g}$ in $\mathbf{x}$-space: indeed, the naive approach we have used so far (see e.g. (I.4.46)) to bound

$$\int d\mathbf{x}\ |\mathbf{x}^m\bar{g}_{h,\omega}(\mathbf{x})|$$

would require a bound on $\partial^n_{\mathbf{k}}\hat{\bar{g}}_{h,\omega}(\mathbf{k})$ with $n > m+3$ (we recall that the integral over $\mathbf{x}$ is 3-dimensional), which would in turn require an estimate on

$$\int d\mathbf{x}\ |\mathbf{x}^n\bar{g}_{h',\omega}(\mathbf{x})|$$

for $h' > h$, which we do not have (and if we tried to prove it by induction, we would immediately find that the estimate would be required to be uniform in n, which we cannot expect to be true).

In order to overcome both of the previously mentioned difficulties, we will expand $\hat{W}_2^{(h')}$ at first order around $\mathbf{p}^{\omega}_{F,0}$. The contributions up to first order in $\mathbf{k}-\mathbf{p}^{\omega}_{F,0}$ will be called the *local part* of $\hat{W}_2^{(h')}$. Through symmetry considerations, we will write the local part in terms of constants which we can control, and then use (I.7.10) to bound the remainder. In particular, we will prove that $\hat{W}_2^{(h)}(\mathbf{p}^{\omega}_{F,0}) = 0$ from which we will deduce an improved bound for (I.7.14). Furthermore, since the $\mathbf{k}$-dependance of the local part is explicit, we will be able to bound all of its derivatives and bound $\bar{g}$ in $\mathbf{x}$-space.

1 - Local and irrelevant contributions. We define a *localization* operator:

$$\mathcal{L} : \bar{A}_{h,\omega}(\mathbf{x}) \longmapsto \delta(\mathbf{x})\int d\mathbf{y}\ \bar{A}_{h,\omega}(\mathbf{y}) - \partial_{\mathbf{x}}\delta(\mathbf{x})\cdot\int d\mathbf{y}\ \mathbf{y}\bar{A}_{h,\omega}(\mathbf{y}) \tag{I.7.15}$$

where $\delta(x_0,x_1,x_2) := \delta(x_0)\delta_{x_1,0}\delta_{x_2,0}$ and in the second term, as usual, the derivative with respect to x_1 and x_2 is discrete; as well as the corresponding *irrelevator*:

$$\mathcal{R} := \mathbb{1} - \mathcal{L}. \tag{I.7.16}$$

The action of $\mathcal{L}$ on functions on $\mathbf{k}$-space is (up to finite size corrections coming from the fact that $L<\infty$ that do not change the dimensional estimates computed in this section and that we neglect for the sake simplicity)

$$\mathcal{L}\hat{\bar{A}}_{h,\omega}(\mathbf{k}) = \hat{\bar{A}}_{h,\omega}(\mathbf{p}^{\omega}_{F,0}) + (\mathbf{k}-\mathbf{p}^{\omega}_{F,0})\cdot\partial_{\mathbf{k}}\hat{\bar{A}}_{h,\omega}(\mathbf{p}^{\omega}_{F,0}). \tag{I.7.17}$$

Remark: The reason why $\mathcal{L}$ is defined as the first order Taylor expansion, is that its role is to separate the *relevant* and *marginal* parts of $\hat{W}_2^{(h')}$ from the *irrelevant* ones. Indeed, we recall the definition of the *scaling dimension* associated to a kernel $\hat{W}_2^{(h')}$ (see one of the remarks after lemma I.5.2)

$$c_k - (c_k - c_g) = 1$$

which, roughly, means that $\hat{W}_2^{(h')}$ is bounded by $2^{(c_k-(c_k-c_g))h'} = 2^{h'}$. As was remarked above, this bound is insufficient since it does not constrain $\sum_{h'\geqslant h}\hat{W}_2^{(h')}$ to be smaller than $2^h \sim \hat{g}^{-1}$. Note that, while $\hat{W}_2^{(h')}(\mathbf{k})$ is bounded by $2^{h'}$, irrespective of $\mathbf{k}$, $(\mathbf{k}-\mathbf{p}^{\omega}_{F,0})\cdot\partial_{\mathbf{k}}\hat{W}_2^{(h')}(\mathbf{k})$ has an improved dimensional bound, proportional to $2^{h-h'}2^{h'}$, where $2^h \sim |\mathbf{k}-\mathbf{p}^{\omega}_{F,0}|$; in this sense, we can think of the operator $(\mathbf{k}-\mathbf{p}^{\omega}_{F,0})\cdot\partial_{\mathbf{k}}$ as scaling like $2^{h-h'}$. Therefore, the remainder of the first order Taylor expansion is bounded by $2^{2(h-h')}2^{h'} = 2^{2h-h'}$ and thereby has a scaling dimension of -1 (with respect to h'). Thus, by defining $\mathcal{L}$ as the first order Taylor expansion, we take the focus away from the remainder, which can be bounded easily because it is irrelevant (i.e., it has negative scaling dimension), and concentrate our attention on the relevant and marginal contributions of $\hat{W}_2^{(h')}$. See [BG95, chapter 8] for details.

We then rewrite (I.7.13) as

$$\hat{\bar{g}}_{h,\omega}(\mathbf{k}) = f_{h,\omega}(\mathbf{k})\left(\mathcal{L}\hat{\bar{A}}_{h,\omega}(\mathbf{k})\right)^{-1}\left(\mathbb{1} + \left(\mathcal{R}\hat{\bar{A}}_{h,\omega}(\mathbf{k})\right)\left(\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega}(\mathbf{k})\right)\right)^{-1} \tag{I.7.18}$$

where $\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega}$ is a shorthand for

$$\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega}(\mathbf{k}) := (f_{\leqslant h+1,\omega}(\mathbf{k}) - f_{\leqslant h-2,\omega}(\mathbf{k}))\left(\mathcal{L}\hat{\bar{A}}_{h,\omega}(\mathbf{k})\right)^{-1}$$

(we can put in the $(f_{\leqslant h+1,\omega}(\mathbf{k}) - f_{\leqslant h-2,\omega}(\mathbf{k}))$ factor for free because of the initial $f_{h,\omega}(\mathbf{k})$).

2 - Local part. We first compute $\mathcal{L}\hat{\bar{A}}_{h,\omega}(\mathbf{k})$.

2-1 - Non-interacting components. As a first step, we write the local part of the free inverse propagator as

$$\mathcal{L}\hat{A}(\mathbf{k}) = -\begin{pmatrix} ik_0 & \gamma_1 & 0 & \xi^* \\ \gamma_1 & ik_0 & \xi & 0 \\ 0 & \xi^* & ik_0 & \gamma_3\xi \\ \xi & 0 & \gamma_3\xi^* & ik_0 \end{pmatrix} \tag{I.7.19}$$

where

$$\xi := \frac{3}{2}(ik'_x + \omega k'_y). \tag{I.7.20}$$

2-2 - Interacting components. We now turn to the terms coming from the interaction. We first note that $\mathcal{V}^{(h')}$ satisfies the same symmetries as the *initial* potential $\mathcal{V}$ (I.2.20), listed in section I.2.3. Indeed, $\mathcal{V}^{(h')}$ is a function of $\mathcal{V}$ and a quantity similar to (I.2.30) but with an extra cutoff function, which satisfies the symmetries (I.2.32) through (I.2.38). Therefore

$$\begin{aligned}\hat{W}_2^{(h')}(\mathbf{k}) = \hat{W}_2^{(h')}(-\mathbf{k})^* = \hat{W}_2^{(h')}(R_v\mathbf{k}) = \sigma_1\hat{W}_2^{(h')}(R_h\mathbf{k})\sigma_1 = -\sigma_3\hat{W}_2^{(h')}(I\mathbf{k})\sigma_3 \\ = \hat{W}_2^{(h')}(P\mathbf{k})^T = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathcal{T}_{\mathbf{k}}^\dagger \end{pmatrix}\hat{W}_2^{(h')}(T^{-1}\mathbf{k})\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathcal{T}_{\mathbf{k}} \end{pmatrix}.\end{aligned} \tag{I.7.21}$$

This imposes a number of restrictions on $\mathcal{L}\hat{W}_2^{(h')}$: indeed, it follows from propositions I.A6.1 and I.A6.2 (see appendix I.A6) that, since

$$\mathbf{p}_{F,0}^{\omega} = -\mathbf{p}_{F,0}^{-\omega} = R_v\mathbf{p}_{F,0}^{-\omega} = R_h\mathbf{p}_{F,0}^{\omega} = I\mathbf{p}_{F,0}^{\omega} = P\mathbf{p}_{F,0}^{-\omega} = T\mathbf{p}_{F,0}^{\omega} \tag{I.7.22}$$

in which R_v, R_h, I, P and T were defined in section I.2.3, we have

$$\mathcal{L}\hat{W}_2^{(h')}(\mathbf{k}' + \mathbf{p}_{F,0}^{\omega}) = -\begin{pmatrix} i\tilde{\zeta}_{h'}k_0 & \gamma_1\tilde{\mu}_{h'} & 0 & \nu_{h'}\xi^* \\ \gamma_1\tilde{\mu}_{h'} & i\tilde{\zeta}_{h'}k_0 & \nu_{h'}\xi & 0 \\ 0 & \nu_{h'}\xi^* & i\zeta_{h'}k_0 & \gamma_3\tilde{\nu}_{h'}\xi \\ \nu_{h'}\xi & 0 & \gamma_3\tilde{\nu}_{h'}\xi^* & i\zeta_{h'}k_0 \end{pmatrix}, \tag{I.7.23}$$

with $(\tilde{\zeta}_{h'}, \tilde{\mu}_{h'}, \tilde{\nu}_{h'}, \zeta_{h'}, \nu_{h'}) \in \mathbb{R}^5$. Furthermore, it follows from (I.7.10) that if $h' \leqslant \mathfrak{h}_0$, then

$$\begin{aligned}|\tilde{\zeta}_{h'}| \leqslant (\text{const.})\ |U|2^{h'}, \quad |\zeta_{h'}| \leqslant (\text{const.})\ |U|2^{h'}, \quad |\tilde{\mu}_{h'}| \leqslant (\text{const.})\ |U|2^{2h'-h_\epsilon}, \\ |\nu_{h'}| \leqslant (\text{const.})\ |U|2^{h'}, \quad |\tilde{\nu}_{h'}| \leqslant (\text{const.})\ |U|2^{h'-h_\epsilon}.\end{aligned} \tag{I.7.24}$$

Injecting (I.7.19) and (I.7.23) into (I.4.14), we find that

$$\mathcal{L}\hat{\bar{A}}_{h,\omega}(\mathbf{k}' + \mathbf{p}_{F,0}^{\omega}) = -\begin{pmatrix} i\tilde{z}_hk_0 & \gamma_1\tilde{m}_h & 0 & v_h\xi^* \\ \gamma_1\tilde{m}_h & i\tilde{z}_hk_0 & v_h\xi & 0 \\ 0 & v_h\xi^* & iz_hk_0 & \gamma_3\tilde{v}_h\xi \\ v_h\xi & 0 & \gamma_3\tilde{v}_h\xi^* & iz_hk_0 \end{pmatrix} \tag{I.7.25}$$

where

$$\tilde z_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\tilde\zeta_{h'},\quad \tilde m_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\tilde\mu_{h'},\quad \tilde v_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\tilde\nu_{h'},$$
$$z_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\zeta_{h'},\quad v_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\nu_{h'}. \tag{I.7.26}$$

By injecting (I.7.24) into (I.7.26), we find

$$|\tilde m_h-1|\leqslant(\text{const.})\ |U|,\quad |\tilde z_h-1|\leqslant(\text{const.})\ |U|,\quad |z_h-1|\leqslant(\text{const.})\ |U|,$$
$$|\tilde v_h-1|\leqslant(\text{const.})\ |U|,\quad |v_h-1|\leqslant(\text{const.})\ |U|. \tag{I.7.27}$$

2-3 - Dominant part of $\mathcal{L}\hat{\bar A}_{h,\omega}$ Furthermore, we notice that the terms proportional to $\tilde m_h$ or $\tilde v_h$ are sub-dominant:

$$\mathcal{L}\hat{\bar A}_{h,\omega}(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega})=\mathfrak{L}\hat{\bar A}_{h,\omega}(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega})(\mathbb{1}+\sigma_1(\mathbf{k}')) \tag{I.7.28}$$

where

$$\mathfrak{L}\hat{\bar A}_{h,\omega}(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega})=-\begin{pmatrix} i\tilde z_hk_0 & 0 & 0 & v_h\xi^* \\ 0 & i\tilde z_hk_0 & v_h\xi & 0 \\ 0 & v_h\xi^* & iz_hk_0 & 0 \\ v_h\xi & 0 & 0 & iz_hk_0\end{pmatrix} \tag{I.7.29}$$

Before bounding σ_1, we compute the inverse of (I.7.29): using proposition I.A2.1 (see appendix I.A2), we find that if we define

$$\bar k_0:=z_hk_0,\quad \tilde k_0:=\tilde z_hk_0,\quad \bar\xi:=v_h\xi \tag{I.7.30}$$

then

$$\det\mathfrak{L}\hat{\bar A}_{h,\omega}^{-1}(\mathbf{k})(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega})=\left(\tilde k_0\bar k_0+|\bar\xi|^2\right)^2 \tag{I.7.31}$$

and

$$\mathfrak{L}\hat{\bar A}_{h,\omega}^{-1}(\mathbf{k})(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega})=-\frac{(\tilde k_0\bar k_0+|\bar\xi|^2)}{\det\mathfrak{L}\hat{\bar A}_{h,\omega}}\begin{pmatrix} -i\bar k_0 & 0 & 0 & \bar\xi^* \\ 0 & -i\bar k_0 & \bar\xi & 0 \\ 0 & \bar\xi^* & -i\tilde k_0 & 0 \\ \bar\xi & 0 & 0 & -i\tilde k_0\end{pmatrix}. \tag{I.7.32}$$

In particular, this implies that

$$|\mathfrak{L}\hat{\bar A}_{h,\omega}^{-1}(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega})|\leqslant(\text{const.})\ 2^{-h} \tag{I.7.33}$$

which in turn implies

$$|\sigma_1(\mathbf{k}')|\leqslant(\text{const.})\ 2^{h_\epsilon-h}. \tag{I.7.34}$$

3 - Irrelevant part. We now focus on the remainder term $\mathcal{R}\hat{\bar A}_{h,\omega}(\mathbf{k})\,\mathbb{L}\hat{\bar{\mathfrak g}}_{[h],\omega}(\mathbf{k})$ in (I.7.18), which we now show to be small. The estimates are carried out in $\mathbf{x}$ space. We have

$$\int d\mathbf{x}\left|\mathcal{R}W_{2,\omega}^{(h')}*\mathbb{L}\bar{\mathfrak g}_{[h],\omega}(\mathbf{x})\right|$$
$$=\int d\mathbf{x}\left|\int d\mathbf{y}\ W_{2,\omega}^{(h')}(\mathbf{y})\left(\mathbb{L}\bar{\mathfrak g}_{[h],\omega}(\mathbf{x}-\mathbf{y})-\mathbb{L}\bar{\mathfrak g}_{[h],\omega}(\mathbf{x})+\mathbf{y}\partial_{\mathbf{x}}\mathbb{L}\bar{\mathfrak g}_{[h],\omega}(\mathbf{x})\right)\right|$$

which, by Taylor's theorem, yields

$$\int d\mathbf{x}\left|\mathcal{R}W_{2,\omega}^{(h')}*\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right|\leqslant\frac{9}{2}\max_{i,j}\int d\mathbf{y}\ \left|y_iy_jW_{2,\omega}^{(h')}(\mathbf{y})\right|\cdot$$

$$\cdot\max_{i,j}\int d\mathbf{x}\ \left|\partial_{x_i}\partial_{x_j}\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right|$$

in which we inject (I.7.10) and (I.4.49) to find,

$$\int d\mathbf{x}\left|\mathcal{R}W_{2,\omega}^{(h')}*\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right|\leqslant 2^h\ (\text{const.})\ |U|. \tag{I.7.35}$$

Similarly, we find that for all $m\leqslant 3$,

$$\int d\mathbf{x}\left|\mathbf{x}^m\mathcal{R}W_{2,\omega}^{(h')}*\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right|\leqslant 2^h2^{-mh}(\text{const.})\ |U|. \tag{I.7.36}$$

This follows in a straightforward way from

$$\int d\mathbf{y}\ \mathbf{y}\mathcal{R}W_{2,\omega}^{(h')}(\mathbf{y})\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x}-\mathbf{y})=\int d\mathbf{y}\ \mathbf{y}W_{2,\omega}^{(h')}(\mathbf{y})\left(\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x}-\mathbf{y})-\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right)$$

and, for $2\leqslant m\leqslant 3$,

$$\int d\mathbf{y}\ \mathbf{y}^m\mathcal{R}W_{2,\omega}^{(h')}(\mathbf{y})\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x}-\mathbf{y})=\int d\mathbf{y}\ \mathbf{y}^mW_{2,\omega}^{(h')}(\mathbf{y})\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x}-\mathbf{y}).$$

Remark: The estimate (I.7.36), as compared to the dimensional estimate without $\mathcal{R}$, is better by a factor $2^{2(h-h')}$. This is a fairly general argument, and could be repeated with $\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}$ replaced by the inverse Fourier transform of $f_{h,\omega}$:

$$\int d\mathbf{x}\left|\mathbf{x}^m\mathcal{R}W_{2,\omega}^{(h',1)}*\check{f}_{h,\omega}(\mathbf{x})\right|\leqslant 2^{2h-mh}\ (\text{const.})\ |U|. \tag{I.7.37}$$

Finally, using (I.7.36) and the explicit expression of $\hat{g}$, we obtain

$$\int d\mathbf{x}\left|\mathbf{x}^m\mathcal{R}\hat{\bar{A}}^{(h,\omega)}*\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right|\leqslant 2^h2^{-mh}(\text{const.})\ (1+|U||h|). \tag{I.7.38}$$

4 - Conclusion of the proof. The proof of the first of (I.7.1) is then completed by injecting (I.7.29), (I.7.34), (I.7.28), (I.7.27) and (I.7.38) into (I.7.18) and its corresponding **x**-space representation. The second of (I.7.1) follows from the first.

I.7.3. Two-point Schwinger function

We now compute the dominant part of the two-point Schwinger function for $\mathbf{k}$ *well inside* the first regime, i.e.

$$\mathbf{k}\in\mathcal{B}_{\mathrm{I}}^{(\omega)}:=\bigcup_{h=\mathfrak{h}_1+1}^{\bar{\mathfrak{h}}_0-1}\text{supp}f_{h,\omega}.$$

Let

$$h_{\mathbf{k}}:=\max\{h:f_{h,\omega}(\mathbf{k})\neq 0\}$$

so that if $h\notin\{h_{\mathbf{k}},h_{\mathbf{k}}-1\}$, then $f_{h,\omega}(\mathbf{k})=0$.

1 - Schwinger function in terms of dressed propagators. Since $h_{\mathbf{k}}\leqslant\bar{\mathfrak{h}}_0$, the source term $\hat{J}^+_{\mathbf{k},\alpha_1}\hat{\psi}^-_{\mathbf{k},\alpha_1}+\hat{\psi}^+_{\mathbf{k},\alpha_2}\hat{J}^-_{\mathbf{k},\alpha_2}$ is constant with respect to the ultraviolet fields, so that the effective source term $\mathcal{X}^{(h)}$ defined in (I.5.27) is given, for $h=\bar{\mathfrak{h}}_0$, by

$$\mathcal{X}^{(\bar{\mathfrak{h}}_0)}(\psi,\hat{J}_{\mathbf{k},\underline{\alpha}})=\hat{J}^+_{\mathbf{k},\alpha_1}\hat{\psi}^-_{\mathbf{k},\alpha_1}+\hat{\psi}^+_{\mathbf{k},\alpha_2}\hat{J}^-_{\mathbf{k},\alpha_2} \tag{I.7.39}$$

which implies that $\mathcal{X}^{(\bar{\mathfrak{h}}_0)}$ is in the form (I.5.28) with

$$q^{\pm(\bar{\mathfrak{h}}_0)}=\mathbb{1},\quad s^{(\bar{\mathfrak{h}}_0)}(\mathbf{k})=0,\quad \bar{G}^{\pm(\bar{\mathfrak{h}}_0)}=0.$$

Therefore, we can compute $\mathcal{X}^{(h)}$ for $h\in\{\mathfrak{h}_1,\cdots,\bar{\mathfrak{h}}_0-1\}$ inductively using lemma I.5.3. By using the fact that the support of $\hat{\bar{g}}_{h,\omega}$ is compact, we find that $\bar{G}^{(h)}(\mathbf{k})$ no longer depends on h as soon as $h\leqslant h_{\mathbf{k}}-2$, i.e., $\bar{G}^{(h)}(\mathbf{k})=\bar{G}^{(h_{\mathbf{k}}-2)}$, $\forall h\leqslant h_{\mathbf{k}}-2$. Moreover, if $h\leqslant h_{\mathbf{k}}-2$, the iterative equation for $s^{(h)}(\mathbf{k})$ (I.5.31) simplifies into

$$s^{(h)}_{\alpha_1,\alpha_2}(\mathbf{k}):=s^{(h+1)}_{\alpha_1,\alpha_2}(\mathbf{k})-\sum_{\alpha',\alpha''}\bar{G}^{+(h_{\mathbf{k}}-2)}_{\alpha_1,\alpha'}(\mathbf{k})\hat{W}^{(h)}_{2,(\alpha',\alpha'')}(\mathbf{k})\bar{G}^{-(h_{\mathbf{k}}-2)}_{\alpha'',\alpha_2}(\mathbf{k}). \tag{I.7.40}$$

We can therefore write out (I.5.31) quite explicitly: for $\mathfrak{h}_1\leqslant h\leqslant h_{\mathbf{k}}-2$

$$\begin{aligned}s^{(h)}(\mathbf{k})=&\ \hat{\bar{g}}_{h_{\mathbf{k}},\omega}-\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\hat{W}^{(h_{\mathbf{k}}-1)}_2\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\\&+\left(\mathbb{1}-\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\hat{W}^{(h_{\mathbf{k}}-1,\omega)}_2\right)\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}\left(\mathbb{1}-\hat{W}^{(h_{\mathbf{k}}-1)}_2\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\right)\\&-\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}-\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\hat{W}^{(h_{\mathbf{k}}-1)}_2\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}\right)\left(\sum_{h'=h}^{h_{\mathbf{k}}-2}\hat{W}^{(h')}_2\right)\cdot\\&\cdot\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}-\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}\hat{W}^{(h_{\mathbf{k}}-1)}_2\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\right)\end{aligned} \tag{I.7.41}$$

where all the functions in the right side are evaluated at $\mathbf{k}$. Note that in order to get the two-point function defined in section I.1, we must integrate down to $h=\mathfrak{h}_\beta$: $s_2(\mathbf{k})=s^{(\mathfrak{h}_\beta)}(\mathbf{k})$. This requires an analysis of the second and third regimes (see sections I.8.3 and I.9.3 below). We thus find

$$s_2(\mathbf{k})=\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}(\mathbf{k})+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}(\mathbf{k})\right)\left(\mathbb{1}-\sigma(\mathbf{k})-\sigma_{<h_{\mathbf{k}}}(\mathbf{k})\right) \tag{I.7.42}$$

where

$$\sigma(\mathbf{k}):=\hat{W}^{(h_{\mathbf{k}}-1)}_2\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega})^{-1}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\hat{W}^{(h_{\mathbf{k}}-1)}_2\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}(\mathbb{1}-\hat{W}^{(h_{\mathbf{k}}-1)}_2\hat{\bar{g}}_{h_{\mathbf{k}},\omega}) \tag{I.7.43}$$

and

$$\begin{aligned}\sigma_{<h_{\mathbf{k}}}(\mathbf{k}):=&\left(\mathbb{1}-\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}\right)^{-1}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\hat{W}^{(h_{\mathbf{k}}-1)}_2\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}\right)\left(\sum_{h'=\mathfrak{h}_\beta}^{h_{\mathbf{k}}-2}\hat{W}^{(h')}_2\right)\cdot\\&\cdot\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}-\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}\hat{W}^{(h_{\mathbf{k}}-1)}_{2,\omega}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\right)\end{aligned} \tag{I.7.44}$$

in which $\hat{W}^{(h')}_2$ with $h'\in\{\bar{\mathfrak{h}}_2+1,\cdots,\mathfrak{h}_2-1\}\cup\{\bar{\mathfrak{h}}_1+1,\cdots,\mathfrak{h}_1-1\}$ should be interpreted as 0.

2 - Bounding the error terms. We then use (I.7.1), (I.7.10) as well as the bound

$$|(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega})^{-1}|\leqslant(\text{const.})\ 2^{h_{\mathbf{k}}} \tag{I.7.45}$$

which follows from (I.7.29) and (I.7.27), in order to bound $\sigma(\mathbf{k})$:

$$|\sigma(\mathbf{k})|\leqslant(\text{const.})\ 2^{h_{\mathbf{k}}}|U|. \tag{I.7.46}$$

Furthermore, if we assume that

$$\left|\sum_{h'=\mathfrak{h}_\beta}^{\bar{\mathfrak{h}}_1}\hat W_2^{(h')}(\mathbf{k})\right|\leqslant(\mathrm{const.})\ 2^{2h_\epsilon}|U|\tag{I.7.47}$$

which will be proved when studying the second and third regimes (I.8.42) and (I.9.63), then

$$|\sigma_{<h_{\mathbf{k}}}(\mathbf{k})|\leqslant(\mathrm{const.})\ 2^{h_{\mathbf{k}}}|U|.\tag{I.7.48}$$

3 - Dominant part of the dressed propagators. Furthermore, it follows from (I.7.32) that

$$\hat{\bar g}_{h_{\mathbf{k}},\omega}(\mathbf{k})+\hat{\bar g}_{h_{\mathbf{k}}-1,\omega}(\mathbf{k})=-\frac{1}{\tilde k_0\bar k_0+|\bar\xi|^2}\begin{pmatrix}-i\bar k_0&0&0&\bar\xi^*\\0&-i\bar k_0&\bar\xi&0\\0&\bar\xi^*&-i\tilde k_0&0\\\bar\xi&0&0&-i\tilde k_0\end{pmatrix}(\mathbb{1}+\sigma')\tag{I.7.49}$$

where we recall (I.7.30)

$$\bar k_0:=z_{h_{\mathbf{k}}}k_0,\quad \tilde k_0:=\tilde z_{h_{\mathbf{k}}}k_0,\quad \bar\xi:=v_{h_{\mathbf{k}}}\xi\tag{I.7.50}$$

in which $\tilde z_{h_{\mathbf{k}}}$, $z_{h_{\mathbf{k}}}$ and $v_{h_{\mathbf{k}}}$ were defined in (I.7.26) and satisfy (see (I.7.27))

$$|1-\tilde z_{h_{\mathbf{k}}}|\leqslant\tilde C_1^{(z)}|U|,\quad |1-z_{h_{\mathbf{k}}}|\leqslant C_1^{(z)}|U|,\quad |1-v_{h_{\mathbf{k}}}|\leqslant C_1^{(v)}|U|$$

where $\tilde C_1^{(z)}$, $C_1^{(z)}$ and $C_1^{(v)}$ are constants (independent of $h_{\mathbf{k}}$, U and ϵ). Finally the error term σ' is bounded using (I.7.38) and (I.7.34)

$$|\sigma'(\mathbf{k})|\leqslant(\mathrm{const.})\ ((1+|U||h_{\mathbf{k}}|)2^{h_{\mathbf{k}}}+2^{h_\epsilon-h_{\mathbf{k}}}).\tag{I.7.51}$$

4 - Proof of theorem I.1.2 We now conclude the proof of theorem I.1.2, *under the assumption* (I.7.47): we define

$$z_1:=z_{\bar{\mathfrak{h}}_1},\quad \tilde z_1:=\tilde z_{\bar{\mathfrak{h}}_1},\quad v_1:=v_{\bar{\mathfrak{h}}_1}$$

and use (I.7.24) to bound

$$|z_{h_{\mathbf{k}}}-z_1|\leqslant(\mathrm{const.})\ |U|2^{h_{\mathbf{k}}},\quad |\tilde z_{h_{\mathbf{k}}}-\tilde z_1|\leqslant(\mathrm{const.})\ |U|2^{h_{\mathbf{k}}},\quad |v_{h_{\mathbf{k}}}-v_1|\leqslant(\mathrm{const.})\ |U|2^{h_{\mathbf{k}}}$$

which we inject into (I.7.49), which, in turn, combined with (I.7.42), (I.7.46), (I.7.48) and (I.7.51) yields (I.1.14).

I.7.4. Intermediate regime: first to second

1 - Integration over the intermediate regime. The integration over the intermediate regime between scales $\mathfrak{h}_1$ and $\bar{\mathfrak{h}}_1$ can be performed in a way that is entirely analogous to that in the bulk of the first regime, with the difference that it is performed in a single step. The outcome is that, in particular, the effective potential on scale $\bar{\mathfrak{h}}_1$ satisfies an estimate analogous to (I.7.10) (details are left to the reader):

$$\left\{\begin{array}{l}\displaystyle\int d\mathbf{x}\ \left|\mathbf{x}^mW_{2,\omega,\underline{\alpha}}^{(\bar{\mathfrak{h}}_1)}(\mathbf{x})\right|\leqslant\bar C_12^{2\bar{\mathfrak{h}}_1}2^{-m\bar{\mathfrak{h}}_1}|U|\\ \displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_4)^mW_{4,\underline{\omega},\underline{\alpha}}^{(\bar{\mathfrak{h}}_1)}(\underline{\mathbf{x}})\right|\leqslant\bar C_12^{-\bar{\mathfrak{h}}_1m}|U|\\ \displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^mW_{2l,\underline{\omega},\underline{\alpha}}^{(\bar{\mathfrak{h}}_1)}(\underline{\mathbf{x}})\right|\leqslant2^{\bar{\mathfrak{h}}_1(3-2l+2\theta-m)}(\bar C_1|U|)^{l-1}\end{array}\right.\tag{I.7.52}$$

for $l \geqslant 3$ and $m \leqslant 3$.

2 - Improved estimate on inter-layer terms. In order to treat the second regime, we will need an improved estimate on

$$\int d\mathbf{x}\ \mathbf{x}^m W^{(h')}_{2,\omega,(\alpha,\alpha')}(\mathbf{x}) \tag{I.7.53}$$

where (i,j) are in *different layers*, i.e. $(\alpha,\alpha') \in \{a,b\}\times\{\tilde a,\tilde b\}$ or $(\alpha,\alpha')\in\{\tilde a,\tilde b\}\times\{a,b\}$, $h' \geqslant \bar{\mathfrak{h}}_1$. Note that since $W^{(M)}_{4,(\alpha_1,\alpha_1',\alpha_2,\alpha_2')}$ is proportional to $\delta_{\alpha_1,\alpha_1'}\delta_{\alpha_2,\alpha_2'}$, any contribution to $W^{(h',2)}_{2,\omega,(\alpha,\alpha')}$ must contain at least one propagator between different layers, i.e. $\bar g_{(h'',\omega),(\bar\alpha,\bar\alpha')}$ with $h' < h'' \leqslant \bar{\mathfrak{h}}_0$ or $g_{(h''),(\bar\alpha,\bar\alpha')}$ with $h'' \geqslant 0$, and $(\bar\alpha,\bar\alpha')\in\{a,b\}\times\{\tilde a,\tilde b\}\cup\{\tilde a,\tilde b\}\times\{a,b\}$. This can be easily proved using the fact that if the inter-layer hoppings were neglected (i.e. $\gamma_1=\gamma_3=0$), then the system would be symmetric under

$$\psi_{\mathbf{k},a}\mapsto\psi_{\mathbf{k},a},\quad \psi_{\mathbf{k},\tilde b}\mapsto-\psi_{\mathbf{k},\tilde b},\quad \psi_{\mathbf{k},\tilde a}\mapsto-\psi_{\mathbf{k},\tilde a},\quad \psi_{\mathbf{k},b}\mapsto\psi_{\mathbf{k},b}$$

which would imply that $W^{(h')}_{2,\omega,(\alpha,\alpha')}=0$. The presence of at least one propagator between different layers allows us to obtain a dimensional gain, induced by an improved estimate on each such propagator. To prove an improved estimate on the inter-layer propagator, let us start by considering the bare one, $g_{(h'',\omega),(\bar\alpha,\bar\alpha')}$ with $(\bar\alpha,\bar\alpha')\in\{a,b\}\times\{\tilde a,\tilde b\}\cup\{\tilde a,\tilde b\}\times\{a,b\}$ and $h'<h''\leqslant\bar{\mathfrak{h}}_0$ (similar considerations are valid for the ultraviolet counterpart): using the explicit expression (I.2.17) it is straightforward to check that it is bounded as in (I.4.50), (I.4.49), times an extra factor $\epsilon 2^{-h''}$. We now proceed as in section I.7.1 and prove by induction that the same dimensional gain is associated with the *dressed* propagator $\bar g_{(h'',\omega),(\bar\alpha,\bar\alpha')}$, with $(\bar\alpha,\bar\alpha')\in\{a,b\}\times\{\tilde a,\tilde b\}\cup\{\tilde a,\tilde b\}\times\{a,b\}$, and, therefore, with (I.7.53) itself.

2-1 - Trees with a single endpoint. We first consider the contributions $\mathfrak{A}^{(h')}_{2,\omega,(\alpha,\alpha')}$ to $W^{(h')}_{2,\omega,(\alpha,\alpha')}$ from trees $\tau\in\mathcal{T}^{(h)}_1$ with a single endpoint. The $\mathfrak{F}_h(\underline{m})$ factor in the estimate (I.5.23) can be removed for these contributions using the fact that they have an empty spanning tree (i.e. $\mathbf{T}(\tau)=\emptyset$), which implies that the $\mathbf{z}^m$'s in the right side of (I.5.22) are all $\underline{\mathbf{z}}^{(v)}$'s and not $\mathbf{z}_\ell$'s, and can be estimated dimensionally by a constant instead of $\mathfrak{F}_h(\underline{m})$. Therefore, combining this fact with the gain associated to the propagator, we find that for all $m\leqslant 3$,

$$\int d\mathbf{x}\ \left|\mathbf{x}^m\mathfrak{A}^{(h')}_{2,\omega,(\alpha,\alpha')}(\mathbf{x})\right| \leqslant (\text{const.})\ \epsilon 2^{h'}|U|. \tag{I.7.54}$$

2-2 - Trees with at least two endpoints. We now consider the contributions $\mathfrak{B}^{(h')}_{2,\omega,(\alpha,\alpha')}$ to $W^{(h')}_{2,\omega,(\alpha,\alpha')}$ from trees with $\geqslant 2$ endpoints. Let v^*_τ be the node that has at least two children that is closest to the root and let h^*_τ be its scale. Repeating the reasoning leading to (I.7.9), and using the fact that the $\mathbf{x}^m$ falls on a node on scale $\geqslant h^*_\tau$, we find

$$\int d\mathbf{x}\ \left|\mathbf{x}^m\mathfrak{B}^{(h')}_{2,\omega,(\alpha,\alpha')}(\mathbf{x})\right| \leqslant (\text{const.})\ \epsilon\sum_{h^*_\tau=h'+1}^{0} 2^{-mh^*_\tau}2^{(h'-h^*_\tau)}2^{2\theta h^*_\tau}|U|^2$$

for any $\theta\in(0,1)$, so that

$$\int d\mathbf{x}\ \left|\mathbf{x}^m\mathfrak{B}^{(h')}_{2,\omega,(\alpha,\alpha')}(\mathbf{x})\right| \leqslant (\text{const.})\ \epsilon 2^{\theta' h'+\min(0,1-m)h'}|U|^2 \tag{I.7.55}$$

where $\theta' := 2\theta-1>1$.

Combining (I.7.54) and (I.7.55), and repeating the argument in section I.7.2, we conclude the proof of the desired improvement on the estimate of $\bar g$, and that

$$\int d\mathbf{x}\ \left|\mathbf{x}^m W^{(h')}_{2,\omega,(\alpha,\alpha')}(\mathbf{x})\right| \leqslant (\text{const.})\ \epsilon 2^{\theta' h}|U|(1+2^{\min(0,1-m)h}|U|) \tag{I.7.56}$$

for $m\leqslant 3$.

I.8. Second regime

We now perform the multiscale integration in the second regime. As in the first regime, we shall inductively prove that $\bar g_{h,\omega}$ satisfies the same estimate as $g_{h,\omega}$ (see (I.4.53) and (I.4.51)): for all $m\leqslant 3$,

$$\left\{\begin{array}{l}\displaystyle\int d\mathbf{x}\ |x_0^{m_0}x^{m_k}\bar g_{h,\omega}(\mathbf{x})| \leqslant (\text{const.})\ 2^{-h-m_0h-m_k\frac{h+h_\epsilon}{2}}\\ \displaystyle\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}^{(h,\omega)}_{\beta,L}}|\hat{\bar g}_{h,\omega}(\mathbf{k})| \leqslant (\text{const.})\ 2^{h+h_\epsilon}\end{array}\right. \tag{I.8.1}$$

which in terms of the hypotheses of lemma I.5.2 means

$$c_k=2,\ c_g=1,\quad \mathfrak{F}_h(m_0,m_1,m_2)=2^{-m_0h-(m_1+m_2)\frac{h+h_\epsilon}{2}},$$

$C_g=(\text{const.})$ and $C_G=(\text{const.})\ 2^{h_\epsilon}$.

Remark: As can be seen from (I.3.19), different components of $g_{h,\omega}$ scale in different ways. In order to highlight this fact, we call the $\{a,\tilde b\}$ components *massive* and the $\{\tilde a,b\}$ components *massless*. It follows from (I.3.19) that the L_1 norm of the massive-massive sub-block of $g_{h,\omega}(\mathbf{x})$ is bounded by (const.) 2^{-h_ϵ} (instead of 2^{-h}, compare with (I.8.1)) and that the massive-massless sub-blocks are bounded by (const.) $2^{-(h+h_\epsilon)/2}$. In the following, in order to simplify the discussion, we will ignore these improvements, even though the bounds we will thus derive for the non-local corrections may not be optimal.

In addition, in order to apply lemma I.5.2, we have to ensure that hypothesis (I.5.12) is satisfied, so we will also prove a bound on the 4-field kernels by induction ($\ell_0=3$ in this regime, so (I.5.12) must be satisfied by the 4-field kernels): for all $m\leqslant 3$,

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ |(\underline{\mathbf{x}}-\mathbf{x}_4)^m W^{(h)}_{4,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})| \leqslant C'_\mu|U|\mathfrak{F}_h(\underline{m}) \tag{I.8.2}$$

where C'_μ is a constant that will be defined below. Note that in this regime,

$$\ell_0=3>\frac{c_k}{c_k-c_g}=2$$

as desired.

I.8.1. Power counting in the second regime

1 - Power counting estimate. It follows from lemma I.5.2 and (I.7.52) that for all $m\leqslant 3$ and some $c_1,c_2>0$,

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right|\leqslant 2^{h(2-l)}\mathfrak{F}_h(\underline{m})2^{-lh_\epsilon}\sum_{N=1}^\infty\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,3}\\|P_{v_0}|=2l}}$$
$$(c_12^{-h_\epsilon})^{N-1}\prod_{v\in\mathfrak{V}(\tau)}2^{(2-\frac{|P_v|}{2})}\prod_{v\in\mathfrak{E}(\tau)}(c_22^{h_\epsilon})^{l_v}|U|^{l_v-1}2^{\mathbf{1}_{l_v>2}(2-l_v+\theta')h_\epsilon}$$

where $\mathbf{1}_{l_v>2}$ is equal to 1 if $l_v>2$ and 0 otherwise, and $\theta':=2\theta-1>0$, so that

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right|\leqslant 2^{h(2-l)}\mathfrak{F}_h(\underline{m})2^{-(l-1)h_\epsilon}\cdot$$
$$\cdot\sum_{N=1}^\infty\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,3}\\|P_{v_0}|=2l}}c_1^{N-1}2^{Nh_\epsilon}\prod_{v\in\mathfrak{V}(\tau)}2^{(2-\frac{|P_v|}{2})}\prod_{v\in\mathfrak{E}(\tau)}c_2^{l_v}|U|^{l_v-1}2^{\mathbf{1}_{l_v>2}\theta'h_\epsilon}.\tag{I.8.3}$$

2 - Bounding the sum on trees. By repeating the computation that leads to (I.6.11), noticing that if $\ell_0=3$, then for $v\in\mathfrak{V}(\tau)$ we have $2-|P_v|/2\leqslant-|P_v|/6$, we bound

$$\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\substack{\mathbf{P}\in\mathcal{P}^{(h)}_{\tau,\underline{l}_\tau,3}\\|P_{v_0}|=2l}}\prod_{v\in\mathfrak{V}(\tau)}2^{2-\frac{|P_v|}{2}}\leqslant c_3^N\tag{I.8.4}$$

for some constant $c_3>0$. Thus (I.8.3) becomes

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right|\leqslant 2^{h(2-l)}\mathfrak{F}_h(\underline{m})2^{-(l-1)h_\epsilon}\cdot$$
$$\cdot\sum_{N\geqslant1}\sum_{\substack{l_1,\dots,l_N\geqslant2\\\sum_{i=1}^N(l_i-1)\geqslant l-1+\delta_{N,1}}}2^{Nh_\epsilon}(c_4|U|)^{\sum_{i=1}^N(l_i-1)}\tag{I.8.5}$$

for some $c_4>0$. Note that, if $l=2$, the contribution with $N=1$ to the left side admits an improved bound of the form $c_4\mathfrak{F}_h(\underline{m})2^{\theta'h}|U|^2$, which is better than the corresponding term in the right side of (I.8.5). This implies

$$\int d\mathbf{x}\left|\mathbf{x}^mW^{(h)}_{2,\underline{\omega},\underline{\alpha}}(\mathbf{x})\right|\leqslant c_52^{h+h_\epsilon}\mathfrak{F}_h(\underline{m})|U|\tag{I.8.6}$$

and

$$\left\{\begin{array}{l}\displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_4)^m B^{(h)}_{4,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right|\leqslant c_5\mathfrak{F}_h(\underline{m})(2^{h_\epsilon}+2^{\theta'h})|U|^2\\[2ex]\displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right|\leqslant 2^{(h+h_\epsilon)(2-l)}\mathfrak{F}_h(\underline{m})(c_5|U|)^{l-1}\end{array}\right.\tag{I.8.7}$$

for some $c_5>0$, with $l\geqslant3$. By summing the previous two inequalities, we find

$$\left\{\begin{array}{l}\displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_4)^m W^{(h)}_{4,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right|\leqslant C_\mu\mathfrak{F}_h(\underline{m})|U|(1+c_6|U|(\epsilon(h_\epsilon-h)+\epsilon^{\theta'}))\\[2ex]\displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m W^{(h)}_{2l,\underline{\omega},\underline{\alpha}}(\underline{\mathbf{x}})\right|\leqslant 2^{(h+h_\epsilon)(2-l)}\mathfrak{F}_h(\underline{m})(c_6|U|)^{l-1}\end{array}\right.\tag{I.8.8}$$

for some $c_6 > 0$, which, in particular, recalling that in the second regime $h_\epsilon - h \leqslant -2h_\epsilon + C$, for some constant C independent of ϵ, implies (I.8.2) with

$$C'_\mu := C_\mu(1 + c_7 \sup_{|U|<U_0,\epsilon<\epsilon_0} |U|(\epsilon|\log\epsilon| + \epsilon^{\theta'}))$$

for some $c_7 > 0$.

Remark: The estimates (I.8.3) and (I.8.4) imply the convergence of the tree expansion (I.5.8), thus providing a convergent expansion of $W^{(h)}_{2l,\underline{\omega},\underline{\alpha}}$ in U.

Remark: The first of (I.8.8) exhibits a tendency to grow *logarithmically* in 2^{-h}. This is not an artifact of the bounding procedure: indeed the second-order flow, computed in [Va10], exhibits the same logarithmic growth. However, the presence of the ϵ factor in front of $(h_\epsilon - h) \leqslant 2|\log\epsilon|$ ensures this growth is benign: it is cut off before it has a chance to be realized.

I.8.2. The dressed propagator

We now turn to the inductive proof of (I.8.1). We recall that (see (I.4.18))

$$\hat{\bar{g}}_{h,\omega}(\mathbf{k}) = f_{h,\omega}(\mathbf{k})\hat{\bar{A}}^{-1}_{h,\omega}(\mathbf{k}) \tag{I.8.9}$$

where

$$\hat{\bar{A}}_{h,\omega}(\mathbf{k}) := \hat{A}(\mathbf{k}) + f_{\leqslant h,\omega}(\mathbf{k})\hat{W}_2^{(h)}(\mathbf{k}) + \sum_{h'=h+1}^{\bar{\mathfrak{h}}_1} \hat{W}_2^{(h')}(\mathbf{k}) + \sum_{h'=\mathfrak{h}_1}^{\bar{\mathfrak{h}}_0} \hat{W}_2^{(h')}(\mathbf{k}).$$

We will separate the *local* part of $\bar{A}$ from the remainder by using the localization operator defined in (I.7.15) (see the remark at the end of this section for an explanation of why we can choose the same localization operator as in the first regime even though the scaling dimension is different) and rewrite (I.8.9) as

$$\hat{\bar{g}}_{h,\omega}(\mathbf{k}) = f_{h,\omega}(\mathbf{k})\left(\mathcal{L}\hat{\bar{A}}_{h,\omega}(\mathbf{k})\right)^{-1}\left(\mathbb{1} + \mathcal{R}\hat{\bar{A}}_{h,\omega}(\mathbf{k})\left(\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega}(\mathbf{k})\right)\right)^{-1} \tag{I.8.10}$$

where $\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega}$ is a shorthand for

$$\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega}(\mathbf{k}) := (f_{\leqslant h+1,\omega}(\mathbf{k}) - f_{\leqslant h-2,\omega}(\mathbf{k}))\left(\mathcal{L}\hat{\bar{A}}_{h,\omega}(\mathbf{k})\right)^{-1}.$$

Similarly to the first regime, we now compute $\mathcal{L}\hat{\bar{A}}_{h,\omega}(\mathbf{k})$ and bound $\mathcal{R}\hat{\bar{A}}_{h,\omega}(\mathbf{k})\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega}(\mathbf{k})$. We first write the local part of the non-interacting contribution:

$$\mathcal{L}\hat{A}(\mathbf{k}) = -\begin{pmatrix} ik_0 & \gamma_1 & 0 & \xi^* \\ \gamma_1 & ik_0 & \xi & 0 \\ 0 & \xi^* & ik_0 & \gamma_3\xi \\ \xi & 0 & \gamma_3\xi^* & ik_0 \end{pmatrix} \tag{I.8.11}$$

where

$$\xi := \frac{3}{2}(ik'_x + \omega k'_y). \tag{I.8.12}$$

1 - Local part. The symmetries discussed in the first regime (see (I.7.21) and (I.7.22)) still hold in this regime, so that (I.7.23) still holds:

$$\mathcal{L}\hat{W}_2^{(h')}(\mathbf{k}' + \mathbf{p}^\omega_{F,0}) = -\begin{pmatrix} i\tilde{\zeta}_{h'}k_0 & \gamma_1\tilde{\mu}_{h'} & 0 & \nu_{h'}\xi^* \\ \gamma_1\tilde{\mu}_{h'} & i\tilde{\zeta}_{h'}k_0 & \nu_{h'}\xi & 0 \\ 0 & \nu_{h'}\xi^* & i\zeta_{h'}k_0 & \gamma_3\tilde{\nu}_{h'}\xi \\ \nu_{h'}\xi & 0 & \gamma_3\tilde{\nu}_{h'}\xi^* & i\zeta_{h'}k_0 \end{pmatrix}, \tag{I.8.13}$$

with $(\tilde\zeta_{h'},\tilde\mu_{h'},\tilde\nu_{h'},\zeta_{h'},\nu_{h'})\in\mathbb R^5$. Furthermore, it follows from (I.8.6) that if $h'\leqslant\bar{\mathfrak h}_1$, then

$$\begin{array}{c}|\tilde\zeta_{h'}|\leqslant(\mathrm{const.})\ |U|2^{h_\epsilon},\quad |\zeta_{h'}|\leqslant(\mathrm{const.})\ |U|2^{h_\epsilon},\quad |\tilde\mu_{h'}|\leqslant(\mathrm{const.})\ |U|2^{h'},\\ |\nu_{h'}|\leqslant(\mathrm{const.})\ |U|2^{\frac{h'}{2}+\frac{h_\epsilon}{2}},\quad |\tilde\nu_{h'}|\leqslant(\mathrm{const.})\ |U|2^{\frac{h'}{2}-\frac{h_\epsilon}{2}}.\end{array}\tag{I.8.14}$$

If $\mathfrak h_1\leqslant h'\leqslant\bar{\mathfrak h}_0$, then it follows from (I.7.10) that

$$|\tilde\zeta_{h'}|\leqslant(\mathrm{const.})\ |U|2^{h'},\quad |\zeta_{h'}|\leqslant(\mathrm{const.})\ |U|2^{h'},\quad |\nu_{h'}|\leqslant(\mathrm{const.})\ |U|2^{h'},\tag{I.8.15}$$

and from (I.7.56) that

$$|\tilde\mu_{h'}|\leqslant(\mathrm{const.})\ 2^{\theta h'}|U|,\quad |\tilde\nu_{h'}|\leqslant(\mathrm{const.})\ 2^{\theta' h'}|U|\tag{I.8.16}$$

for some $\theta'\in(0,1)$. Injecting (I.8.11) and (I.8.13) into (I.4.18), we find that

$$\mathcal L\hat{\bar A}_{h,\omega}(\mathbf k'+\mathbf p^\omega_{F,0})=-\left(\begin{array}{cccc}i\tilde z_hk_0&\gamma_1\tilde m_h&0&v_h\xi^*\\ \gamma_1\tilde m_h&i\tilde z_hk_0&v_h\xi&0\\ 0&v_h\xi^*&iz_hk_0&\gamma_3\tilde v_h\xi\\ v_h\xi&0&\gamma_3\tilde v_h\xi^*&iz_hk_0\end{array}\right)\tag{I.8.17}$$

where

$$\begin{array}{c}\displaystyle\tilde z_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\tilde\zeta_{h'},\quad \tilde m_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\tilde\mu_{h'},\quad \tilde v_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\tilde\nu_{h'},\\ \displaystyle z_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\zeta_{h'},\quad v_h:=1+\sum_{h'=h}^{\bar{\mathfrak h}_0}\nu_{h'}\end{array}\tag{I.8.18}$$

in which $\tilde\zeta_{h'}$, $\tilde\mu_{h'}$, $\tilde\nu_{h'}$, $\zeta_{h'}$ and $\nu_{h'}$ with $h'\in\{\bar{\mathfrak h}_1+1,\cdots,\mathfrak h_1-1\}$ are to be interpreted as 0. By injecting (I.8.14) through (I.8.16) into (I.8.18), we find

$$\begin{array}{c}|\tilde m_h-1|\leqslant(\mathrm{const.})\ |U|,\quad |\tilde z_h-1|\leqslant(\mathrm{const.})\ |U|,\quad |z_h-1|\leqslant(\mathrm{const.})\ |U|,\\ |\tilde v_h-1|\leqslant(\mathrm{const.})\ |U|,\quad |v_h-1|\leqslant(\mathrm{const.})\ |U|.\end{array}\tag{I.8.19}$$

2 - Dominant part of $\mathcal L\hat{\bar A}_{h,\omega}$ Furthermore, we notice that the terms proportional to $\tilde z_h$ or $\tilde v_h$ are sub-dominant:

$$\mathcal L\hat{\bar A}_{h,\omega}(\mathbf k'+\mathbf p^\omega_{F,0})=\mathfrak L\hat{\bar A}_{h,\omega}(\mathbf k'+\mathbf p^\omega_{F,0})(\mathbb 1+\sigma_3(\mathbf k'))\tag{I.8.20}$$

where

$$\mathfrak L\hat{\bar A}_{h,\omega}(\mathbf k'+\mathbf p^\omega_{F,0})=-\left(\begin{array}{cccc}0&\gamma_1\tilde m_h&0&v_h\xi^*\\ \gamma_1\tilde m_h&0&v_h\xi&0\\ 0&v_h\xi^*&iz_hk_0&0\\ v_h\xi&0&0&iz_hk_0\end{array}\right)\tag{I.8.21}$$

Before bounding σ_3, we compute the inverse of (I.8.21), which is elementary once it is put in block-diagonal form: using proposition I.A3.1 (see appendix I.A3), we find that if we define

$$\bar\gamma_1:=\tilde m_h\gamma_1,\quad \bar k_0:=z_hk_0,\quad \bar\xi:=v_h\xi\tag{I.8.22}$$

then

$$\left(\mathfrak L\hat{\bar A}_{h,\omega}(\mathbf k)\right)^{-1}=\left(\begin{array}{cc}\mathbb 1&\bar M_h(\mathbf k)^\dagger\\ 0&\mathbb 1\end{array}\right)\left(\begin{array}{cc}\bar a_h^{(M)}&0\\ 0&\bar a_h^{(m)}(\mathbf k)\end{array}\right)\left(\begin{array}{cc}\mathbb 1&0\\ \bar M_h(\mathbf k)&\mathbb 1\end{array}\right)\tag{I.8.23}$$

where

$$\bar a_h^{(M)}:=-\left(\begin{array}{cc}0&\bar\gamma_1^{-1}\\ \bar\gamma_1^{-1}&0\end{array}\right),\quad \bar a_h^{(m)}(\mathbf p^\omega_{F,0}+\mathbf k'):=\frac{\bar\gamma_1}{\bar\gamma_1^2\bar k_0^2+|\bar\xi|^4}\left(\begin{array}{cc}i\bar\gamma_1\bar k_0&(\bar\xi^*)^2\\ \bar\xi^2&i\bar\gamma_1\bar k_0\end{array}\right)\tag{I.8.24}$$

and

$$\bar{M}_h(\mathbf{p}^\omega_{F,0}+\mathbf{k}') := -\frac{1}{\bar{\gamma}_1}\left(\begin{array}{cc} \bar{\xi}^* & 0 \\ 0 & \bar{\xi} \end{array}\right). \tag{I.8.25}$$

In particular, this implies that

$$|\mathfrak{L}\hat{\bar{A}}^{-1}_{h,\omega}(\mathbf{k}'+\mathbf{p}^\omega_{F,0})| \leqslant (\text{const.})\left(\begin{array}{cc} 2^{-h_\epsilon} & 2^{-\frac{h+h_\epsilon}{2}} \\ 2^{-\frac{h+h_\epsilon}{2}} & 2^{-h} \end{array}\right) \tag{I.8.26}$$

in which the bound should be understood as follows: the upper-left element in (I.8.26) is the bound on the upper-left 2×2 block of $\mathfrak{L}\hat{\bar{A}}^{-1}_{h,\omega,0}$, and similarly for the upper-right, lower-left and lower-right. In turn, (I.8.26) implies

$$|\sigma_3(\mathbf{k}')| \leqslant (\text{const.})\ \left(2^{\frac{h-h_\epsilon}{2}}+2^{\frac{3h_\epsilon-h}{2}}\right). \tag{I.8.27}$$

3 - Irrelevant part. The irrelevant part is bounded in the same way as in the first regime (see (I.7.36)): using (I.8.17) and the bounds (I.8.14) through (I.8.16), we find that for $m\leqslant 3$ and $\mathfrak{h}_2\leqslant h\leqslant h'\leqslant \bar{\mathfrak{h}}_1$,

$$\int d\mathbf{x}\left|\mathbf{x}^m\mathcal{R}W^{(h')}_{2,\omega}*\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right| \leqslant 2^{h_\epsilon}\mathfrak{F}_h(\underline{m})(\text{const.})\ |U| \tag{I.8.28}$$

and for $\mathfrak{h}_2\leqslant h\leqslant \mathfrak{h}_1\leqslant h'\leqslant \bar{\mathfrak{h}}_0$,

$$\int d\mathbf{x}\left|\mathbf{x}^m\mathcal{R}W^{(h')}_{2,\omega}*\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right| \leqslant 2^{h_\epsilon}\mathfrak{F}_h(\underline{m})(\text{const.})\ |U|. \tag{I.8.29}$$

Therefore, using the fact that

$$\int d\mathbf{x}\ \left|\mathbf{x}^m\mathcal{R}(g^{-1})*\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right| \leqslant 2^{h_\epsilon}\mathfrak{F}_h(\underline{m})(\text{const.})$$

we find

$$\int d\mathbf{x}\left|\mathbf{x}^m\mathcal{R}\bar{A}_{h,\omega}*\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega}(\mathbf{x})\right| \leqslant 2^{h_\epsilon}\mathfrak{F}_h(\underline{m})(\text{const.})\ (1+|h||U|). \tag{I.8.30}$$

4 - Conclusion of the proof. The proof of (I.8.1) is then concluded by injecting (I.8.21), (I.8.27), (I.8.20) and (I.8.30) into (I.8.10).

Remark: By following the rationale explained in the remark following (I.7.17), one may notice that the "correct" localization operator in the second regime is different from that in the first. Indeed, in the second regime, $(k-p^\omega_{F,0})\partial_k$ scales like $2^{\frac{1}{2}(h-h')}$ instead of $2^{h-h'}$ in the first. This implies that the remainder of the first order Taylor expansion of $\hat{W}^{(h')}_2$ is bounded by 2^h instead of $2^{2h-h'}$ in the first regime, and is therefore *marginal*. However, this is not a problem in this case since the effect of the "marginality" of the remainder is to produce the $|h|$ factor in (I.8.30), which, since the second regime is cut off at scale $3h_\epsilon$ and the integration over the super-renormalizable first regime produced an extra 2^{h_ϵ} (see (I.8.30)), is of little consequence. If one were to do things "right", one would define the localization operator for the *massless* fields as the Taylor expansion to *second* order in k and first order in k_0, and find that the $|h|$ factor in (I.8.30) can be dropped. We have not taken this approach here, since it complicates the definition of $\mathcal{L}$ (which would differ between massive and massless blocks) as well as the symmetry discussion that we used in (I.8.17).

I.8.3. Two-point Schwinger function

We now compute the dominant part of the two-point Schwinger function for $\mathbf{k}$ *well inside* the second regime, i.e.

$$\mathbf{k}\in\mathcal{B}_{\mathrm{II}}^{(\omega)}:=\bigcup_{h=\mathfrak{h}_2+1}^{\bar{\mathfrak{h}}_1-1}\mathrm{supp}f_{h,\omega}.$$

Let

$$h_{\mathbf{k}}:=\max\{h:f_{h,\omega}(\mathbf{k})\neq 0\}$$

so that if $h\notin\{h_{\mathbf{k}},h_{\mathbf{k}}-1\}$, then $f_{h,\omega}(\mathbf{k})=0$.

1 - Schwinger function in terms of dressed propagators. Recall that the two-point Schwinger function can be computed in terms of the effective source term $\mathcal{X}^{(h)}$ defined in (I.5.27), see the comment after lemma I.5.3. Since $h_{\mathbf{k}}\leqslant\bar{\mathfrak{h}}_1$, $\mathcal{X}^{(h)}$ is left invariant by the integration over the ultraviolet and the first regime, in the sense that $\mathcal{X}^{(\bar{\mathfrak{h}}_1)}=\mathcal{X}^{(\bar{\mathfrak{h}}_0)}$, with $\mathcal{X}^{(\bar{\mathfrak{h}}_0)}$ given by (I.7.39). We can therefore compute $\mathcal{X}^{(h)}$ for $h\in\{\mathfrak{h}_2,\cdots,\bar{\mathfrak{h}}_1-1\}$ inductively using lemma I.5.3, and find, similarly to (I.7.42), that

$$s_2(\mathbf{k})=\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}(\mathbf{k})+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}(\mathbf{k})\right)\left(\mathbb{1}-\sigma(\mathbf{k})-\sigma_{<h_{\mathbf{k}}}(\mathbf{k})\right) \tag{I.8.31}$$

where

$$\sigma(\mathbf{k}):=\hat{W}_2^{(h_{\mathbf{k}}-1)}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega})^{-1}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\hat{W}_2^{(h_{\mathbf{k}}-1)}\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}(\mathbb{1}-\hat{W}_2^{(h_{\mathbf{k}}-1)}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}) \tag{I.8.32}$$

and

$$\begin{aligned}\sigma_{<h_{\mathbf{k}}}(\mathbf{k}):=\left(\mathbb{1}-\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}\right)^{-1}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\hat{W}_2^{(h_{\mathbf{k}}-1)}\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}\right)\left(\sum_{h'=\mathfrak{h}_\beta}^{h_{\mathbf{k}}-2}\hat{W}_2^{(h')}\right)\cdot\\ \cdot\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}-\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}\hat{W}_{2,\omega}^{(h_{\mathbf{k}}-1)}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\right).\end{aligned} \tag{I.8.33}$$

2 - Bounding the error terms. We now bound $\sigma(\mathbf{k})$ and $\sigma_{<h_{\mathbf{k}}}(\mathbf{k})$. We first note that

$$\left|(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega})^{-1}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}\right|\leqslant(\text{const.}) \tag{I.8.34}$$

which can be proved as follows: using (I.8.9), we write $\hat{\bar{g}}_{h_{\mathbf{k}},\omega}=f_{h_{\mathbf{k}}}\hat{\bar{A}}_{h_{\mathbf{k}},\omega}^{-1}$ and

$$\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}=f_{h_{\mathbf{k}}-1}\hat{\bar{A}}_{h_{\mathbf{k}},\omega}^{-1}(\mathbb{1}+f_{\leqslant h_{\mathbf{k}}-1}\hat{W}_2^{(h_{\mathbf{k}}-1)}\hat{\bar{A}}_{h_{\mathbf{k}},\omega}^{-1})^{-1}$$

Therefore, noting that $f_{h_{\mathbf{k}}}(\mathbf{k})+f_{h_{\mathbf{k}-1}}(\mathbf{k})=1$, we obtain

$$(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega})^{-1}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}=f_{h_{\mathbf{k}}}\left[\mathbb{1}+f_{h_{\mathbf{k}}-1}\left((\mathbb{1}+f_{\leqslant h_{\mathbf{k}}-1}\hat{W}_2^{(h_{\mathbf{k}}-1)}\hat{\bar{A}}_{h_{\mathbf{k}},\omega}^{-1})^{-1}-\mathbb{1}\right)\right]^{-1}. \tag{I.8.35}$$

Now, by (I.8.6), we see that $|\hat{W}_2^{(h_{\mathbf{k}}-1)}(\mathbf{k})\hat{\bar{A}}_{h_{\mathbf{k}},\omega}^{-1}(\mathbf{k})|\leqslant(\text{const.})\ 2^{h_\epsilon}$, which implies (I.8.34). By inserting (I.8.34), (I.8.6) and (I.8.1) into (I.8.32), we obtain

$$|\sigma(\mathbf{k})|\leqslant(\text{const.})\ 2^{h_\epsilon}|U|. \tag{I.8.36}$$

Moreover, if we assume that

$$\left|\sum_{h'=\mathfrak{h}_\beta}^{\mathfrak{h}_2}\hat{W}_2^{(h')}(\mathbf{k})\right|\leqslant(\text{const.})\ 2^{4h_\epsilon}|U| \tag{I.8.37}$$

which will be proved after studying the third regime (I.9.63), then, since $3h_\epsilon\leqslant\mathfrak{h}_2\leqslant h_{\mathbf{k}}$,

$$|\sigma_{<h_{\mathbf{k}}}(\mathbf{k})|\leqslant(\text{const.})\ 2^{h_\epsilon}|U|. \tag{I.8.38}$$

3 - Dominant part of the dressed propagators. We now compute $\hat{\bar{g}}_{h_{\mathbf{k}},\omega}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}$: it follows from (I.8.10), (I.8.20) and (I.8.23) that

$$\begin{array}{l}\hat{\bar{g}}_{h_{\mathbf{k}},\omega}(\mathbf{k})+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}(\mathbf{k})\\ \quad=\left(\begin{array}{cc}\mathbb{1}&\bar{M}^\dagger_{h_{\mathbf{k}}}(\mathbf{k})\\0&\mathbb{1}\end{array}\right)\left(\begin{array}{cc}\bar{a}^{(M)}_{h_{\mathbf{k}}}&0\\0&\bar{a}^{(m)}_{h_{\mathbf{k}}}(\mathbf{k})\end{array}\right)\left(\begin{array}{cc}\mathbb{1}&0\\\bar{M}_{h_{\mathbf{k}}}(\mathbf{k})&\mathbb{1}\end{array}\right)(\mathbb{1}+\sigma'(\mathbf{k}))\end{array} \tag{I.8.39}$$

where $\bar{M}_{h_{\mathbf{k}}}$, $\bar{a}^{(M)}_{h_{\mathbf{k}}}$ and $\bar{a}^{(m)}_{h_{\mathbf{k}}}$ were defined in (I.8.25) and (I.8.24), and the error term σ' can be bounded using (I.8.30) and (I.8.27):

$$|\sigma'(\mathbf{k})|\leqslant(\text{const.})\ \left(2^{\frac{h_{\mathbf{k}}-h_\epsilon}{2}}+2^{\frac{3h_\epsilon-h_{\mathbf{k}}}{2}}+|U||h_\epsilon|2^{h_\epsilon}\right). \tag{I.8.40}$$

4 - Proof of theorem I.1.3 We now conclude the proof of theorem I.1.3, *under the assumption* (I.8.37). We define

$$B_{h_{\mathbf{k}}}(\mathbf{k}):=(\mathbb{1}+\sigma'(\mathbf{k}))\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega}(\mathbf{k})+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega}(\mathbf{k})\right)^{-1}$$

(i.e. the inverse of the matrix on the right side of (I.8.39), whose explicit expression is similar to the right side of (I.8.21)), and

$$\tilde{m}_2:=\tilde{m}_{\mathfrak{h}_2},\quad z_2:=z_{\mathfrak{h}_2},\quad v_2:=v_{\mathfrak{h}_2}$$

and use (I.8.14) to bound

$$|\tilde{m}_{h_{\mathbf{k}}}-\tilde{m}_2|\leqslant(\text{const.})\ |U|2^{h_{\mathbf{k}}},\quad |z_{h_{\mathbf{k}}}-z_2|\leqslant(\text{const.})\ |U||h_\epsilon|2^{h_\epsilon},$$
$$|v_{h_{\mathbf{k}}}-v_2|\leqslant(\text{const.})\ |U|2^{\frac12(h_{\mathbf{k}}+h_\epsilon)}$$

so that

$$\left|(B_{\mathfrak{h}_2}(\mathbf{k})-B_{h_{\mathbf{k}}}(\mathbf{k}))B^{-1}_{\mathfrak{h}_2}(\mathbf{k})\right|\leqslant(\text{const.})\ |U||h_\epsilon|2^{h_\epsilon}$$

which implies

$$B^{-1}_{h_{\mathbf{k}}}(\mathbf{k})=B^{-1}_{\mathfrak{h}_2}(\mathbf{k})(\mathbb{1}+O(|U||h_\epsilon|2^{h_\epsilon})). \tag{I.8.41}$$

We inject (I.8.41) into (I.8.39), which we then combine with (I.8.31), (I.8.36), (I.8.38) and (I.8.40), and find an expression for s_2 which is similar to the right side of (I.8.39) but with $h_{\mathbf{k}}$ replaced by $\mathfrak{h}_2$. This concludes the proof of (I.1.18). Furthermore, the estimate (I.1.23) follows from (I.8.19), which concludes the proof of theorem I.1.3.

5 - Partial proof of (I.7.47) Before moving on to the third regime, we bound part of the sum on the left side of (I.7.47), which we recall was assumed to be true to prove (I.1.14) (see section I.7.3). It follows from (I.8.6) that

$$\left|\sum_{h'=\mathfrak{h}_2}^{\bar{\mathfrak{h}}_1}\hat{W}_2^{(h')}(\mathbf{k})\right|\leqslant(\text{const.})\ 2^{2h_\epsilon}|U|. \tag{I.8.42}$$

I.8.4. Intermediate regime: second to third

In the intermediate regime, we integrate over the first scales for which the effect of the extra Fermi points $\mathbf{p}_{F,j}^{\omega}$ cannot be neglected. As a consequence, the local part of $\hat{\bar{A}}_{\mathfrak{h}_2,\omega}(\mathbf{k})$ is not dominant, so that the proof of the inductive assumption (I.8.1) for $h=\mathfrak{h}_2$ must be discussed anew. In addition, we will see that dressing the propagator throughout the integrations over the first and second regimes will have shifted the Fermi points away from $\mathbf{p}_{F,j}^{\omega}$ by a small amount. Such an effect has not been seen so far because the position of $\mathbf{p}_{F,0}^{\omega}$ is fixed by symmetry.

1 - Power counting estimate. We first prove that

$$\int d\mathbf{x}\ |\mathbf{x}^m\bar{g}_{\mathfrak{h}_2,\omega}(\mathbf{x})|\leqslant(\text{const.})\ 2^{-\mathfrak{h}_2}\mathfrak{F}_{\mathfrak{h}_2}(\underline{m}). \tag{I.8.43}$$

The proof is slightly different from the proof in section I.8.2: instead of splitting $\hat{\bar{g}}_{\mathfrak{h}_2,\omega}$ according to (I.8.10), we rewrite it as

$$\hat{\bar{g}}_{\mathfrak{h}_2,\omega}(\mathbf{k})=f_{\mathfrak{h}_2,\omega}(\mathbf{k})\left(\hat{A}(\mathbf{k})+\mathcal{L}\hat{\bar{\mathfrak{W}}}_{\mathfrak{h}_2,\omega}(\mathbf{k})\right)^{-1}\left(\mathbb{1}+\left(\mathcal{R}\hat{\bar{\mathfrak{W}}}_{\mathfrak{h}_2,\omega}(\mathbf{k})\right)\left(\mathbb{L}\hat{\mathfrak{g}}_{[\mathfrak{h}_2],\omega}(\mathbf{k})\right)\right)^{-1} \tag{I.8.44}$$

(this decomposition suggests that the dominant part of $\hat{\bar{A}}_{\mathfrak{h}_2,\omega}$ is $\hat{A}+\mathcal{L}\hat{\bar{\mathfrak{W}}}_{\mathfrak{h}_2,\omega}$ instead of $\mathcal{L}\hat{\bar{A}}_{\mathfrak{h}_2,\omega}$) in which we recall that $\hat{A}\equiv\hat{\bar{A}}_{\mathfrak{h}_2,\omega}|_{U=0}$,

$$\hat{\bar{\mathfrak{W}}}_{\mathfrak{h}_2,\omega}(\mathbf{k}):=\hat{\bar{A}}_{\mathfrak{h}_2,\omega}(\mathbf{k})-\hat{A}(\mathbf{k})$$

and

$$\mathbb{L}\hat{\mathfrak{g}}_{[\mathfrak{h}_2],\omega}(\mathbf{k}):=\left(f_{\leqslant\mathfrak{h}_2+1,\omega}(\mathbf{k})-\sum_{j\in\{0,1,2,3\}}f_{\leqslant\mathfrak{h}_2-2,\omega,j}(\mathbf{k})\right)\left(\hat{A}(\mathbf{k})+\mathcal{L}\hat{\bar{\mathfrak{W}}}_{\mathfrak{h}_2,\omega}(\mathbf{k})\right)^{-1}(\mathbf{k}).$$

We want to estimate the behavior of (I.8.44) in $\mathcal{B}_{\beta,L}^{(\mathfrak{h}_2,\omega)}$, which we recall is a ball with four holes around each $\mathbf{p}_{F,j}^{\omega}$, $j=0,1,2,3$. The splitting in (I.8.44) is convenient in that it is easy to see that $\hat{A}(\mathbf{k})+\mathcal{L}\hat{\bar{\mathfrak{W}}}_{\mathfrak{h}_2,\omega}(\mathbf{k})$ satisfies the same estimates as $\hat{A}(\mathbf{k})$; in particular, via proposition I.A2.1 (see appendix I.A2), we see that $\det(\hat{A}(\mathbf{k})+\mathcal{L}\hat{\bar{\mathfrak{W}}}_{\mathfrak{h}_2,\omega}(\mathbf{k}))\geqslant\det\hat{A}(\mathbf{k})\cdot(1+O(U))$ on $\mathcal{B}_{\beta,L}^{(\mathfrak{h}_2,\omega)}$, so that for all $n\leqslant 7$ and $\mathbf{k}\in\mathcal{B}_{\beta,L}^{(\mathfrak{h}_2,\omega)}$,

$$\left|\partial_{\mathbf{k}}^{n}\left(\hat{A}(\mathbf{k})+\mathcal{L}\hat{\bar{\mathfrak{W}}}_{\mathfrak{h}_2,\omega}(\mathbf{k})\right)^{-1}\right|\leqslant(\text{const.})\ 2^{-\mathfrak{h}_2}\mathfrak{F}_{\mathfrak{h}_2}(\underline{n}) \tag{I.8.45}$$

and, moreover, for $m\leqslant 3$,

$$\int d\mathbf{x}\ \left|\mathbf{x}^m\mathcal{R}\bar{\mathfrak{W}}_{\mathfrak{h}_2,\omega}*\mathbb{L}\mathfrak{g}_{[\mathfrak{h}_2],\omega}(\mathbf{x})\right|\leqslant(\text{const.})\ |U||h_\epsilon|2^{h_\epsilon}\mathfrak{F}_{\mathfrak{h}_2}(\underline{m}). \tag{I.8.46}$$

The proof of (I.8.43) is then concluded by injecting (I.8.45) and (I.8.46) into (I.8.44). We can then use the discussion in section I.8.1 to bound

$$\left\{\begin{array}{l}\displaystyle\int d\mathbf{x}\ \left|\mathbf{x}^m W_{2,\omega,\underline{\alpha}}^{(\bar{\mathfrak{h}}_2)}(\mathbf{x})\right|\leqslant\bar{C}_2 2^{\bar{\mathfrak{h}}_2+h_\epsilon}\mathfrak{F}_{\bar{\mathfrak{h}}_2}(\underline{m})|U|\\ \displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_4)^m W_{4,\underline{\omega},\underline{\alpha}}^{(\bar{\mathfrak{h}}_2)}(\underline{\mathbf{x}})\right|\leqslant\bar{C}_2\mathfrak{F}_{\bar{\mathfrak{h}}_2}(\underline{m})|U|\\ \displaystyle\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m W_{2l,\underline{\omega},\underline{\alpha}}^{(\bar{\mathfrak{h}}_2)}(\underline{\mathbf{x}})\right|\leqslant 2^{(\bar{\mathfrak{h}}_2+h_\epsilon)(2-l)}\mathfrak{F}_{\bar{\mathfrak{h}}_2}(\underline{m})(\bar{C}_2|U|)^{l-1}\end{array}\right. \tag{I.8.47}$$

for some constant $\bar{C}_2>1$.

2 - Shift in the Fermi points. We now discuss the shift of the Fermi points, and show that $\hat{\bar{g}}_{\leqslant\mathfrak{h}_2,\omega}$ has *at least* 8 singularities: $\mathbf{p}_{F,0}^{\omega}$ and $\tilde{\mathbf{p}}_{F,j}^{(\omega,\mathfrak{h}_2)}$ for $j\in\{1,2,3\}$ where

$$\tilde{\mathbf{p}}_{F,1}^{(\omega,\mathfrak{h}_2)}=\mathbf{p}_{F,1}^{\omega}+(0,0,\omega\Delta_{\mathfrak{h}_2}) \tag{I.8.48}$$

and

$$\tilde{\mathbf{p}}_{F,2}^{(\omega,\mathfrak{h}_2)}=T^{-\omega}\tilde{\mathbf{p}}_{F,1}^{(\omega,\mathfrak{h}_2)},\quad \tilde{\mathbf{p}}_{F,3}^{(\omega,\mathfrak{h}_2)}=T^{\omega}\tilde{\mathbf{p}}_{F,1}^{(\omega,\mathfrak{h}_2)} \tag{I.8.49}$$

in which $T^{\pm}$ denotes the spatial rotation by $\pm2\pi/3$; and that

$$|\Delta_{\mathfrak{h}_2}|\leqslant(\text{const.})\ \epsilon^2|U| \tag{I.8.50}$$

(note that (I.8.49) follows immediately from the rotation symmetry (I.2.33), so we can restrict our discussion to $j=1$).

Remark: Actually, we could prove in this section that $\hat{\bar{g}}_{\leqslant\mathfrak{h}_2,\omega}$ has *exactly* 8 singularities, but this fact follows automatically from the discussion in section I.9, for the same reason that the proof that the splittings (I.7.18) and (I.8.10) are well defined in the first and second regimes implies that no additional singularity can appear in those regimes. Since the third regime extends to $h\to-\infty$, proving that the splitting (I.8.10) is well defined in the third regime will imply that there are 8 Fermi points.

We will be looking for $\tilde{\mathbf{p}}_{F,1}^{(\omega,\mathfrak{h}_2)}$ in the form (I.8.48). In particular, its k_0 component vanishes, so that, by corollary I.A2.2 (see appendix I.A2), $\Delta_{\mathfrak{h}_2}$ solves

$$\hat{\bar{D}}_{\mathfrak{h}_2,\omega}(\Delta_{\mathfrak{h}_2}):=\hat{\bar{A}}^{2}_{\mathfrak{h}_2,\omega,(b,a)}(\tilde{\mathbf{p}}_{F,1}^{(\omega,\mathfrak{h}_2)})-\hat{\bar{A}}_{\mathfrak{h}_2,\omega,(\tilde{b},a)}(\tilde{\mathbf{p}}_{F,1}^{(\omega,\mathfrak{h}_2)})\hat{\bar{A}}_{\mathfrak{h}_2,\omega,(b,\tilde{a})}(\tilde{\mathbf{p}}_{F,1}^{(\omega,\mathfrak{h}_2)})=0. \tag{I.8.51}$$

In order to solve (I.8.51), we can use a Newton iteration, so we expand $\hat{\bar{D}}_{\mathfrak{h}_2,\omega}$ around 0: it follows from the symmetries (I.2.35) and (I.2.36) that

$$\hat{\bar{D}}_{\mathfrak{h}_2,\omega}(\Delta_{\mathfrak{h}_2})=M_{\mathfrak{h}_2}+\omega Y_{\mathfrak{h}_2}\Delta_{\mathfrak{h}_2}+\Delta_{\mathfrak{h}_2}^2R^{(2)}_{\mathfrak{h}_2,\omega}(\Delta_{\mathfrak{h}_2}) \tag{I.8.52}$$

with $(M_{\mathfrak{h}_2},Y_{\mathfrak{h}_2})\in\mathbb{R}^2$, independent of ω. Furthermore by injecting (I.7.10) and (I.8.6) into (I.8.51), we find that

$$Y_{\mathfrak{h}_2}=\frac{3}{2}\gamma_1\gamma_3+O(\epsilon^2|U|)+O(\epsilon^4),\quad M_{\mathfrak{h}_2}=O(\epsilon^4|U|) \tag{I.8.53}$$

and

$$\left|R^{(2)}_{\mathfrak{h}_2,\omega}(\Delta_{\mathfrak{h}_2})\right|\leqslant(\text{const.})\ . \tag{I.8.54}$$

Therefore, by using a Newton scheme, one finds a root $\Delta_{\mathfrak{h}_2}$ of (I.8.51) and, by (I.8.53) and (I.8.54),

$$|\Delta_{\mathfrak{h}_2}|\leqslant(\text{const.})\ \epsilon^2|U|. \tag{I.8.55}$$

This concludes the proof of (I.8.48) and (I.8.50).

I.9. Third regime

Finally, we perform the multiscale integration in the third regime. Similarly to the first and second regimes, we prove by induction that $\bar{g}_{h,\omega,j}$ satisfies the same estimate as $g_{h,\omega,j}$ (see (I.4.56) and (I.4.54)): for all $m\leqslant3$,

$$\left\{\begin{array}{l}\displaystyle\int d\mathbf{x}\ |x_0^{m_0}x^{m_k}\bar{g}_{h,\omega,j}(\mathbf{x})|\leqslant(\text{const.})\ 2^{-h-m_0h-m_k(h-h_\epsilon)}\\[2ex] \displaystyle\frac{1}{\beta|\Lambda|}\sum_{\mathbf{k}\in\mathcal{B}_{\beta,L}^{(h,\omega,j)}}|\hat{\bar{g}}_{h,\omega,j}(\mathbf{k})|\leqslant(\text{const.})\ 2^{2h-2h_\epsilon}.\end{array}\right. \tag{I.9.1}$$

which in terms of the hypotheses of lemma I.5.2 means

$$c_k = 3,\ c_g = 1,\quad \mathfrak{F}_h(m_0, m_1, m_2) = 2^{-m_0 h-(m_1+m_2)(h-h_\epsilon)},$$

$C_g = (\text{const.})$ and $C_G = (\text{const.})\ 2^{-2h_\epsilon}$.

Remark: As in the second regime, the estimates (I.9.1) are not optimal because the massive components scale differently from the massless ones.

Like in the first regime,

$$\ell_0 = 2 > \frac{c_k}{c_k - c_g} = \frac{3}{2}.$$

I.9.1. Power counting in the third regime

1 - Power counting estimate. By lemma I.5.2 and (I.8.47), we find that for all $m \leqslant 3$

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\omega},\underline{j},\underline{\alpha}}(\underline{\mathbf{x}})\right| \leqslant 2^{h(3-2l)}\mathfrak{F}_h(\underline{m})2^{2lh_\epsilon}\sum_{N=1}^{\infty}\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\ |P_{v_0}|=2l}} (c_1 2^{2h_\epsilon})^{N-1}\prod_{v\in\mathfrak{V}(\tau)}2^{(3-|P_v|)}\prod_{v\in\mathfrak{E}(\tau)}(c_2 2^{-2h_\epsilon})^{l_v}|U|^{\max(1,l_v-1)}2^{(2l_v-1)h_\epsilon}$$

so that

$$\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m B^{(h)}_{2l,\underline{\omega},\underline{j},\underline{\alpha}}(\underline{\mathbf{x}})\right| \leqslant 2^{h(3-2l)}\mathfrak{F}_h(\underline{m})2^{2(l-1)h_\epsilon}\cdot \sum_{N=1}^{\infty}\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\underline{l}_\tau}\sum_{\substack{\mathbf{P}\in\tilde{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\ |P_{v_0}|=2l}} c_1^{N-1}2^{Nh_\epsilon}\prod_{v\in\mathfrak{V}(\tau)}2^{(3-|P_v|)}\prod_{v\in\mathfrak{E}(\tau)}c_2^{l_v}|U|^{\max(1,l_v-1)}. \tag{I.9.2}$$

2 - Bounding the sum of trees. We then bound the sum over trees as in the first regime (see (I.7.4) and (I.7.8)): if $l\geqslant 2$ then for $\theta\in(0,1)$ and recalling that $\bar{\mathfrak{h}}_2 = 3h_\epsilon + \text{const}$,

$$\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\substack{\mathbf{P}\in\bar{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\ |P_{v_0}|=2l}}\prod_{v\in\mathfrak{V}(\tau)\setminus\{v_0\}}2^{(3-|P_v|)} \leqslant 2^{2\theta(h-3h_\epsilon)}C_T^N\prod_{i=1}^N C_P^{2l_i}. \tag{I.9.3}$$

and if $l=1$ then

$$\sum_{\tau\in\mathcal{T}_N^{(h)}}\sum_{\substack{\mathbf{P}\in\bar{\mathcal{P}}^{(h)}_{\tau,\underline{l}_\tau,2}\\ |P_{v_0}|=2}}\prod_{v\in\mathfrak{V}(\tau)\setminus\{v_0\}}2^{(3-|P_v|)} \leqslant 2^{h-3h_\epsilon}C_T^N\prod_{i=1}^N C_P^{2l_i}. \tag{I.9.4}$$

Therefore, proceeding as in the proof of (I.7.10) and (I.7.12) we find that

$$\int d\mathbf{x}\ \left|\mathbf{x}^m W^{(h)}_{2,\omega,j,\underline{\alpha}}(\mathbf{x})\right| \leqslant 2^{2(h-h_\epsilon)}\mathfrak{F}_h(\underline{m})C_1|U| \tag{I.9.5}$$

and

$$\left\{\begin{array}{l}\frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_4)^m W^{(h)}_{4,\underline{\omega},\underline{j},\underline{\alpha}}(\underline{\mathbf{x}})\right| \leqslant \mathfrak{F}_h(\underline{m})C_1|U| \\ \frac{1}{\beta|\Lambda|}\int d\underline{\mathbf{x}}\ \left|(\underline{\mathbf{x}}-\mathbf{x}_{2l})^m W^{(h)}_{2l,\underline{\omega},\underline{j},\underline{\alpha}}(\underline{\mathbf{x}})\right| \leqslant 2^{(3-2l)h+2\theta(h-3h_\epsilon)+(2l-1)h_\epsilon}\mathfrak{F}_h(\underline{m})(C_1|U|)^{l-1}\end{array}\right. \tag{I.9.6}$$

for $l\geqslant 3$ and $m\leqslant 3$.

Remark: The estimates (I.9.2), (I.9.3) and (I.9.4) imply the convergence of the tree expansion (I.5.8), thus providing a convergent expansion of $W^{(h)}_{2l,\underline{\omega},\underline{\alpha}}$ in U.

I.9.2. The dressed propagator

We now prove (I.9.1). We recall that (see (I.4.23))

$$\hat{\bar{g}}_{h,\omega,j}(\mathbf{k}) = f_{h,\omega,j}(\mathbf{k})\hat{\bar{A}}^{-1}_{h,\omega,j}(\mathbf{k}) \tag{I.9.7}$$

where

$$\hat{\bar{A}}_{h,\omega,j}(\mathbf{k}) = \hat{A}(\mathbf{k}) + f_{\leqslant h,\omega,j}(\mathbf{k})\hat{W}^{(h)}_2(\mathbf{k}) + \sum_{h'=h+1}^{\bar{\mathfrak{h}}_2}\hat{W}^{(h')}_2(\mathbf{k}) + \sum_{h'=\mathfrak{h}_2}^{\bar{\mathfrak{h}}_1}\hat{W}^{(h')}_2(\mathbf{k}) + \sum_{h'=\mathfrak{h}_1}^{\bar{\mathfrak{h}}_0}\hat{W}^{(h')}_2(\mathbf{k}).$$

1 - $j=0$ case. We first study the $j=0$ case, which is similar to the discussion in the second regime. We use the localization operator defined in (I.7.15) and split $\hat{\bar{g}}_{h,\omega,0}$ in the same way as in (I.8.10). We then compute $\mathcal{L}\hat{W}^{(h')}_2$ and bound $\mathcal{R}\hat{\bar{A}}_{h,\omega,0}\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega,0}$.

1-1 - Local part. The symmetry considerations of the first and second regime still hold (see (I.7.21) and (I.7.22)) so that (I.7.23) still holds:

$$\mathcal{L}\hat{W}^{(h')}_2(\mathbf{k}'+\mathbf{p}^\omega_{F,0}) = -\begin{pmatrix} i\tilde{\zeta}_{h'}k_0 & \gamma_1\tilde{\mu}_{h'} & 0 & \nu_{h'}\xi^* \\ \gamma_1\tilde{\mu}_{h'} & i\tilde{\zeta}_{h'}k_0 & \nu_{h'}\xi & 0 \\ 0 & \nu_{h'}\xi^* & i\zeta_{h'}k_0 & \gamma_3\tilde{\nu}_{h'}\xi \\ \nu_{h'}\xi & 0 & \gamma_3\tilde{\nu}_{h'}\xi^* & i\zeta_{h'}k_0\end{pmatrix}, \tag{I.9.8}$$

with $(\tilde{\zeta}_{h'},\tilde{\mu}_{h'},\tilde{\nu}_{h'},\zeta_{h'},\nu_{h'})\in\mathbb{R}^5$. The estimates (I.8.14) through (I.8.16) hold, and it follows from (I.9.5) that if $h'\leqslant\bar{\mathfrak{h}}_2$, then

$$\begin{array}{c}|\tilde{\zeta}_{h'}|\leqslant(\text{const.})\ |U|2^{h'-2h_\epsilon},\quad |\zeta_{h'}|\leqslant(\text{const.})\ |U|2^{h'-2h_\epsilon},\quad |\tilde{\mu}_{h'}|\leqslant(\text{const.})\ |U|2^{2h'-3h_\epsilon},\\ |\nu_{h'}|\leqslant(\text{const.})\ |U|2^{h'-h_\epsilon},\quad |\tilde{\nu}_{h'}|\leqslant(\text{const.})\ |U|2^{h'-2h_\epsilon}.\end{array} \tag{I.9.9}$$

Therefore

$$\mathcal{L}\hat{\bar{A}}_{h,\omega,0}(\mathbf{k}'+\mathbf{p}^\omega_{F,0}) = -\begin{pmatrix} i\tilde{z}_hk_0 & \gamma_1\tilde{m}_h & 0 & v_h\xi^* \\ \gamma_1\tilde{m}_h & i\tilde{z}_hk_0 & v_h\xi & 0 \\ 0 & v_h\xi^* & iz_hk_0 & \gamma_3\tilde{v}_h\xi \\ v_h\xi & 0 & \gamma_3\tilde{v}_h\xi^* & iz_hk_0\end{pmatrix} \tag{I.9.10}$$

where z_h, $\tilde{z}_h$, m_h, v_h and $\tilde{v}_h$ are defined as in (I.8.18). and are bounded as in (I.8.19):

$$\begin{array}{c}|\tilde{m}_h-1|\leqslant(\text{const.})\ |U|,\quad |\tilde{z}_h-1|\leqslant(\text{const.})\ |U|,\quad |z_h-1|\leqslant(\text{const.})\ |U|,\\ |\tilde{v}_h-1|\leqslant(\text{const.})\ |U|,\quad |v_h-1|\leqslant(\text{const.})\ |U|.\end{array} \tag{I.9.11}$$

1-2 - Dominant part of $\mathcal{L}\hat{\bar{A}}_{h,\omega,0}$ Furthermore, we notice that the terms proportional to $\tilde{z}_h$ are sub-dominant:

$$\mathcal{L}\hat{\bar{A}}_{h,\omega,0}(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega}) = \mathfrak{L}\hat{\bar{A}}_{h,\omega,0}(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega})(\mathbb{1}+\sigma_4(\mathbf{k}')) \tag{I.9.12}$$

where

$$\mathfrak{L}\hat{\bar{A}}_{h,\omega,0}(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega}) := -\left(\begin{array}{cccc} 0 & \gamma_1\tilde{m}_h & 0 & v_h\xi^* \\ \gamma_1\tilde{m}_h & 0 & v_h\xi & 0 \\ 0 & v_h\xi^* & iz_hk_0 & \gamma_3\tilde{v}_h\xi \\ v_h\xi & 0 & \gamma_3\tilde{v}_h\xi^* & iz_hk_0 \end{array}\right). \tag{I.9.13}$$

Before bounding σ_4, we compute the inverse of (I.9.13) by block-diagonalizing it using proposition I.A3.1 (see appendix I.A3): if we define

$$\bar{k}_0 := z_hk_0, \quad \bar{\gamma}_1 := \tilde{m}_h\gamma_1, \quad \tilde{\bar{\xi}} := \tilde{v}_h\xi, \quad \bar{\xi} := v_h\xi \tag{I.9.14}$$

then for $\mathbf{k}\in\mathcal{B}_{\beta,L}^{(h,\omega,0)}$,

$$\left(\mathfrak{L}\hat{\bar{A}}_{h,\omega,0}(\mathbf{k})\right)^{-1} = \left(\begin{array}{cc} \mathbb{1} & \bar{M}_{h,0}^{\dagger}(\mathbf{k}) \\ 0 & \mathbb{1} \end{array}\right)\left(\begin{array}{cc} \bar{a}_{h,0}^{(M)} & 0 \\ 0 & \bar{a}_{h,0}^{(m)}(\mathbf{k}) \end{array}\right)\left(\begin{array}{cc} \mathbb{1} & 0 \\ \bar{M}_{h,0}(\mathbf{k}) & \mathbb{1} \end{array}\right)(\mathbb{1}+O(2^{h-3h_\epsilon})) \tag{I.9.15}$$

where

$$\bar{a}_{h,0}^{(M)} := -\left(\begin{array}{cc} 0 & \bar{\gamma}_1^{-1} \\ \bar{\gamma}_1^{-1} & 0 \end{array}\right), \quad \bar{a}_{h,0}^{(m)}(\mathbf{p}_{F,0}^{\omega}+\mathbf{k}') := -\frac{1}{\bar{k}_0^2+\gamma_3^2|\tilde{\bar{\xi}}|^2}\left(\begin{array}{cc} -i\bar{k}_0 & \gamma_3\tilde{\bar{\xi}} \\ \gamma_3\tilde{\bar{\xi}}^* & -i\bar{k}_0 \end{array}\right) \tag{I.9.16}$$

(the $O(2^{h-3h_\epsilon})$ term comes from the terms we neglected from $\bar{a}^{(m)}$ that are of order 2^{-3h_ϵ}) and

$$\bar{M}_{h,0}(\mathbf{p}_{F,0}^{\omega}+\mathbf{k}') := -\frac{1}{\bar{\gamma}_1}\left(\begin{array}{cc} \bar{\xi}^* & 0 \\ 0 & \bar{\xi} \end{array}\right). \tag{I.9.17}$$

In particular, this implies that, if $(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega})\in\mathcal{B}_{\beta,L}^{(h,\omega,0)}$, then

$$|\mathfrak{L}\hat{\bar{A}}_{h,\omega,0}^{-1}(\mathbf{k}'+\mathbf{p}_{F,0}^{\omega})| \leqslant (\text{const.})\left(\begin{array}{cc} 2^{-h_\epsilon} & 2^{-2h_\epsilon} \\ 2^{-2h_\epsilon} & 2^{-h} \end{array}\right) \tag{I.9.18}$$

in which the bound should be understood as follows: the upper-left element in (I.9.18) is the bound on the upper-left 2×2 block of $\mathfrak{L}\hat{\bar{A}}_{h,\omega,0}^{-1}$, and similarly for the upper-right, lower-left and lower-right. In turn, (I.9.18) implies

$$|\sigma_4(\mathbf{k}')| \leqslant (\text{const.})\ 2^{h-2h_\epsilon}. \tag{I.9.19}$$

1-3 - Irrelevant part. We now bound $\mathcal{R}W_{2,\omega,0}^{(h')} * \mathbb{L}\bar{\mathfrak{g}}_{[h,\omega,0]}$ in the same way as in the second regime, and find that for $m\leqslant 3$, if $h\leqslant h'\leqslant\mathfrak{h}_0$, then

$$\int d\mathbf{x}\left|\mathbf{x}^m\mathcal{R}W_{2,\omega,0}^{(h')} * \mathbb{L}\bar{\mathfrak{g}}_{[h],\omega,0}(\mathbf{x})\right| \leqslant 2^{h-2h_\epsilon}\mathfrak{F}_h(\underline{m})(\text{const.})\ |U| \tag{I.9.20}$$

so that

$$\int d\mathbf{x}\left|\mathbf{x}^m\mathcal{R}\bar{A}_{h,\omega,0} * \mathbb{L}\bar{\mathfrak{g}}_{[h],\omega,0}(\mathbf{x})\right| \leqslant 2^{h-2h_\epsilon}\mathfrak{F}_h(\underline{m})(\text{const.})\ (1+|h||U|). \tag{I.9.21}$$

This concludes the proof of (I.9.1) for $j=0$.

2 - $j=1$ case. We now turn to the case $j=1$ ($j=2,3$ will then follow by using the $2\pi/3$-rotation symmetry). Again, we split $\hat{\bar{g}}_{h,\omega,1}$ in the same way as in (I.8.10), then we compute

$\mathcal{L}\hat{W}_2^{(h')}$ and bound $\mathcal{R}\hat{\bar{A}}_{h,\omega,1}\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega,1}$. Before computing $\mathcal{L}\hat{\bar{A}}_{h,\omega,1}$ and bounding $\mathcal{R}\hat{\bar{A}}_{h,\omega,1}\mathbb{L}\hat{\bar{\mathfrak{g}}}_{[h],\omega,1}$, we first discuss the shift in the Fermi points $\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}$ (i.e., the singularities of $\hat{\bar{A}}_{h,\omega,1}^{-1}(\mathbf{k})$ in the vicinity of $\mathbf{p}_{F,1}^{(\omega,h)}$), due to the renormalization group flow.

2-1 - Shift in the Fermi points. We compute the position of the shifted Fermi points in the form

$$\tilde{\mathbf{p}}_{F,1}^{(\omega,h)} = \mathbf{p}_{F,1}^{\omega} + (0,0,\omega\Delta_h) \tag{I.9.22}$$

and show that

$$|\Delta_h| \leqslant (\text{const.})\ \epsilon^2|U|. \tag{I.9.23}$$

The proof goes along the same lines as that in section I.8.4.

Similarly to (I.8.51), Δ_h is a solution of

$$\hat{\bar{D}}_{h,\omega,1}(\Delta_h) := \hat{\bar{A}}_{h,\omega,1,(b,a)}^2(\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}) - \hat{\bar{A}}_{h,\omega,1,(\tilde{b},a)}(\tilde{\mathbf{p}}_{F,1}^{(\omega,h)})\hat{\bar{A}}_{h,\omega,1,(b,\tilde{a})}(\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}) = 0. \tag{I.9.24}$$

We expand $\hat{\bar{D}}_{h,\omega,1}$ around Δ_{h+1}: it follows from the symmetries (I.2.35) and (I.2.36) that

$$\hat{\bar{D}}_{h,\omega,1}(\Delta_h) = M_h + \omega Y_h(\Delta_h - \Delta_{h+1}) + (\Delta_h - \Delta_{h+1})^2 R_{h,\omega,1}^{(2)}(\Delta_h) \tag{I.9.25}$$

with $(M_h, Y_h) \in \mathbb{R}^2$, independent of ω. Furthermore,

$$M_h = \hat{\bar{D}}_{h,\omega,1}(\Delta_{h+1}) = \hat{\bar{D}}_{h,\omega,1}(\Delta_{h+1}) - \hat{\bar{D}}_{h+1,\omega,1}(\Delta_{h+1})$$

so that, by injecting (I.9.5), (I.7.10) and (I.8.6) into (I.9.24) and using the symmetry structure of $\hat{\bar{A}}_{h,\omega,1}(\mathbf{k})$ (which imply, in particular, that $|\hat{\bar{A}}_{h,\omega,1}(\mathbf{k})| \leqslant (\text{const.})\ \epsilon$ in $\mathcal{B}_{\beta,L}^{(\leqslant h,\omega,1)}$), we find

$$|M_h| \leqslant (\text{const.})\ 2^{2h-3h_\epsilon}\epsilon^2|U| \tag{I.9.26}$$

and

$$Y_h = \frac{3}{2}\gamma_1\gamma_3 + O(\epsilon^2|U|) + O(\epsilon^4). \tag{I.9.27}$$

as well as

$$\left|R_{h,\omega,1}^{(2)}(\Delta_h)\right| \leqslant (\text{const.})\ (1 + \epsilon|U||h|). \tag{I.9.28}$$

Therefore, by using a Newton scheme, we compute Δ_h satisfying (I.9.24) and, by (I.9.26), (I.9.27) and (I.9.28),

$$|\Delta_h - \Delta_{h+1}| \leqslant (\text{const.})\ 2^{2h-3h_\epsilon}|U|. \tag{I.9.29}$$

This concludes the proof of (I.9.22) and (I.9.23).

2-2 - Local part. We now compute $\mathcal{L}\hat{\bar{A}}_{h,\omega,1}$. The computation is similar to the $j=0$ case, though it is complicated slightly by the presence of constant terms in $\hat{\bar{A}}_{h,\omega,1}$. Recall the $\mathbf{x}$-space representation of $\bar{A}_{h,\omega,1}$ (I.4.42). The localization operator has the same definition as (I.7.15), but because of the shift by $\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}$ in the Fourier transform, its action in $\mathbf{k}$-space becomes

$$\mathcal{L}\hat{\bar{A}}_{h,\omega,1}(\mathbf{k}) = \hat{\bar{A}}_{h,\omega,1}(\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}) + (\mathbf{k} - \tilde{\mathbf{p}}_{F,1}^{(\omega,h)}) \cdot \partial_{\mathbf{k}}\hat{\bar{A}}_{h,\omega,1}(\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}).$$

In order to avoid confusion, we will denote the localization operator in $\mathbf{k}$ space around $\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}$ by $\hat{\mathcal{L}}_h$.

2-2-1 - Non-interacting local part. As a preliminary step, we discuss the action of $\hat{\mathcal{L}}$ on the *undressed* inverse propagator $\hat{A}(\mathbf{k})$. Let us first split $\hat{A}(\mathbf{k})$ into 2×2 blocks:

$$\hat{A}(\mathbf{k}) =: \left(\begin{array}{cc} \hat{A}^{\xi\xi}(\mathbf{k}) & \hat{A}^{\xi\phi}(\mathbf{k}) \\ \hat{A}^{\phi\xi}(\mathbf{k}) & \hat{A}^{\phi\phi}(\mathbf{k}) \end{array}\right)$$

in terms of which

$$\hat{\mathcal{L}}_h\hat{A}^{\xi\xi}(\mathbf{k}'_1+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}) = -\left(\begin{array}{cc} ik_0 & \gamma_1 \\ \gamma_1 & ik_0 \end{array}\right)$$

$$\begin{array}{l}\hat{\mathcal{L}}_h\hat{A}^{\xi\phi}(\mathbf{k}'_1+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}) = \hat{\mathcal{L}}_h\hat{A}^{\phi\xi}(\mathbf{k}'_1+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}) \\ \qquad = -\left(\begin{array}{cc} 0 & m_h^{(0)}+(-iv_h^{(0)}k'_{1,x}+\omega w_h^{(0)}k'_{1,y}) \\ m_h^{(0)}+(iv_h^{(0)}k'_{1,x}+\omega w_h^{(0)}k'_{1,y}) & 0 \end{array}\right)\end{array} \tag{I.9.30}$$

$$\begin{array}{l}\hat{\mathcal{L}}_h\hat{A}^{\phi\phi}(\mathbf{k}'_1+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}) \\ \qquad = -\left(\begin{array}{cc} ik'_{\omega,1,0} & \gamma_3(m_h^{(0)}+(i\tilde{v}_h^{(0)}k'_{1,x}+\omega w_h^{(0)}k'_{1,y})) \\ \gamma_3(m_h^{(0)}+(-i\tilde{v}_h^{(0)}k'_{1,x}+\omega w_h^{(0)}k'_{1,y})) & ik'_{\omega,1,0} \end{array}\right)\end{array}$$

where

$$\begin{array}{c} m_h^{(0)} = \gamma_1\gamma_3 + O(\Delta_h), \quad v_h^{(0)} = \frac{3}{2}+O(\epsilon^2,\Delta_h), \\ \tilde{v}_h^{(0)} = \frac{3}{2}+O(\epsilon^2,\Delta_h), \quad w_h^{(0)} = \frac{3}{2}+O(\epsilon^2,\Delta_h). \end{array} \tag{I.9.31}$$

2-2-2 - Local part of $\hat{W}_2$. We now turn our attention to $\hat{\mathcal{L}}_h\hat{W}_2^{(h')}$. In order to reduce the size of the coming equations, we split $\hat{W}_2^{(h')}$ into 2×2 blocks:

$$\hat{W}_2^{(h')} =: \left(\begin{array}{cc} \hat{W}_2^{(h')\xi\xi} & \hat{W}_2^{(h')\xi\phi} \\ \hat{W}_2^{(h')\phi\xi} & \hat{W}_2^{(h')\phi\phi} \end{array}\right).$$

The symmetry structure around $\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}$ is slightly different from that around $\mathbf{p}_{F,0}^{\omega}$. Indeed (I.7.21) still holds, but $\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}$ is not invariant under rotations, so that (I.7.22) becomes

$$\tilde{\mathbf{p}}_{F,1}^{(\omega,h)} = -\tilde{\mathbf{p}}_{F,1}^{(-\omega,h)} = R_v\tilde{\mathbf{p}}_{F,1}^{(-\omega,h)} = R_h\tilde{\mathbf{p}}_{F,1}^{(\omega,h)} = I\tilde{\mathbf{p}}_{F,1}^{(\omega,h)} = P\tilde{\mathbf{p}}_{F,1}^{(-\omega,h)}. \tag{I.9.32}$$

It then follows from proposition I.A6.1 (see appendix I.A6) that for all $(f,f')\in\{\xi,\phi\}^2$,

$$\begin{array}{l}\hat{\mathcal{L}}_h\hat{W}_2^{(h')ff'}(\mathbf{k}'_1+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)}) \\ \qquad = -\left(\begin{array}{cc} i\zeta_{h',1}^{ff'}k_0 & \mu_{h',1}^{ff'}+(i\nu_{h',1}^{ff'}k'_{1,x}+\omega\varpi_{h',1}^{ff'}k'_{1,y}) \\ \mu_{h',1}^{ff'}+(-i\nu_{h',1}^{ff'}k'_{1,x}+\omega\varpi_{h',1}^{ff'}k'_{1,y}) & i\zeta_{h',1}^{ff'}k_0 \end{array}\right)\end{array} \tag{I.9.33}$$

with $(\mu_{h',1}^{ff'},\zeta_{h',1}^{ff'},\nu_{h',1}^{ff'},\varpi_{h',1}^{ff'})\in\mathbb{R}^4$. In addition, by using the parity symmetry, it follows from (I.A6.10) (see appendix I.A6) that the $\xi\phi$ block is equal to the $\phi\xi$ block. Furthermore, it follows from (I.9.5) that for $h'\leqslant\bar{\mathfrak{h}}_2$,

$$\begin{array}{c} |\mu_{h',1}^{ff'}|\leqslant(\mathrm{const.})\ |U|2^{2(h'-h_\epsilon)}, \quad |\zeta_{h',1}^{ff'}|\leqslant(\mathrm{const.})\ |U|2^{h'-2h_\epsilon}, \\ |\nu_{h',1}^{ff'}|\leqslant(\mathrm{const.})\ |U|2^{h'-h_\epsilon}, \quad |\varpi_{h',1}^{ff'}|\leqslant(\mathrm{const.})\ |U|2^{h'-h_\epsilon}. \end{array} \tag{I.9.34}$$

If $\mathfrak{h}_2 \leqslant h' \leqslant \bar{\mathfrak{h}}_1$, then it follows from (I.8.6) that

$$\begin{gathered}|\zeta^{ff'}_{h',1}| \leqslant (\text{const.})\ |U|2^{h_\epsilon},\\ |\nu^{ff'}_{h',1}| \leqslant (\text{const.})\ |U|2^{\frac12(h'+h_\epsilon)}, \quad |\varpi^{ff'}_{h',1}| \leqslant (\text{const.})\ |U|2^{\frac12(h'+h_\epsilon)}\end{gathered} \tag{I.9.35}$$

and because $\hat W_2^{(h')}(\mathbf{p}^\omega_{F,0})=0$ and $|\tilde{\mathbf{p}}^{(\omega,h)}_{F,1}-\mathbf{p}^\omega_{F,0}| \leqslant (\text{const.})\ 2^{2h_\epsilon}$, by expanding $\hat W_2^{(h')}$ to first order around $\mathbf{p}^\omega_{F,0}$, we find that it follows from (I.8.6) that

$$|\mu^{ff'}_{h',1}| \leqslant (\text{const.})\ 2^{\frac12(h'+h_\epsilon)}2^{2h_\epsilon}|U|. \tag{I.9.36}$$

Finally, if $\mathfrak{h}_1 \leqslant h' \leqslant \bar{\mathfrak{h}}_0$, then it follows from (I.7.10) that

$$\begin{gathered}|\zeta^{ff'}_{h',1}| \leqslant (\text{const.})\ |U|2^{h'},\\ |\nu^{ff'}_{h',1}| \leqslant (\text{const.})\ |U|2^{h'}, \quad |\varpi^{ff'}_{h',1}| \leqslant (\text{const.})\ |U|2^{h'}\end{gathered} \tag{I.9.37}$$

and by expanding $\hat W_2^{(h')}$ to first order around $\mathbf{p}^\omega_{F,0}$, we find that

$$|\mu^{ff'}_{h',1}| \leqslant (\text{const.})\ |U|2^{h'+2h_\epsilon}. \tag{I.9.38}$$

By using the improved estimate (I.7.56), we can refine these estimates for the inter-layer components, thus finding:

$$\begin{gathered}|\mu^{ff}_{h',1}| \leqslant (\text{const.})\ |U|2^{\theta h'+3h_\epsilon},\\ |\nu^{ff}_{h',1}| \leqslant (\text{const.})\ |U|2^{\theta h'+h_\epsilon}, \quad |\varpi^{ff}_{h',1}| \leqslant (\text{const.})\ |U|2^{\theta h'+h_\epsilon},\\ |\zeta^{\phi\xi}_{h',1}| = |\zeta^{\xi\phi}_{h',1}| \leqslant (\text{const.})\ |U|2^{\theta h'+h_\epsilon}\end{gathered} \tag{I.9.39}$$

for all $f\in\{\phi,\xi\}$.

2-2-3 - Interacting local part. Therefore, putting (I.9.33) together with (I.9.30), we find

$$\hat{\mathcal{L}}_h\hat{\bar A}_{h,\omega,1}(\mathbf{k}'_1+\tilde{\mathbf{p}}^{(\omega,h)}_{F,1})$$

$$= -\begin{pmatrix} iz^{\xi\xi}_{h,1}k_0 & \gamma_1(m^{\xi\xi}_{h,1}+K^{*\xi\xi}_{h,1}) & iz^{\xi\phi}_{h,1}k_0 & m^{\xi\phi}_{h,1}+K^{*\xi\phi}_{h,1}\\ \gamma_1(m^{\xi\xi}_{h,1}+K^{\xi\xi}_{h,1}) & iz^{\xi\xi}_{h,1}k_0 & m^{\xi\phi}_{h,1}+K^{\xi\phi}_{h,1} & iz^{\xi\phi}_{h,1}k_0\\ iz^{\xi\phi}_{h,1}k_0 & m^{\xi\phi}_{h,1}+K^{*\xi\phi}_{h,1} & iz^{\phi\phi}_{h,1}k_0 & \gamma_3(m^{\phi\phi}_{h,1}+K^{\phi\phi}_{h,1})\\ m^{\xi\phi}_{h,1}+K^{\xi\phi}_{h,1} & iz^{\xi\phi}_{h,1}k_0 & \gamma_3(m^{\phi\phi}_{h,1}+K^{*\phi\phi}_{h,1}) & iz^{\phi\phi}_{h,1}k_0\end{pmatrix} \tag{I.9.40}$$

with

$$K^{ff'}_{h,1} := iv^{ff'}_{h,1}k'_{1,x}+\omega w^{ff'}_{h,1}k'_{1,y}$$

for $(f,f')\in\{\phi,\xi\}^2$, and

$$\begin{gathered}m^{\phi\phi}_{h,1} := m^{(0)}_h+\frac{1}{\gamma_3}\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\mu^{\phi\phi}_{h',1}, \quad m^{\xi\xi}_{h,1} := 1+\frac{1}{\gamma_1}\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\mu^{\xi\xi}_{h',1}, \quad m^{\xi\phi}_{h,1} := m^{(0)}_h+\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\mu^{\xi\phi}_{h',1},\\ z^{ff}_{h,1} := 1+\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\zeta^{ff}_{h',1}, \quad z^{\xi\phi}_{h,1} := \sum_{h'=h}^{\bar{\mathfrak{h}}_0}\zeta^{\xi\phi}_{h',1},\\ v^{\phi\phi}_{h,1} := \tilde v^{(0)}_h+\frac{1}{\gamma_3}\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\nu^{\phi\phi}_{h',1}, \quad v^{\xi\xi}_{h,1} := -\frac{1}{\gamma_1}\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\nu^{\xi\xi}_{h',1}, \quad v^{\xi\phi}_{h,1} := v^{(0)}_h-\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\nu^{\xi\phi}_{h',1},\\ w^{\phi\phi}_{h,1} := w^{(0)}_h+\frac{1}{\gamma_3}\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\varpi^{\phi\phi}_{h',1}, \quad w^{\xi\xi}_{h,1} := \frac{1}{\gamma_1}\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\varpi^{\xi\xi}_{h',1}, \quad w^{\xi\phi}_{h,1} := w^{(0)}_h+\sum_{h'=h}^{\bar{\mathfrak{h}}_0}\varpi^{\xi\phi}_{h',1}.\end{gathered} \tag{I.9.41}$$

Furthermore, using the bounds (I.9.34) through (I.9.39),

$$
\begin{gathered}
|m_{h,1}^{\phi\phi}-m_h^{(0)}|+|m_{h,1}^{\xi\xi}-1|+|m_{h,1}^{\xi\phi}-m_h^{(0)}|\leqslant(\mathrm{const.})\ \epsilon^2|U|,\\
|z_{h,1}^{ff}-1|\leqslant(\mathrm{const.})\ |U|,\quad |z_{h,1}^{\xi\phi}|\leqslant(\mathrm{const.})\ |\log\epsilon|\epsilon|U|,\\
|v_{h,1}^{\phi\phi}-\tilde v_h^{(0)}|+|v_{h,1}^{\xi\xi}|+|v_{h,1}^{\xi\phi}-v_h^{(0)}|\leqslant(\mathrm{const.})\ |U|,\\
|w_{h,1}^{\phi\phi}-w_h^{(0)}|+|w_{h,1}^{\xi\xi}|+|w_{h,1}^{\xi\phi}-w_h^{(0)}|\leqslant(\mathrm{const.})\ |U|.
\end{gathered}
\tag{I.9.42}
$$

2-2-4 - Dominant part of $\hat{\mathcal{L}}_h\hat{\bar{A}}_{h,\omega,1}$ Finally, we notice that the terms in (I.9.40) that are proportional to $z_{h,1}^{\xi\xi}$, $z_{h,1}^{\xi\phi}$ or $K_{h,1}^{\xi\xi}$ are subdominant:

$$
\hat{\mathcal{L}}_h\hat{\bar{A}}_{h,\omega,1}(\mathbf{k}_1'+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)})=\hat{\mathfrak{L}}_h\hat{\bar{A}}_{h,\omega,1}(\mathbf{k}_1'+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)})(\mathbb{1}+\sigma_{4,1}(\mathbf{k}_1'))
\tag{I.9.43}
$$

where

$$
\hat{\mathfrak{L}}_h\hat{\bar{A}}_{h,\omega,1}(\mathbf{k}_1'+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)})
:=-\left(\begin{array}{cccc}
0 & \gamma_1 m_{h,1}^{\xi\xi} & 0 & m_{h,1}^{\xi\phi}+K_{h,1}^{*\xi\phi}\\
\gamma_1 m_{h,1}^{\xi\xi} & 0 & m_{h,1}^{\xi\phi}+K_{h,1}^{\xi\phi} & 0\\
0 & m_{h,1}^{\xi\phi}+K_{h,1}^{*\xi\phi} & iz_{h,1}^{\phi\phi}k_0 & \gamma_3(m_{h,1}^{\phi\phi}+K_{h,1}^{\phi\phi})\\
m_{h,1}^{\xi\phi}+K_{h,1}^{\xi\phi} & 0 & \gamma_3(m_{h,1}^{\phi\phi}+K_{h,1}^{*\phi\phi}) & iz_{h,1}^{\phi\phi}k_0
\end{array}\right).
\tag{I.9.44}
$$

Before bounding $\sigma_{4,1}$, we compute the inverse of (I.9.44) by block-diagonalizing it using proposition I.A3.1 (see appendix I.A3): if we define

$$
\bar k_0:=z_{h,1}^{\phi\phi}k_0,\quad \bar\gamma_1:=m_{h,1}^{\xi\xi}\gamma_1,\quad \bar\Xi_1:=m_{h,1}^{\xi\phi}+K_{h,1}^{\xi\phi},\quad \bar x_1:=\frac{2m_{h,1}^{\xi\phi}}{\bar\gamma_1\gamma_3}K_{h,1}^{\xi\phi}-K_{h,1}^{\phi\phi}
\tag{I.9.45}
$$

then for $\mathbf{k}\in\mathcal{B}_{\beta,L}^{(h,\omega,1)}$,

$$
\left(\mathfrak{L}\hat{\bar{A}}_{h,\omega,1}(\mathbf{k})\right)^{-1}=\left(\begin{array}{cc}\mathbb{1} & \bar M_{h,1}^\dagger(\mathbf{k})\\ 0 & \mathbb{1}\end{array}\right)\left(\begin{array}{cc}\bar a_{h,1}^{(M)} & 0\\ 0 & \bar a_{h,1}^{(m)}(\mathbf{k})\end{array}\right)\left(\begin{array}{cc}\mathbb{1} & 0\\ \bar M_{h,1}(\mathbf{k}) & \mathbb{1}\end{array}\right)(\mathbb{1}+O(2^{h-3h_\epsilon}))
\tag{I.9.46}
$$

where

$$
\bar a_{h,1}^{(M)}:=-\left(\begin{array}{cc}0 & \bar\gamma_1^{-1}\\ \bar\gamma_1^{-1} & 0\end{array}\right),\quad \bar a_{h,1}^{(m)}(\mathbf{p}_{F,1}^\omega+\mathbf{k}_1'):=\frac{1}{\bar k_0^2+\gamma_3^2|\bar x_1|^2}\left(\begin{array}{cc}i\bar k_0 & \gamma_3\bar x_1^*\\ \gamma_3\bar x_1 & i\bar k_0\end{array}\right)
\tag{I.9.47}
$$

(the $O(2^{h-3h_\epsilon})$ term comes from the terms in $\bar a^{(m)}$ of order 2^{-3h_ϵ}) and

$$
\bar M_{h,1}(\mathbf{p}_{F,1}^\omega+\mathbf{k}_1'):=-\frac{1}{\bar\gamma_1}\left(\begin{array}{cc}\bar\Xi_1^* & 0\\ 0 & \bar\Xi_1\end{array}\right).
\tag{I.9.48}
$$

In particular, this implies that if $(\mathbf{k}_1'+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)})\in\mathcal{B}_{\beta,L}^{(h,\omega,1)}$, then

$$
|[\hat{\mathfrak{L}}_h\hat{\bar{A}}_{h,\omega,1}(\mathbf{k}_1'+\tilde{\mathbf{p}}_{F,1}^{(\omega,h)})]^{-1}|\leqslant(\mathrm{const.})\left(\begin{array}{cc}2^{2h_\epsilon-h} & 2^{h_\epsilon-h}\\ 2^{h_\epsilon-h} & 2^{-h}\end{array}\right)
\tag{I.9.49}
$$

in which the bound should be understood as follows: the upper-left element in (I.9.49) is the bound on the upper-left 2×2 block of $\hat{\mathfrak{L}}_h\hat{\bar{A}}_{h,\omega,1}^{-1}$, and similarly for the upper-right, lower-left and lower-right. In turn, using (I.9.49) we obtain

$$
|\sigma_{4,1}(\mathbf{k}_1')|\leqslant(\mathrm{const.})\ \epsilon(1+|\log\epsilon||U|).
\tag{I.9.50}
$$

2-3 - Irrelevant part. Finally, we are left with bounding $\mathcal{R}\bar{A}_{h,\omega,1}\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega,1}$, which we show is small. The bound is identical to (I.9.21): indeed, it follows from (I.9.46) and (I.9.49) that for all $m\leqslant 3$,

$$\int d\mathbf{x}\ \left|\mathbf{x}^m\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega,1}(\mathbf{x})\right|\leqslant(\text{const.})\ 2^{-h}\mathfrak{F}_h(\underline{m})$$

so that

$$\int d\mathbf{x}\left|\mathbf{x}^m\mathcal{R}\bar{A}_{h,\omega,1}*\mathbb{L}\bar{\mathfrak{g}}_{[h],\omega,1}(\mathbf{x})\right|\leqslant 2^{h-2h_\epsilon}\mathfrak{F}_h(\underline{m})(\text{const.})\ (1+|h||U|). \tag{I.9.51}$$

3 - $j=2,3$ cases. The cases with $j=2,3$ follow from the $2\pi/3$-rotation symmetry (I.2.33):

$$\hat{\bar{g}}_{h,\omega,j}(\mathbf{k}'_j+\tilde{\mathbf{p}}_{F,j}^{(\omega,h)})=\left(\begin{array}{cc}\mathbb{1}&0\\0&\mathcal{T}_{T\mathbf{k}'_j+\tilde{\mathbf{p}}_{F,j-\omega}^{(\omega,h)}}\end{array}\right)\hat{\bar{g}}_{h,\omega,j}(T\mathbf{k}'_j+\tilde{\mathbf{p}}_{F,j-\omega}^{(\omega,h)})\left(\begin{array}{cc}\mathbb{1}&0\\0&\mathcal{T}^\dagger_{T\mathbf{k}'_j+\tilde{\mathbf{p}}_{F,j-\omega}^{(\omega,h)}}\end{array}\right) \tag{I.9.52}$$

where T and $\mathcal{T}_\mathbf{k}$ were defined above (I.2.33), and $\tilde{\mathbf{p}}_{F,4}^{(-,h)}\equiv\tilde{\mathbf{p}}_{F,1}^{(-,h)}$.

I.9.3. Two-point Schwinger function

We now compute the dominant part of the two-point Schwinger function for $\mathbf{k}$ *well inside* the third regime, i.e.

$$\mathbf{k}\in\mathcal{B}_{\mathrm{III}}^{(\omega,j)}:=\bigcup_{h=\mathfrak{h}_\beta+1}^{\bar{\mathfrak{h}}_2-1}\mathrm{supp}f_{h,\omega,j}.$$

Let

$$h_\mathbf{k}:=\max\{h:\ f_{h,\omega,j}(\mathbf{k})\neq 0\}$$

so that if $h\notin\{h_\mathbf{k},h_\mathbf{k}-1\}$, then $f_{h,\omega,j}(\mathbf{k})=0$.

1 - Schwinger function in terms of dressed propagators. Recall that the two-point Schwinger function can be computed in terms of the effective source term $\mathcal{X}^{(h)}$, see (I.5.27) and comment after Lemma I.5.3. Since $h_\mathbf{k}\leqslant\bar{\mathfrak{h}}_2$, $\mathcal{X}^{(h)}$ is left invariant by the integration over the ultraviolet, the first and the second regimes, in the sense that $\mathcal{X}^{(\bar{\mathfrak{h}}_2)}=\mathcal{X}^{(\bar{\mathfrak{h}}_0)}$, with $\mathcal{X}^{(\bar{\mathfrak{h}}_0)}$ given by (I.7.39). Therefore, we can compute $\mathcal{X}^{(h)}$ for $h\in\{\mathfrak{h}_\beta,\cdots,\bar{\mathfrak{h}}_2-1\}$ inductively using lemma I.5.3, and find, similarly to (I.7.42) and (I.8.31), that

$$s_2(\mathbf{k})=\left(\hat{\bar{g}}_{h_\mathbf{k},\omega,j}(\mathbf{k})+\hat{\bar{g}}_{h_\mathbf{k}-1,\omega,j}(\mathbf{k})\right)(\mathbb{1}-\sigma(\mathbf{k})-\sigma_{<h_\mathbf{k}}(\mathbf{k})) \tag{I.9.53}$$

where

$$\sigma(\mathbf{k}):=\hat{W}_2^{(h_\mathbf{k}-1)}\hat{\bar{g}}_{h_\mathbf{k},\omega,j}+(\hat{\bar{g}}_{h_\mathbf{k},\omega,j}+\hat{\bar{g}}_{h_\mathbf{k}-1,\omega,j})^{-1}\hat{\bar{g}}_{h_\mathbf{k},\omega,j}\hat{W}_2^{(h_\mathbf{k}-1)}\hat{\bar{g}}_{h_\mathbf{k}-1,\omega,j}(\mathbb{1}-\hat{W}_2^{(h_\mathbf{k}-1)}\hat{\bar{g}}_{h_\mathbf{k},\omega,j}) \tag{I.9.54}$$

and

$$\begin{aligned}\sigma_{<h_\mathbf{k}}(\mathbf{k}):=&\left(\mathbb{1}-\left(\hat{\bar{g}}_{h_\mathbf{k},\omega,j}+\hat{\bar{g}}_{h_\mathbf{k}-1,\omega,j}\right)^{-1}\hat{\bar{g}}_{h_\mathbf{k},\omega,j}\hat{W}_2^{(h_\mathbf{k}-1)}\hat{\bar{g}}_{h_\mathbf{k}-1,\omega,j}\right)\left(\sum_{h'=\mathfrak{h}_\beta}^{h_\mathbf{k}-2}\hat{W}_2^{(h')}\right)\cdot\\&\cdot\left(\hat{\bar{g}}_{h_\mathbf{k},\omega,j}+\hat{\bar{g}}_{h_\mathbf{k}-1,\omega,j}-\hat{\bar{g}}_{h_\mathbf{k}-1,\omega,j}\hat{W}_{2,\omega}^{(h_\mathbf{k}-1)}\hat{\bar{g}}_{h_\mathbf{k},\omega,j}\right).\end{aligned} \tag{I.9.55}$$

Similarly to (I.8.34), we have

$$\left|(\hat{\bar{g}}_{h_\mathbf{k},\omega,j}+\hat{\bar{g}}_{h_\mathbf{k}-1,\omega,j})^{-1}\hat{\bar{g}}_{h_\mathbf{k},\omega,j}\right|\leqslant(\text{const.}) \tag{I.9.56}$$

and, by (I.9.5) and (I.9.1), we have

$$\left\{\begin{array}{l}|\sigma(\mathbf{k})|\leqslant(\mathrm{const.})\ 2^{h_{\mathbf{k}}-2h_\epsilon}|U|\\ |\sigma_{<h_{\mathbf{k}}}(\mathbf{k})|\leqslant(\mathrm{const.})\ 2^{h_{\mathbf{k}}-2h_\epsilon}|U|.\end{array}\right. \tag{I.9.57}$$

2 - Dominant part of the dressed propagators. We now compute $\hat{\bar{g}}_{h_{\mathbf{k}},\omega,j}+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega,j}$.

2-1 - $j=0$ case. We first treat the case $j=0$. It follows from (the analogue of) (I.8.10), (I.9.12) and (I.9.15), that

$$\hat{\bar{g}}_{h_{\mathbf{k}},\omega,0}(\mathbf{k})+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega,0}(\mathbf{k})\\ =\left(\begin{array}{cc}\mathbb{1}&\bar M^\dagger_{h_{\mathbf{k}},0}(\mathbf{k})\\0&\mathbb{1}\end{array}\right)\left(\begin{array}{cc}\bar a^{(M)}_{h_{\mathbf{k}},0}&0\\0&\bar a^{(m)}_{h_{\mathbf{k}},0}(\mathbf{k})\end{array}\right)\left(\begin{array}{cc}\mathbb{1}&0\\ \bar M_{h_{\mathbf{k}},0}(\mathbf{k})&\mathbb{1}\end{array}\right)(\mathbb{1}+\sigma'_0(\mathbf{k})) \tag{I.9.58}$$

where $\bar M_{h_{\mathbf{k}},0}$, $\bar a^{(M)}_{h_{\mathbf{k}},0}$ and $\bar a^{(m)}_{h_{\mathbf{k}},0}$ were defined in (I.9.17) and (I.9.16), and the error term σ'_0 can be bounded using (I.9.21) and (I.9.19):

$$|\sigma'_0(\mathbf{k})|\leqslant(\mathrm{const.})\ 2^{h_{\mathbf{k}}-2h_\epsilon}(2^{-h_\epsilon}+|h_{\mathbf{k}}||U|). \tag{I.9.59}$$

2-2 - $j=1$ case. We now consider $j=1$. It follows from (I.9.43) and (I.9.46) that

$$\hat{\bar{g}}_{h_{\mathbf{k}},\omega,0}(\mathbf{k})+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega,0}(\mathbf{k})=\\ \left(\begin{array}{cc}\mathbb{1}&\bar M^\dagger_{h_{\mathbf{k}},1}(\mathbf{k})\\0&\mathbb{1}\end{array}\right)\left(\begin{array}{cc}\bar a^{(M)}_{h_{\mathbf{k}},1}&0\\0&\bar a^{(m)}_{h_{\mathbf{k}},1}(\mathbf{k})\end{array}\right)\left(\begin{array}{cc}\mathbb{1}&0\\ \bar M_{h_{\mathbf{k}},1}(\mathbf{k})&\mathbb{1}\end{array}\right)(\mathbb{1}+\sigma'_1(\mathbf{k})) \tag{I.9.60}$$

where $\bar M_{h_{\mathbf{k}},1}$, $\bar a^{(M)}_{h_{\mathbf{k}},1}$ and $\bar a^{(m)}_{h_{\mathbf{k}},1}$ were defined in (I.9.48) and (I.9.47), and the error term σ'_1 can be bounded using (I.9.51) and (I.9.50):

$$|\sigma'_1(\mathbf{k})|\leqslant(\mathrm{const.})\ \left(2^{h_\epsilon}(1+|h_\epsilon||U|)+2^{h_{\mathbf{k}}-2h_\epsilon}(2^{-h_\epsilon}+|h_{\mathbf{k}}||U|)\right). \tag{I.9.61}$$

2-3 - $j=2,3$ cases. The cases with $j=2,3$ follow from the $2\pi/3$-rotation symmetry (I.2.33) (see (I.9.52)).

3 - Proof of theorem I.1.4 We now conclude the proof of theorem I.1.4. We focus our attention on $j=0,1$ since the cases with $j=2,3$ follow by symmetry. Similarly to section I.8.3, we define

$$B_{h_{\mathbf{k}},j}(\mathbf{k}):=(\mathbb{1}+\sigma'_j(\mathbf{k}))\left(\hat{\bar{g}}_{h_{\mathbf{k}},\omega,j}(\mathbf{k})+\hat{\bar{g}}_{h_{\mathbf{k}}-1,\omega,j}(\mathbf{k})\right)^{-1}$$

(i.e. the inverse of the matrix on the right side of (I.9.58) for $j=0$, (I.9.60) for $j=1$, whose explicit expression is similar to the right side of (I.9.13) and (I.9.44)), and

$$\tilde m_{3,0}:=\tilde m_{\mathfrak{h}_\beta},\quad z_{3,0}:=z_{\mathfrak{h}_2},\quad v_{3,0}:=v_{\mathfrak{h}_2},\quad \tilde v_{3,0}:=\tilde v_{\mathfrak{h}_2},\\ \tilde m_{3,1}:=m^{\xi\xi}_{\mathfrak{h}_\beta,1},\quad \bar m_{3,1}:=m^{\phi\phi}_{\mathfrak{h}_\beta,1},\quad m_{3,1}:=m^{\xi\phi}_{\mathfrak{h}_\beta,1},\quad z_{3,1}:=z^{\phi\phi}_{\mathfrak{h}_\beta,1},\\ \bar v_{3,1}:=v^{\xi\phi}_{\mathfrak{h}_\beta,1},\quad \bar w_{3,1}:=w^{\xi\phi}_{\mathfrak{h}_\beta,1},\quad \tilde v_{3,1}:=v^{\phi\phi}_{\mathfrak{h}_\beta,1},\quad \tilde w_{3,1}:=w^{\phi\phi}_{\mathfrak{h}_\beta,1}$$

and use (I.9.9) and (I.9.34) to bound

$$|\tilde m_{h_{\mathbf{k}}}-\tilde m_{3,0}|+|m^{\xi\xi}_{h_{\mathbf{k}},1}-\tilde m_{3,1}|+|m^{\phi\phi}_{h_{\mathbf{k}},1}-\bar m_{3,1}|\leqslant(\mathrm{const.})\ |U|2^{2h_{\mathbf{k}}-3h_\epsilon},\\ |m^{\xi\phi}_{h_{\mathbf{k}},1}-m_{3,1}|\leqslant(\mathrm{const.})\ |U|2^{2h_{\mathbf{k}}-2h_\epsilon},\\ |z_{h_{\mathbf{k}}}-z_{3,1}|+|z^{\phi\phi}_{h_{\mathbf{k}},1}-z_{3,0}|\leqslant(\mathrm{const.})\ |U|2^{h_{\mathbf{k}}-2h_\epsilon},\\ |v_{h_{\mathbf{k}}}-v_{3,0}|+|v^{\xi\phi}_{h_{\mathbf{k}},1}-v_{3,1}|+|w^{\xi\phi}_{h_{\mathbf{k}},1}-w_{3,1}|\leqslant(\mathrm{const.})\ |U|2^{h_{\mathbf{k}}-h_\epsilon},\\ |\tilde v_{h_{\mathbf{k}}}-\tilde v_{3,0}|+|v^{\phi\phi}_{h_{\mathbf{k}},1}-\tilde v_{3,1}|+|w^{\phi\phi}_{h_{\mathbf{k}},1}-\tilde w_{3,1}|\leqslant(\mathrm{const.})\ |U|2^{h_{\mathbf{k}}-2h_\epsilon}$$

so that

$$\left|(B_{\mathfrak{h}_2,j}(\mathbf{k})-B_{h_\mathbf{k},j}(\mathbf{k}))B^{-1}_{\mathfrak{h}_2,j}(\mathbf{k})\right|\leqslant(\text{const.})\ |U|2^{h_\mathbf{k}-2h_\epsilon}$$

which implies

$$B^{-1}_{h_\mathbf{k}}(\mathbf{k})=B^{-1}_{\mathfrak{h}_2}(\mathbf{k})(\mathbb{1}+O(|U|2^{h_\mathbf{k}-2h_\epsilon})). \tag{I.9.62}$$

We inject (I.9.62) into (I.9.58) and (I.9.60), which we then combine with (I.9.53), (I.9.57), (I.9.59) and (I.9.61), and find an expression for s_2 which is similar to the right side of (I.9.58) and (I.9.60) but with $h_\mathbf{k}$ replaced by $\mathfrak{h}_2$. This concludes the proof of (I.1.24). Furthermore, the estimate (I.1.29) follows from (I.9.11) and (I.9.42) as well as (I.9.31) and (I.9.23), which concludes the proof of theorem I.1.4.

4 - Proof of (I.7.47) and (I.8.37) In order to conclude the proofs of theorems I.1.2 and I.1.3 as well as theorem I.1.1, we still have to bound the sums on the left side of (I.7.47) and of (I.8.37), which we recall were assumed to be true to prove (I.1.14) and (I.1.18) (see sections I.7.3 and I.8.3). It follows from (I.9.5) that

$$\left|\sum_{h'=\mathfrak{h}_\beta}^{\bar{\mathfrak{h}}_2}\hat{W}_2^{(h')}(\mathbf{k})\right|\leqslant(\text{const.})\ 2^{4h_\epsilon}|U|. \tag{I.9.63}$$

This, along with (I.8.42) concludes the proofs of (I.7.47) and (I.8.37), and thus concludes the proof of theorems I.1.2, I.1.3 and I.1.4 as well as theorem I.1.1.

I.10. Conclusion

We considered a tight-binding model of bilayer graphene describing spin-less Fermions hopping on two hexagonal layers in Bernal stacking, in the presence of short range interactions. We assumed that only three hopping parameters are different from zero (those usually called γ_0,γ_1 and γ_3 in the literature), in which case the Fermi surface at half-filling degenerates to a collection of 8 Fermi points. Under a smallness assumption on the interaction strength U and on the transverse hopping ϵ, we proved by rigorous RG methods that the specific ground state energy and correlation functions in the thermodynamic limit are analytic in U, uniformly in ϵ. Our proof requires a detailed analysis of the crossover regimes from one in which the two layers are effectively decoupled, to one where the effective dispersion relation is approximately parabolic around the central Fermi points (and the inter-particle interaction is effectively marginal), to the deep infrared one, where the effective dispersion relation is approximately conical around each Fermi points (and the inter-particle interaction is effectively irrelevant). Such an analysis, in which the influence of the flow of the effective constants in one regime has crucial repercussions in lower regimes, is, to our knowledge, novel.

We expect our proof to be adaptable without substantial efforts to the case where γ_4 and Δ are different from zero, as in (I.1.5), the intra-layer next-to-nearest neighbor hopping γ_0' is $O(\epsilon)$, the chemical potential is $O(\epsilon^3)$, and the temperature is larger than (const.)ϵ^4. At smaller scales, the Fermi set becomes effectively one-dimensional, which thoroughly changes the scaling properties. In particular, the effective inter-particle interaction becomes marginal, again, and its flow tends to grow logarithmically. Perturbative analysis thus breaks down at exponentially small temperatures in ϵ and in U, and it should be possible to rigorously control the system down to such temperatures. Such an analysis could prove difficult, because it requires fine control on the geometry of the Fermi surface, as in [BGM06] and in [FKT04, FKT04b, FKT04c], where the Fermi liquid behavior of a system of interacting electrons was proved, respectively down to exponentially small and zero temperatures, under different physical conditions. We hope to come back to this issue in the future.

Another possible extension would be the study of crossover effects on other physical observables, such as the conductivity, in the spirit of [Ma11]. In addition, it would be interesting to study the case of three-dimensional Coulomb interactions, which is physically interesting in describing *clean* bilayer graphene samples, i.e. where screening effects are supposedly negligible. It may be possible to draw inspiration from the analysis of [GMP10, GMP11b] to construct the ground state, order by order in renormalized perturbation theory. The construction of the theory in the second and third regimes would pave the way to understanding the universality of the conductivity in the deep infrared, beyond the regime studied in [Ma11].

Appendices

I.A1. Computation of the Fermi points

In this appendix, we prove (I.3.2).

Proposition I.A1.1

Given

$$\Omega(k) := 1 + 2e^{-\frac{3}{2}ik_x}\cos\left(\frac{\sqrt{3}}{2}k_y\right),$$

the solutions for $k \in \hat{\Lambda}_\infty$ (see (I.2.4) and following lines for the definition of $\hat{\Lambda}$ and $\hat{\Lambda}_\infty$) of

$$\Omega^2(k) - \gamma_1\gamma_3\Omega^*(k)e^{-3ik_x} = 0 \tag{I.A1.1}$$

with

$$0 < \gamma_1\gamma_3 < 2$$

are

$$\left\{\begin{array}{l}
p_{F,0}^\omega := \left(\frac{2\pi}{3}, \omega\frac{2\pi}{3\sqrt{3}}\right)\\
p_{F,1}^\omega := \left(\frac{2\pi}{3}, \omega\frac{2}{\sqrt{3}}\arccos\left(\frac{1-\gamma_1\gamma_3}{2}\right)\right)\\
p_{F,2}^\omega := \left(\frac{2\pi}{3} + \frac{2}{3}\arccos\left(\frac{\sqrt{1+\gamma_1\gamma_3}(2-\gamma_1\gamma_3)}{2}\right), \omega\frac{2}{\sqrt{3}}\arccos\left(\frac{1+\gamma_1\gamma_3}{2}\right)\right)\\
p_{F,3}^\omega := \left(\frac{2\pi}{3} - \frac{2}{3}\arccos\left(\frac{\sqrt{1+\gamma_1\gamma_3}(2-\gamma_1\gamma_3)}{2}\right), \omega\frac{2}{\sqrt{3}}\arccos\left(\frac{1+\gamma_1\gamma_3}{2}\right)\right)
\end{array}\right. \tag{I.A1.2}$$

for $\omega \in \{+, -\}$.

Proof: We define

$$C := \cos\left(\frac{3}{2}k_x\right), \quad S := \sin\left(\frac{3}{2}k_x\right), \quad Y := \cos\left(\frac{\sqrt{3}}{2}k_y\right), \quad G := \gamma_1\gamma_3$$

in terms of which (I.A1.1) becomes

$$\left\{\begin{array}{l}
4(2C^2-1)Y^2 + 2C(2-G)Y + 1 - G(2C^2-1) = 0\\
-2S(C(4Y^2-G) + Y(2-G)) = 0.
\end{array}\right. \tag{I.A1.3}$$

1 - If $S = \sin((3/2)k_x) = 0$, then $k_x \in \{0, 2\pi/3\}$. Furthermore, since $k \in \hat{\Lambda}_\infty$, if $k_x = 0$ then $k_y = 0$, which is not a solution of (I.A1.1) as long as $G < 3$. Therefore $k_x = 2\pi/3$, so that $C = -1$, and Y solves

$$4Y^2 - 2(2-G)Y + 1 - G = 0$$

so that

$$Y = \frac{2 - G \pm G}{4}$$

and therefore

$$k_y = \pm\frac{2\pi}{3\sqrt{3}} \quad \text{or} \quad k_y = \pm\frac{2}{\sqrt{3}}\arccos\left(\frac{1-G}{2}\right).$$

2 - If $S \neq 0$, then

$$C(4Y^2 - G) = -Y(2 - G)$$

so that the first of (I.A1.3) becomes $4Y^2 = 1 + G$, which implies

$$Y = \pm\frac{\sqrt{1+G}}{2}, \quad C = \mp\frac{\sqrt{1+G}(2-G)}{2}$$

so that

$$k_x = \frac{2\pi}{3} + \frac{2}{3}\arccos\left(\frac{\sqrt{1+G}(2-G)}{2}\right), \quad k_y = \pm\frac{2}{\sqrt{3}}\arccos\left(\frac{\sqrt{1+G}}{2}\right)$$

or

$$k_x = \frac{2\pi}{3} - \frac{2}{3}\arccos\left(\frac{\sqrt{1+G}(2-G)}{2}\right), \quad k_y = \pm\frac{2}{\sqrt{3}}\arccos\left(\frac{\sqrt{1+G}}{2}\right).$$

□

I.A2. 4×4 matrix inversions

In this appendix, we give the explicit expression of the determinant and the inverse of matrices that have the form of the inverse free propagator. The result is collected in the following proposition and corollary, whose proofs are straightforward, brute force, computations.

Proposition I.A2.1

Given a matrix

$$A = \begin{pmatrix} i\mathfrak{x} & \mathfrak{a}^* & 0 & \mathfrak{b}^* \\ \mathfrak{a} & i\mathfrak{x} & \mathfrak{b} & 0 \\ 0 & \mathfrak{b}^* & i\mathfrak{z} & \mathfrak{c} \\ \mathfrak{b} & 0 & \mathfrak{c}^* & i\mathfrak{z} \end{pmatrix} \tag{I.A2.1}$$

with $(\mathfrak{x}, \mathfrak{z}) \in \mathbb{R}^2$ and $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \in \mathbb{C}^3$. We have

$$\det A = (|\mathfrak{b}|^2 + \mathfrak{z}\mathfrak{x})^2 + |\mathfrak{a}|^2\mathfrak{z}^2 + |\mathfrak{c}|^2(\mathfrak{x}^2 + |\mathfrak{a}|^2) - 2\mathcal{R}e(\mathfrak{a}^*\mathfrak{b}^2\mathfrak{c}) \tag{I.A2.2}$$

and

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} \mathfrak{g}_{a,a} & \mathfrak{g}_{a,\tilde{b}} & \mathfrak{g}_{a,\tilde{a}} & \mathfrak{g}_{a,b} \\ \mathfrak{g}^+_{a,\tilde{b}} & \mathfrak{g}_{a,a} & \mathfrak{g}^+_{a,b} & \mathfrak{g}^+_{a,\tilde{a}} \\ \mathfrak{g}^+_{a,\tilde{a}} & \mathfrak{g}_{a,b} & \mathfrak{g}_{\tilde{a},\tilde{a}} & \mathfrak{g}_{\tilde{a},b} \\ \mathfrak{g}^+_{a,b} & \mathfrak{g}_{a,\tilde{a}} & \mathfrak{g}^+_{\tilde{a},b} & \mathfrak{g}_{\tilde{a},\tilde{a}} \end{pmatrix}$$

with

$$\left\{\begin{array}{l} \mathfrak{g}_{a,a} = -i\mathfrak{z}|\mathfrak{b}|^2 - i\mathfrak{x}(\mathfrak{z}^2 + |\mathfrak{c}|^2) \\ \\ \mathfrak{g}_{a,\tilde{b}} = \mathfrak{z}^2\mathfrak{a}^* - \mathfrak{c}^*((\mathfrak{b}^*)^2 - \mathfrak{a}^*\mathfrak{c}) \\ \\ \mathfrak{g}_{a,\tilde{a}} = i\mathfrak{z}\mathfrak{a}^*\mathfrak{b} + i\mathfrak{x}\mathfrak{b}^*\mathfrak{c}^* \\ \\ \mathfrak{g}_{a,b} = \mathfrak{b}((\mathfrak{b}^*)^2 - \mathfrak{a}^*\mathfrak{c}) + \mathfrak{z}\mathfrak{x}\mathfrak{b}^* \\ \\ \mathfrak{g}_{\tilde{a},b} = -\mathfrak{a}((\mathfrak{b}^*)^2 - \mathfrak{a}^*\mathfrak{c}) + \mathfrak{x}^2\mathfrak{c} \\ \\ \mathfrak{g}_{\tilde{a},\tilde{a}} = -i\mathfrak{z}|\mathfrak{a}|^2 - i\mathfrak{x}(\mathfrak{x}\mathfrak{z} + |\mathfrak{b}|^2). \end{array}\right. \tag{I.A2.3}$$

and given a function $\mathfrak{g}(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{x},\mathfrak{z})$,

$$\mathfrak{g}^+(\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{x},\mathfrak{z}) := \mathfrak{g}^*(\mathfrak{a},\mathfrak{b},\mathfrak{c},-\mathfrak{x},-\mathfrak{z}).$$

Corollary I.A2.2

If $\mathfrak{z}=\mathfrak{x}=0$, then

$$\det A = \left|\mathfrak{b}^2 - \mathfrak{a}\mathfrak{c}^*\right|^2 \geqslant 0. \tag{I.A2.4}$$

In particular, A is invertible if and only if $\mathfrak{b}^2 \neq \mathfrak{a}\mathfrak{c}^*$.

I.A3. Block diagonalization

In this appendix, we give the formula for block-diagonalizing 4×4 matrices, which is useful to separate the massive block from the massless one. The result is collected in the following proposition, whose proof is straightforward.

Proposition I.A3.1

Given a 4×4 complex matrix B, which can be written in block-form as

$$B = \left(\begin{array}{cc} B^{\xi\xi} & B^{\xi\phi} \\ B^{\xi\phi} & B^{\phi\phi} \end{array}\right) \tag{I.A3.1}$$

in which $B^{\xi\xi}$, $B^{\xi\phi}$ and $B^{\phi\phi}$ are 2×2 complex matrices and $B^{\xi\xi}$ and $B^{\phi\phi}$ are invertible, we have

$$\left(\begin{array}{cc} \mathbb{1} & 0 \\ -B^{\xi\phi}(B^{\xi\xi})^{-1} & \mathbb{1} \end{array}\right) B \left(\begin{array}{cc} \mathbb{1} & -(B^{\xi\xi})^{-1}B^{\xi\phi} \\ 0 & \mathbb{1} \end{array}\right) = \left(\begin{array}{cc} B^{\xi\xi} & 0 \\ 0 & B^{\phi\phi} - B^{\xi\phi}(B^{\xi\xi})^{-1}B^{\xi\phi} \end{array}\right). \tag{I.A3.2}$$

If $B^{\phi\phi} - B^{\xi\phi}(B^{\xi\xi})^{-1}B^{\xi\phi}$ is invertible then

$$(B^{\phi\phi} - B^{\xi\phi}(B^{\xi\xi})^{-1}B^{\xi\phi})^{-1}$$

is the lower-right block of B^{-1}.

I.A4. Bound of the propagator in the II-III intermediate regime

In this appendix, we prove the assertion between (I.3.38) and (I.3.39), that is that the determinant of the inverse propagator is bounded below by (const.) ϵ^8 in the intermediate regime between the second and third regimes. Using the symmetry under $k_x \mapsto -k_x$ and under $2\pi/3$ rotations, we restrict our discussion to $\omega = +$ and $k_y - p^+_{F,0,y} > 0$. In a coordinate frame centered at $p^+_{F,0}$, we denote with some abuse of notation $\mathbf{k}'_{+,0} = (k_0, k_x, k_y)$ and $p^+_{F,1} = (0, D\bar\epsilon^2)$, where $\bar\epsilon\frac{3}{2}\gamma_3$ and $D = \frac{8}{27}\frac{\gamma_1}{\gamma_3}(1+O(\epsilon^2))$ (see (I.3.3)). Note that $D > 0$ is uniformly bounded away from 0 for $\bar\epsilon$ small (recall that $\gamma_1 = \epsilon$ and $\gamma_3 = 0.33\epsilon$). In these coordinates, we restrict to $k_y > 0$, and the first and third conditions in (I.3.37) read

$$\sqrt{k_0^2 + \bar\epsilon^2(k_x^2+k_y^2)} \geqslant \bar\kappa\bar\epsilon^3, \qquad \sqrt{k_0^2 + \bar\epsilon^2(9k_x^2 + (k_y - D\bar\epsilon^2)^2)} \geqslant \bar\kappa\bar\epsilon^3, \tag{I.A4.1}$$

where $\bar\kappa\bar\kappa_2(\frac{2\epsilon}{3\gamma_3})^3$. The second condition in (I.3.37) implies that $(k_x^2+k_y^2)\leqslant(\text{const.})\ \epsilon^2$, in which case the desired bound (that is, $|\det\hat A|\geqslant(\text{const.})\ \epsilon^8$, with $\det\hat A$ as in (I.3.38)) reads

$$\epsilon^2k_0^2+\frac{81}{16}|(ik_x+k_y)^2-D\bar\epsilon^2(-ik_x+k_y)|^2\geqslant(\text{const.})\ \epsilon^8. \tag{I.A4.2}$$

Therefore, the desired estimate follows from the following Proposition, which is proved below.

Proposition I.A4.1

For all $D,\epsilon>0$, if $(k_0,k_x,k_y)\in\mathbb{R}^3$ satisfies

$$k_y>0,\quad\sqrt{k_0^2+\bar\epsilon^2(k_x^2+k_y^2)}>\bar\kappa\bar\epsilon^3,\quad\sqrt{k_0^2+\bar\epsilon^2(9k_x^2+(k_y-D\bar\epsilon^2)^2)}>\bar\kappa\bar\epsilon^3$$

for some constant $\bar\kappa>0$, then, for all $\alpha>0$, we have

$$\bar\epsilon^2k_0^2+\alpha\left|(ik_x+k_y)^2-D\bar\epsilon^2(-ik_x+k_y)\right|^2>C\bar\epsilon^8, \tag{I.A4.3}$$

where

$$C:=\min\left(1,\frac{\alpha D^2}{12},\frac{\alpha(473-3\sqrt{105})\bar\kappa^2}{288}\right)\frac{\bar\kappa^2}{4}.$$

Proof: We rewrite the left side of (I.A4.3) as

$$l:=\bar\epsilon^2k_0^2+\alpha\left(-k_x^2+k_y^2-D\bar\epsilon^2k_y\right)^2+\alpha k_x^2\left(2k_y+D\bar\epsilon^2\right)^2.$$

If $|k_0|>\bar\kappa\bar\epsilon^3/2$, then $l>\bar\kappa^2\bar\epsilon^8/4$ from which (I.A4.3) follows. If $|k_0|\leqslant\bar\kappa\bar\epsilon^3/2$, then

$$k_x^2+k_y^2>\frac34\bar\kappa^2\bar\epsilon^4,\quad 9k_x^2+(k_y-D\bar\epsilon^2)^2>\frac34\bar\kappa^2\bar\epsilon^4.$$

If $|k_x|>(1/4\sqrt3)\bar\kappa\bar\epsilon^2$, then, using the fact that $k_y>0$, $l>\alpha(1/48)D^2\bar\kappa^2\bar\epsilon^8$ from which (I.A4.3) follows. If $|k_x|\leqslant(1/4\sqrt3)\bar\kappa\bar\epsilon^2$, then

$$k_y>\sqrt{\frac{35}{48}}\bar\kappa\bar\epsilon^2,\quad |k_y-D\bar\epsilon^2|>\frac34\bar\kappa\bar\epsilon^2$$

so that

$$\left|k_y(k_y-D\bar\epsilon^2)\right|-k_x^2>\frac{3\sqrt{105}-1}{48}\bar\kappa^2\bar\epsilon^4$$

and $l>\alpha((3\sqrt{105}-1)^2/2304)\bar\kappa^4\bar\epsilon^8$ from which (I.A4.3) follows. □

I.A5. Symmetries

In this appendix, we prove that the symmetries listed in (I.2.32) through (I.2.38) leave h_0 and $\mathcal V$ invariant. We first recall

$$h_0=-\frac{1}{\chi_0(2^{-M}|k_0|)\beta|\Lambda|}\sum_{\mathbf k\in\mathcal B^*_{\beta,L}}\begin{pmatrix}\hat\xi^+_{\mathbf k}&\hat\phi^+_{\mathbf k}\end{pmatrix}\begin{pmatrix}A^{\xi\xi}(\mathbf k)&A^{\xi\phi}(\mathbf k)\\A^{\phi\xi}(\mathbf k)&A^{\phi\phi}(\mathbf k)\end{pmatrix}\begin{pmatrix}\hat\xi^-_{\mathbf k}\\\hat\phi^-_{\mathbf k}\end{pmatrix} \tag{I.A5.1}$$

with
$$A^{\xi\xi}(\mathbf{k}) := \begin{pmatrix} ik_0 & \gamma_1 \\ \gamma_1 & ik_0 \end{pmatrix}, \quad A^{\xi\phi}(\mathbf{k}) \equiv A^{\phi\xi}(\mathbf{k}) := \begin{pmatrix} 0 & \Omega^*(k) \\ \Omega(k) & 0 \end{pmatrix},$$

$$A^{\phi\phi}(\mathbf{k}) := \begin{pmatrix} ik_0 & \gamma_3\Omega(k)e^{3ik_x} \\ \gamma_3\Omega^*(k)e^{-3ik_x} & ik_0 \end{pmatrix}$$

and
$$\mathcal{V}(\psi) = \frac{U}{(\beta|\Lambda|)^3}\sum_{(\alpha,\alpha')}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3} \hat v_{\alpha,\alpha'}(k_1-k_2)\hat\psi^+_{\mathbf{k}_1,\alpha}\hat\psi^-_{\mathbf{k}_2,\alpha}\hat\psi^+_{\mathbf{k}_3,\alpha'}\hat\psi^-_{\mathbf{k}_1-\mathbf{k}_2+\mathbf{k}_3,\alpha'} \tag{I.A5.2}$$

where
$$\hat v_{\alpha,\alpha'}(k) := \sum_{x\in\Lambda} e^{ik\cdot x}v(x+d_\alpha-d_{\alpha'}).$$

1 - Global $U(1)$. Follows immediately from the fact that there are as many ψ^+ as ψ^- in h_0 and $\mathcal{V}$. □

2 - $2\pi/3$ rotation. We have

$$\Omega(e^{i\frac{2\pi}{3}\sigma_2}k) = e^{il_2\cdot k}\Omega(k), \quad e^{3i(e^{i\frac{2\pi}{3}\sigma_2}k)|_x} = e^{-3il_2\cdot k}e^{3ik_x}$$

so that $\mathcal{T}^\dagger_{\mathbf{k}}A^{\phi\phi}(T^{-1}\mathbf{k})\mathcal{T}_{\mathbf{k}} = A^{\phi\phi}(\mathbf{k})$ and $A^{\xi\phi}(T^{-1}\mathbf{k})\mathcal{T}_{\mathbf{k}} = A^{\xi\phi}(\mathbf{k})$. This, together with $A^{\xi\xi}(T^{-1}\mathbf{k}) = A^{\xi\xi}(\mathbf{k})$, implies that h_0 is invariant under (I.2.33).

Furthermore, interpreting $e^{-i\frac{2\pi}{3}\sigma_2}$ as a rotation in $\mathbb{R}^3$ around the z axis,

$$e^{-i\frac{2\pi}{3}\sigma_2}d_a = d_a, \quad e^{-i\frac{2\pi}{3}\sigma_2}d_{\tilde b} = d_{\tilde b}, \quad e^{-i\frac{2\pi}{3}\sigma_2}d_{\tilde a} = l_2 + d_{\tilde a}, \quad e^{-i\frac{2\pi}{3}\sigma_2}d_b = -l_2 + d_b,$$

which implies, denoting by $\hat v(k)$ the matrix with elements $\hat v_{\alpha,\alpha'}(k)$,

$$\hat v(e^{i\frac{2\pi}{3}\sigma_2}k) = \begin{pmatrix} \hat v_{a,a}(k) & \hat v_{a,\tilde b}(k) & e^{ik\cdot l_2}\hat v_{a,\tilde a}(k) & e^{-ik\cdot l_2}\hat v_{a,b}(k) \\ \hat v_{\tilde b,a}(k) & \hat v_{\tilde b,\tilde b}(k) & e^{ik\cdot l_2}\hat v_{\tilde b,\tilde a}(k) & e^{-ik\cdot l_2}\hat v_{\tilde b,b}(k) \\ e^{-ik\cdot l_2}\hat v_{\tilde a,a}(k) & e^{-ik\cdot l_2}\hat v_{\tilde a,\tilde b}(k) & \hat v_{\tilde a,\tilde a} & e^{-2ik\cdot l_2}\hat v_{\tilde a,b}(k) \\ e^{ik\cdot l_2}\hat v_{b,a}(k) & e^{ik\cdot l_2}\hat v_{b,\tilde b}(k) & e^{2ik\cdot l_2}\hat v_{b,\tilde a}(k) & \hat v_{b,b}(k) \end{pmatrix}$$

furthermore
$$\begin{pmatrix} \hat\xi^+_{\mathbf{k}_1,a}\hat\xi^-_{\mathbf{k}_2,a} \\ \hat\xi^+_{\mathbf{k}_1,\tilde b}\hat\xi^-_{\mathbf{k}_2,\tilde b} \\ (\hat\phi^+_{\mathbf{k}_1}\mathcal{T}^\dagger_{\mathbf{k}_1})_{\tilde a}(\mathcal{T}_{\mathbf{k}_2}\hat\phi^-_{\mathbf{k}_2})_{\tilde a} \\ (\hat\phi^+_{\mathbf{k}_1}\mathcal{T}^\dagger_{\mathbf{k}_1})_b(\mathcal{T}_{\mathbf{k}_2}\hat\phi^-_{\mathbf{k}_2})_b \end{pmatrix} = \begin{pmatrix} \hat\psi^+_{\mathbf{k}_1,a}\hat\psi^-_{\mathbf{k}_1,a} \\ \hat\psi^+_{\mathbf{k}_1,\tilde b}\hat\psi^-_{\mathbf{k}_1,\tilde b} \\ e^{il_2(k_1-k_2)}\hat\psi^+_{\mathbf{k}_1,\tilde a}\hat\psi^-_{\mathbf{k}_1,\tilde a} \\ e^{-il_2(k_1-k_2)}\hat\psi^+_{\mathbf{k}_1,b}\hat\psi^-_{\mathbf{k}_1,b} \end{pmatrix}$$

from which one easily concludes that $\mathcal{V}$ is invariant under (I.2.33). □

3 - Complex conjugation. Follows immediately from $\Omega(-k) = \Omega^*(k)$ and $v(-k) = v^*(k)$. □

4 - Vertical reflection. Follows immediately from $\Omega(R_vk) = \Omega(k)$ and $v(R_vk) = v(k)$ (since the second component of d_α is 0). □

5 - Horizontal reflection. We have $\Omega(R_hk) = \Omega^*(k)$, $\sigma_1A^{\xi\xi}(\mathbf{k})\sigma_1 = A^{\xi\xi}(\mathbf{k})$,

$$\sigma_1A^{\xi\phi}(\mathbf{k})\sigma_1 = \begin{pmatrix} 0 & \Omega(k) \\ \Omega^*(k) & 0 \end{pmatrix}, \quad \sigma_1A^{\phi\phi}(\mathbf{k})\sigma_1 = \begin{pmatrix} ik_0 & \gamma_3\Omega^*(k)e^{-3ik_x} \\ \gamma_3\Omega(k)e^{3ik_x} & ik_0 \end{pmatrix}$$

from which the invariance of h_0 follows immediately. Furthermore

$$v_{\alpha,\alpha'}(R_h k) = v_{\pi_h(\alpha),\pi_h(\alpha')}(k)$$

where π_h is the permutation that exchanges a with $\tilde{b}$ and $\tilde{a}$ with b, from which the invariance of $\mathcal{V}$ follows immediately. $\square$

6 - Parity. We have $\Omega(Pk) = \Omega^*(k)$ so that $[A^{\xi\phi}(P\mathbf{k})]^T = A^{\xi\phi}(\mathbf{k})$, $[A^{\phi\phi}(P\mathbf{k})]^T = A^{\phi\phi}(\mathbf{k})$, $[A^{\xi\xi}(P\mathbf{k})]^T = A^{\xi\xi}(\mathbf{k})$. Therefore h_0 is mapped to

$$h_0 \longmapsto -\frac{1}{\chi_0(2^{-M}|k_0|)\beta|\Lambda|} \sum_{\mathbf{k}\in\mathcal{B}^*_{\beta,L}} \begin{pmatrix} \hat{\xi}^-_{\mathbf{k}} & \hat{\phi}^-_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} A^{\xi\xi}(\mathbf{k}) & A^{\xi\phi}(\mathbf{k}) \\ A^{\phi\xi}(\mathbf{k}) & A^{\phi\phi}(\mathbf{k}) \end{pmatrix}^T \begin{pmatrix} \hat{\xi}^+_{\mathbf{k}} \\ \hat{\phi}^+_{\mathbf{k}} \end{pmatrix}$$

which is equal to h_0 since exchanging $\hat{\psi}^-$ and $\hat{\psi}^+$ adds a minus sign. The invariance of $\mathcal{V}$ follows from the remark that under parity $\hat{\psi}^+_{\mathbf{k}_1,\alpha}\hat{\psi}^-_{\mathbf{k}_2,\alpha} \mapsto \hat{\psi}^+_{P\mathbf{k}_2,\alpha}\hat{\psi}^-_{P\mathbf{k}_1,\alpha}$, and $\hat{v}(k_1-k_2) = \hat{v}(P(k_2-k_1))$. $\square$

7 - Time inversion. We have

$$\sigma_3 A^{\xi\xi}(I\mathbf{k})\sigma_3 = -A^{\xi\xi}(\mathbf{k}), \quad \sigma_3 A^{\xi\phi}(I\mathbf{k})\sigma_3 = -A^{\xi\phi}(\mathbf{k}),$$
$$\sigma_3 A^{\phi\phi}(I\mathbf{k})\sigma_3 = -A^{\phi\phi}(\mathbf{k})$$

from which the invariance of h_0 follows immediately. The invariance of $\mathcal{V}$ is trivial. $\square$

I.A6. Constraints due to the symmetries

In this appendix we discuss some of the consequences of the symmetries listed in section I.2.3 on $\hat{W}_2^{(h)}(\mathbf{k})$ and its derivatives.

We recall the definitions of the symmetry transformations from section I.2.3:

$$T\mathbf{k} := (k_0, e^{-i\frac{2\pi}{3}\sigma_2}k), \quad R_v\mathbf{k} := (k_0, k_x, -k_y), \quad R_h\mathbf{k} := (k_0, -k_x, k_y),$$
$$P\mathbf{k} := (k_0, -k_x, -k_y), \quad I\mathbf{k} := (-k_0, k_x, k_y). \tag{I.A6.1}$$

Furthermore, given a 4×4 matrix $\mathbf{M}$ whose components are indexed by $\{a, \tilde{b}, \tilde{a}, b\}$, we denote the sub-matrix with components in $\{a,\tilde{b}\}^2$ by $\mathbf{M}^{\xi\xi}$, that with $\{\tilde{a}, b\}^2$ by $\mathbf{M}^{\phi\phi}$, with $\{a,\tilde{b}\}\times\{\tilde{a},b\}$ by $\mathbf{M}^{\xi\phi}$ and with $\{\tilde{a},b\}\times\{a,\tilde{b}\}$ by $\mathbf{M}^{\phi\xi}$. In addition, given a complex matrix M, we denote its component-wise complex conjugate by M^* (which is not to be confused with its adjoint $M^\dagger$).

Proposition I.A6.1

Given a 2×2 complex matrix $M(\mathbf{k})$ parametrized by $\mathbf{k}\in\mathcal{B}_\infty$ (we recall that $\mathcal{B}_\infty$ was defined above the statement of theorem I.1.1 in section I.1.3) and a pair of points $(\mathbf{p}^+_F, \mathbf{p}^-_F) \in \mathcal{B}^2_\infty$, if $\forall \mathbf{k}\in\mathcal{B}_\infty$

$$M(\mathbf{k}) = M(-\mathbf{k})^* = M(R_v\mathbf{k}) = \sigma_1 M(R_h\mathbf{k})\sigma_1 = -\sigma_3 M(I\mathbf{k})\sigma_3 \tag{I.A6.2}$$

and

$$\mathbf{p}^\omega_F = -\mathbf{p}^{-\omega}_F = R_v\mathbf{p}^{-\omega}_F = R_h\mathbf{p}^\omega_F = I\mathbf{p}^\omega_F \tag{I.A6.3}$$

for $\omega\in\{-,+\}$, then $\exists(\mu,\zeta,\nu,\varpi)\in\mathbb{R}^4$ such that

$$M(\mathbf{p}^\omega_F) = \mu\sigma_1, \quad \partial_{k_0}M(\mathbf{p}^\omega_F) = i\zeta\mathbb{1},$$
$$\partial_{k_x}M(\mathbf{p}^\omega_F) = \nu\sigma_2, \quad \partial_{k_y}M(\mathbf{p}^\omega_F) = \omega\varpi\sigma_1. \tag{I.A6.4}$$

Proof:

1 - We first prove that $M(\mathbf{p}_F^\omega)=\mu\sigma_1$. We write

$$M(\mathbf{p}_F^\omega)=:t\mathbb{1}+x\sigma_1+y\sigma_2+z\sigma_3$$

where $(t,x,y,z)\in\mathbb{C}^4$. We have

$$M(\mathbf{p}_F^\omega)=M(\mathbf{p}_F^{-\omega})^*=M(\mathbf{p}_F^{-\omega})=\sigma_1M(\mathbf{p}_F^\omega)\sigma_1=-\sigma_3M(\mathbf{p}_F^\omega)\sigma_3.$$

Therefore (t,x,y,z) are independent of ω, $t=y=z=0$ and $x\in\mathbb{R}$.

2 - We now study $\partial_{k_0}M$ which we write as

$$\partial_{k_0}M(\mathbf{p}_F^\omega)=:t_0\mathbb{1}+x_0\sigma_1+y_0\sigma_2+z_0\sigma_3.$$

We have

$$\partial_{k_0}M(\mathbf{p}_F^\omega)=-(\partial_{k_0}M(\mathbf{p}_F^{-\omega}))^*=\partial_{k_0}M(\mathbf{p}_F^{-\omega})=\sigma_1\partial_{k_0}M(\mathbf{p}_F^\omega)\sigma_1=\sigma_3\partial_{k_0}M(\mathbf{p}_F^\omega)\sigma_3.$$

Therefore (t_0,x_0,y_0,z_0) are independent of ω, $x_0=y_0=z_0=0$ and $t_0\in i\mathbb{R}$.

3 - We now turn our attention to $\partial_{k_x}M$:

$$\partial_{k_x}M(\mathbf{p}_F^\omega)=:t_1\mathbb{1}+x_1\sigma_1+y_1\sigma_2+z_1\sigma_3.$$

We have

$$\partial_{k_x}M(\mathbf{p}_F^\omega)=-(\partial_{k_x}M(\mathbf{p}_F^{-\omega}))^*=\partial_{k_x}M(\mathbf{p}_F^{-\omega})=-\sigma_1\partial_{k_x}M(\mathbf{p}_F^\omega)\sigma_1=-\sigma_3\partial_{k_x}M(\mathbf{p}_F^\omega)\sigma_3.$$

Therefore (t_1,x_1,y_1,z_1) are independent of ω, $t_1=x_1=z_1=0$ and $y_1\in\mathbb{R}$.

4 - Finally, we consider $\partial_{k_y}M$:

$$\partial_{k_y}M(\mathbf{p}_F^\omega)=:t_2^{(\omega)}\mathbb{1}+x_2^{(\omega)}\sigma_1+y_2^{(\omega)}\sigma_2+z_2^{(\omega)}\sigma_3.$$

We have

$$\partial_{k_y}M(\mathbf{p}_F^\omega)=-(\partial_{k_y}M(\mathbf{p}_F^{-\omega}))^*=-\partial_{k_y}M(\mathbf{p}_F^{-\omega})=\sigma_1\partial_{k_y}M(\mathbf{p}_F^\omega)\sigma_1=-\sigma_3\partial_{k_y}M(\mathbf{p}_F^\omega)\sigma_3.$$

Therefore $t_2^{(\omega)}=y_2^{(\omega)}=z_2^{(\omega)}=0$, $x_2^{(\omega)}=-x_2^{(-\omega)}\in\mathbb{R}$. □

Proposition I.A6.2

Given a 4×4 complex matrix $\mathbf{M}(\mathbf{k})$ parametrized by $\mathbf{k}\in\mathcal{B}_\infty$ and two points $(\mathbf{p}_F^+,\mathbf{p}_F^-)\in\mathcal{B}_\infty^2$, if $\forall(f,f')\in\{\xi,\phi\}^2$ and $\forall\omega\in\{-,+\}$,

$$\mathbf{M}^{ff'}(\mathbf{p}_F^\omega)=\mu^{ff'}\sigma_1,\quad \partial_{k_0}\mathbf{M}^{ff'}(\mathbf{p}_F^\omega)=i\zeta^{ff'}\mathbb{1},$$
$$\partial_{k_x}\mathbf{M}^{ff'}(\mathbf{p}_F^\omega)=\nu^{ff'}\sigma_2,\quad \partial_{k_y}\mathbf{M}^{ff'}(\mathbf{p}_F^\omega)=\omega\varpi^{ff'}\sigma_1 \tag{I.A6.5}$$

with $(\mu^{ff'},\zeta^{ff'},\nu^{ff'},\varpi^{ff'})\in\mathbb{R}^4$ independent of ω, and $\forall\mathbf{k}\in\mathcal{B}_\infty$

$$\mathbf{M}(\mathbf{k})=\mathbf{M}^T(P\mathbf{k}) \tag{I.A6.6}$$

and

$$\mathbf{p}_F^\omega=P\mathbf{p}_F^{-\omega} \tag{I.A6.7}$$

then

$$\mu^{\phi\xi} = \mu^{\xi\phi}, \quad \zeta^{\phi\xi} = \zeta^{\xi\phi}, \quad \nu^{\phi\xi} = \nu^{\xi\phi}, \quad \varpi^{\phi\xi} = \varpi^{\xi\phi}. \tag{I.A6.8}$$

Furthermore, if $\mathbf{p}_F^\omega = (0, \frac{2\pi}{3}, \omega\frac{2\pi}{3\sqrt{3}})$ and (recalling that $\mathcal{T}_\mathbf{k} = e^{-i(l_2\cdot k)\sigma_3}$, with $l_2 = (3/2, -\sqrt{3}/2)$)

$$\mathbf{M}(\mathbf{k}) = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathcal{T}_\mathbf{k}^\dagger \end{pmatrix} \mathbf{M}(T^{-1}\mathbf{k}) \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathcal{T}_\mathbf{k} \end{pmatrix} \tag{I.A6.9}$$

then

$$\begin{gathered} \nu^{\phi\phi} = -\varpi^{\phi\phi}, \quad \nu^{\xi\phi} = \varpi^{\xi\phi}, \quad \nu^{\phi\xi} = \varpi^{\phi\xi}, \quad \nu^{\xi\xi} = \varpi^{\xi\xi} = 0, \\ \mu^{\phi\phi} = \mu^{\xi\phi} = \mu^{\phi\xi} = 0, \quad \zeta^{\phi\xi} = \zeta^{\xi\phi} = 0. \end{gathered} \tag{I.A6.10}$$

Proof: (I.A6.8) is straightforward, so we immediately turn to the proof of (I.A6.10).

1 - We first focus on $\mathbf{M}^{\phi\phi}$ which satisfies

$$\mathbf{M}^{\phi\phi}(\mathbf{k}) = \mathcal{T}_\mathbf{k}^\dagger \mathbf{M}^{\phi\phi}(T^{-1}\mathbf{k})\mathcal{T}_\mathbf{k}. \tag{I.A6.11}$$

Evaluating this formula at $\mathbf{k} = \mathbf{p}_F^\omega$, recalling that $\mathbf{M}^{\phi\phi}(\mathbf{p}_F^\omega) = \mu^{\phi\phi}\sigma_1$, and noting that $\mathcal{T}_{\mathbf{p}_F^\omega} = -\frac{1}{2}\mathbb{1} - i\omega\frac{\sqrt{3}}{2}\sigma_3$, we obtain $\mu^{\phi\phi} = 0$. Therefore, deriving (I.A6.11) with respect to k_i, $i = 1, 2$, and evaluating at $\mathbf{p}_F^\omega$, we get:

$$\partial_{k_i}\mathbf{M}^{\phi\phi}(\mathbf{p}_F^\omega) = \sum_{j=1}^{2} T_{i,j}\mathcal{T}_{\mathbf{p}_F^\omega}^\dagger \partial_{k_j}\mathbf{M}^{\phi\phi}(\mathbf{p}_F^\omega)\mathcal{T}_{\mathbf{p}_F^\omega}$$

with

$$T = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}.$$

Furthermore, recalling that $\partial_{k_x}\mathbf{M}^{\phi\phi} = \nu^{\phi\phi}\sigma_2$ and $\partial_{k_y}\mathbf{M}^{\phi\phi} = \omega\varpi^{\phi\phi}\sigma_1$,

$$\mathcal{T}_{\mathbf{p}_F^\omega}^\dagger \partial_{k_x}\mathbf{M}^{\phi\phi}\mathcal{T}_{\mathbf{p}_F^\omega} = \nu^{\phi\phi}\Big(-\frac{1}{2}\sigma_2 - \omega\frac{\sqrt{3}}{2}\sigma_1\Big), \quad \mathcal{T}_{\mathbf{p}_F^\omega}^\dagger \partial_{k_y}\mathbf{M}^{\phi\phi}\mathcal{T}_{\mathbf{p}_F^\omega} = \omega\varpi^{\phi\phi}\Big(-\frac{1}{2}\sigma_1 + \omega\frac{\sqrt{3}}{2}\sigma_2\Big),$$

which implies

$$\begin{pmatrix} \nu^{\phi\phi}\sigma_2 \\ \omega\varpi^{\phi\phi}\sigma_1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \nu^{\phi\phi} - 3\varpi^{\phi\phi} & \omega\sqrt{3}(\nu^{\phi\phi} + \varpi^{\phi\phi}) \\ -\sqrt{3}(\nu^{\phi\phi} + \varpi^{\phi\phi}) & \omega(\varpi^{\phi\phi} - 3\nu^{\phi\phi}) \end{pmatrix}\begin{pmatrix} \sigma_2 \\ \sigma_1 \end{pmatrix}$$

so $\nu^{\phi\phi} = -\varpi^{\phi\phi}$.

2 - We now study $\mathbf{M}^{\phi\xi}$ which satisfies

$$\mathbf{M}^{\phi\xi}(\mathbf{k}) = \mathcal{T}_\mathbf{k}^\dagger \mathbf{M}^{\phi\xi}(T^{-1}\mathbf{k}).$$

Evaluating this formula and its derivative with respect to k_0 at $\mathbf{k} = \mathbf{p}_F^\omega$, we obtain $\mu^{\phi\xi} = \zeta^{\phi\xi} = 0$. Evaluating the derivative of this formula with respect to k_i at $\mathbf{k} = \mathbf{p}_F^\omega$, we obtain

$$\partial_{k_i}\mathbf{M}^{\phi\xi}(\mathbf{p}_F^\omega) = \sum_{j=1}^{2} T_{i,j}\mathcal{T}_{\mathbf{p}_F^\omega}^\dagger \partial_{k_j}\mathbf{M}^{\phi\xi}(\mathbf{p}_F^\omega)$$

where we used the following notation $k_1 \equiv k_x$ and $k_2 \equiv k_y$. Furthermore,

$$\mathcal{T}_{\mathbf{p}_F^\omega}^\dagger \partial_{k_x}\mathbf{M}^{\phi\xi} = \nu^{\phi\xi}\Big(-\frac{1}{2}\sigma_2 + \omega\frac{\sqrt{3}}{2}\sigma_1\Big), \quad \mathcal{T}_{\mathbf{p}_F^\omega}^\dagger \partial_{k_y}\mathbf{M}^{\phi\xi} = \omega\varpi^{\phi\xi}\Big(-\frac{1}{2}\sigma_1 - \omega\frac{\sqrt{3}}{2}\sigma_2\Big),$$

which implies

$$\begin{pmatrix} \nu^{\phi\xi}\sigma_2 \\ \omega\varpi^{\phi\xi}\sigma_1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} \nu^{\phi\xi}+3\varpi^{\phi\xi} & -\omega\sqrt{3}(\nu^{\phi\xi}-\varpi^{\phi\xi}) \\ -\sqrt{3}(\nu^{\phi\xi}-\varpi^{\phi\xi}) & \omega(\varpi^{\phi\xi}+3\nu^{\phi\xi}) \end{pmatrix}\begin{pmatrix} \sigma_2 \\ \sigma_1 \end{pmatrix}$$

so that $\nu_h^{\phi\xi} = \varpi_h^{\phi\xi}$. The case of $\mathbf{M}^{\xi\phi}$ is completely analogous and gives $\mu^{\xi\phi} = \zeta^{\xi\phi} = 0$ and $\nu_h^{\xi\phi} = \varpi_h^{\xi\phi}$.

3 - We finally turn to $\mathbf{M}^{\xi\xi}$, which satisfies

$$\mathbf{M}^{\xi\xi}(\mathbf{k}) = \mathbf{M}^{\xi\xi}(T^{-1}\mathbf{k}).$$

Therefore for $i \in \{1,2\}$,

$$\partial_{k_i}\mathbf{M}^{\xi\xi}(\mathbf{p}_F^{\omega}) = \sum_{j=1}^{2} T_{i,j}\partial_{k_j}\mathbf{M}^{\xi\xi}(\mathbf{p}_F^{\omega})$$

where we used the following notation $k_1 \equiv k_x$ and $k_2 \equiv k_y$, so that $\partial_{k_i}\mathbf{M}^{\xi\xi}(\mathbf{p}_F^{\omega}) = 0$, that is $\nu^{\xi\xi} = \varpi^{\xi\xi} = 0$. □

References

[AR98] A. Abdesselam, V. Rivasseau - *Explicit Fermionic tree expansions*, Letters in Mathematical Physics, Vol. 44, n. 1, p. 77-88, 1998, doi:10.1023/A:1007413417112.

[BF84] G.A. Battle, P. Federbush - *A note on cluster expansions, tree graph identities, extra* $1/N!$ *factors!!!*, Letters in Mathematical Physics, Vol. 8, p. 55-57, 1984, doi:10.1007/BF00420041.

[BG90] G. Benfatto, G. Gallavotti - *Perturbation theory of the Fermi surface in a quantum liquid - a general quasiparticle formalism and one-dimensional systems*, Journal of Statistical Physics, Vol. 59, n. 3-4, p. 541-664, 1990, doi:10.1007/BF01025844.

[BG95] G. Benfatto, G. Gallavotti - *Renormalization Group*, Princeton University Press, 1995.

[BM02] G. Benfatto, V. Mastropietro - *On the density-density critical indices in interacting Fermi systems*, Communications in Mathematical Physics, Vol. 231, n. 1, p. 97-134, 2002, doi:10.1007/s00220-002-0671-x.

[BGM06] G. Benfatto, A. Giuliani, V. Mastropietro - *Fermi liquid behavior in the 2D Hubbard model*, Annales Henri Poincaré, Vol. 7, p. 809-898, 2006, doi:10.1007/s00023-006-0270-z.

[BF78] D. Brydges, P. Federbush - *A new form of the Mayer expansion in classical statistical mechanics*, Journal of Mathematical Physics, Vol. 19, p. 2064, 1978, doi:10.1063/1.523586.

[BK87] D.C. Brydges, T. Kennedy - *Mayer expansions and the Hamilton-Jacobi equation*, Journal of Statistical Physics, Vol. 48, n. 1-2, p. 19-49, 1987, doi:10.1007/BF01010398.

[CTV12] V. Cvetkovic, R.E. Throckmorton, O. Vafek - *Electronic multicriticality in bilayer graphene*, Physical Review B, Vol. 86, n. 075467, 2012, doi:10.1103/PhysRevB.86.075467.

[DDe79] R.E. Doezema, W.R. Datars, H. Schaber, A. Van Schyndel - *Far-infrared magnetospectroscopy of the Landau-level structure in graphite*, Physical Review B, Vol. 19, n. 8, p. 4224-4230, 1979, doi:10.1103/PhysRevB.19.4224.

[DD02] M.S. Dresselhaus, G. Dresselhaus - *Intercalation compounds of graphite*, Advances in Physics, Vol. 51, n. 1, p. 1-186, 2002, doi:10.1080/00018730110113644.

[FKT04] J. Feldman, H. Knörrer, E. Trubowitz - *A two dimensional Fermi liquid. Part 1: Overview*, Communications in Mathematical Physics, Vol. 247, n. 1, p. 1-47, 2004, doi:10.1007/s00220-003-0996-0.

[FKT04b] J. Feldman, H. Knörrer, E. Trubowitz - *A two dimensional Fermi liquid. Part 2: Convergence*, Communications in Mathematical Physics, Vol. 247, n. 1, p. 49-111, 2004, doi:10.1007/s00220-003-0997-z.

[FKT04c] J. Feldman, H. Knörrer, E. Trubowitz - *A two dimensional Fermi liquid. Part 3: The Fermi surface*, Communications in Mathematical Physics, Vol. 247, n. 1, p. 113-177, 2004, doi:10.1007/s00220-003-0998-y.

[GN85] G. Gallavotti, F. Nicolò - *Renormalization theory for four dimensional scalar fields I*, Communications in Mathematical Physics, Vol. 100, p. 545-590, 1985, doi:10.1007/BF01217729.

[GN85b] G. Gallavotti, F. Nicolò - *Renormalization theory in four dimensional scalar fields II*, Communications in Mathematical Physics, Vol. 101, n. 2, p. 247-282, 1985, doi:10.1007/BF01218761.

[GN07] A.K. Geim, K.S. Novoselov - *The rise of graphene*, Nature Materials, Vol. 6, p. 183-191, 2007, doi:10.1038/nmat1849.

[Ge11] A.K. Geim - *Nobel Lecture: Random walk to graphene*, Reviews of Modern Physics, Vol. 83, p. 851-862, 2011, doi:10.1103/RevModPhys.83.851.

[GM01] G. Gentile, V. Mastropietro - *Renormalization group for one-dimensional fermions - a review on mathematical results*, Physics Reports, Vol. 352, p. 273-437, 2001, doi:10.1016/S0370-1573(01)00041-2.

[GM10] A. Giuliani, V. Mastropietro - *The two-dimensional Hubbard model on the honeycomb lattice*, Communications in Mathematical Physics, Vol. 293, p. 301-364, 2010, doi:10.1007/s00220-009-0910-5.

[Gi10] A. Giuliani - *The Ground State Construction of the Two-dimensional Hubbard Model on the Honeycomb Lattice*, Quantum Theory from Small to Large Scales, lecture notes of the Les Houches Summer School, Vol. 95, Oxford University Press, 2010.

[GMP10] A. Giuliani, V. Mastropietro, M. Porta - *Lattice gauge theory model for graphene*, Physical Review B, Vol. 82, n. 121418(R), 2010, doi:10.1103/PhysRevB.82.121418.

[GMP11] A. Giuliani, V. Mastropietro, M. Porta - *Absence of interaction corrections in the optical conductivity of graphene*, Physical Review B, Vol. 83, n. 195401, 2011, doi:10.1103/PhysRevB.83.195401.

[GMP11b] A. Giuliani, V. Mastropietro, M. Porta - *Lattice quantum electrodynamics for graphene*, Annals of Physics, Vol. 327, n. 2, p. 461-511, 2011, doi:10.1016/j.aop.2011.10.007.

[GMP12] A. Giuliani, V. Mastropietro, M. Porta - *Universality of conductivity in interacting graphene*, Communications in Mathematical Physics, Vol. 311, n. 2, p. 317-355, 2012, doi:10.1007/s00220-012-1444-9.

[GJ15] A. Giuliani, I. Jauslin - *The ground state construction of bilayer graphene*, preprint, 2015, arXiv:1507.06024.

[Lu13] L. Lu - *Constructive analysis of two dimensional Fermi systems at finite temperature*, PhD thesis, Institute for Theoretical Physics, Heidelberg, http://www.ub.uni-heidelberg.de/archiv/14947, 2013.

[MNe07] L.M. Malard, J. Nilsson, D.C. Elias, J.C. Brant, F. Plentz, E.S. Alves, A.H.C. Neto, M.A. Pimenta - *Probing the electronic structure of bilayer graphene by Raman scattering*, Physical Review B, Vol. 76, n. 201401, 2007, doi:10.1103/PhysRevB.76.201401.

[Ma11] V. Mastropietro - *Conductivity between Luttinger liquids: coupled chains and bilayer graphene*, Physical Review B, Vol. 84, n. 035109, 2011, doi:10.1103/PhysRevB.84.035109.

[MF06] E. McCann, V.I. Fal'ko - *Landau-level degeneracy and Quantum Hall Effect in a graphite bilayer*, Physical Review Letters, Vol. 86, 086805, 2006, doi:10.1103/PhysRevLett.96.086805.

[Mc57] J.W. McClure - *Band structure of graphite and de Haas-van Alphen effect*, Physical review, Vol. 108, p. 612-618, 1957, doi:10.1103/PhysRev.108.612.

[MMD79] A. Misu, E.E. Mendez, M.S. Dresselhaus - *Near Infrared Reflectivity of Graphite under Hydrostatic Pressure*, Journal of the Physical Society of Japan, Vol. 47, n. 1, p. 199-207, 1979, doi:10.1143/JPSJ.47.199.

[NGe04] K.S. Novoselov, A.K. Geim, S. V.Morozov, D. Jiang, Y. Zhang, S.V. Dubonos, I.V. Grigorieva, A.A. Firsov - *Electric field effect in atomically thin carbon films*, Science, vol. 306, p. 666-669, 2004, doi:10.1126/science.1102896.

[NGe05] K.S. Novoselov, A.K. Geim, S.V. Morozov, D. Jiang, M.I. Katsnelson, I.V. Grigorieva, S.V. Dubonos, A.A. Firsov - *Two-dimensional gas of massless Dirac fermions in graphene*, Nature, Vol. 438, n. 10, p. 197-200, 2005, doi:10.1038/nature04233.

[NMe06] K.S. Novoselov, E. McCann, S.V. Morozov, V.I. Fal'ko, M.I. Katsnelson, U. Zeitler, D.Jiang, F. Schedin, A.K. Geim - *Unconventional quantum Hall effect and Berry's phase of π in bilayer graphene*, Nature Physics, Vol. 2, p. 177-180, 2006, doi:10.1038/nphys245.

[PP06] V. Partoens, F.M. Peeters - *From graphene to graphite: electronic structure around the K point*, Physical Review B, Vol. 74, n. 075404, 2006, doi:10.1103/PhysRevB.74.075404.

[SS08] W.A. de S. Pedra, M. Salmhofer - *Determinant bounds and the Matsubara UV problem of many-fermion systems*, Communications in Mathematical Physics, Vol. 282, n. 3, p. 797-818, 2008, doi:10.1007/s00220-008-0463-z.

[Sa13] M. Salmhofer - *Renormalization: an introduction*, Springer Science & Business Media, 2013.

[SW58] J.C. Slonczewski, P.R. Weiss - *Band structure of graphite*, Physical Review, Vol. 109, p. 272-279, 1958, doi:10.1103/PhysRev.109.272.

[TV12] R.E. Throckmorton, O. Vafek - *Fermions on bilayer graphene: symmetry breaking for $B = 0$ and $u = 0$*, Physical Review B, Vol. 86, 115447, 2012, doi:10.1103/PhysRevB.86.115447.

[TDD77] W.W. Toy, M.S. Dresselhaus, G. Dresselhaus - *Minority carriers in graphite and the H-point magnetoreflection spectra*, Physical Review B, Vol. 15, p. 4077-4090, 1977, doi:10.1103/PhysRevB.15.4077.

[TMe92] S.B. Trickey, F. Müller-Plathe, G.H.F. Diercksen, J.C. Boettger - *Interplanar binding and lattice relaxation in a graphite dilayer*, Physical Review B, Vol. 45, p. 4460-4468, 1992, doi:10.1103/PhysRevB.45.4460.

[Va10] O. Vafek - *Interacting Fermions on the honeycomb bilayer: from weak to strong coupling*, Physical Review B, Vol. 82, 205106, 2010, doi:10.1103/PhysRevB.82.205106.

[VY10] O. Vafek, K. Yang - *Many-body instability of Coulomb interacting bilayer graphene: renormalization group approach*, Physical Review B, Vol. 81, 041401, 2010, doi:10.1103/PhysRevB.81.041401.

[Wa47] P.R. Wallace - *The band theory of graphite*, Physical Review, Vol. 71, n. 9, p. 622-634, 1947, doi:10.1103/PhysRev.71.622.

[ZTe05] Y. Zhang, Y. Tan, H.L. Stormer, P. Kim - *Experimental observation of the quantum Hall effect and Berry's phase in graphene*, Nature, Vol. 438, n. 10, p. 201-204, 2005, doi:10.1038/nature04235.

[ZLe08] L.M. Zhang, Z.Q. Li, D.N. Basov, M.M. Fogler, Z. Hao, M.C. Martin - *Determination of the electronic structure of bilayer graphene from infrared spectroscopy*, Physical Review B, Vol. 78, n. 235408, 2008, doi:10.1103/PhysRevB.78.235408.

Part II

Kondo effect in a hierarchical $s - d$ model

We will now turn to the second model studied in this thesis, namely a hierarchical version of the $s - d$ (or *Kondo*) model. The discussion in this part is a closely linked to that appearing in [BGJ15, GJ15], and is the fruit of a collaboration with Giuseppe Benfatto and Giovanni Gallavotti. The model studied in [BGJ15] is a variant of the $s - d$ model, called the *Andrei* model. A short discussion of the hierarchical $s - d$ model was later published in [GJ15]. Here, we reproduce the discussion in [BGJ15], but present the result in [GJ15] on the $s - d$ model rather than the Andrei model.

II.1. Introduction

Although at high temperature the resistivity of most metals is an increasing function of the temperature, experiments carried out since the early XX$^{\text{th}}$ century have shown that in metals containing trace amounts of magnetic impurities (i.e. copper polluted by iron), the resistivity has a minimum at a small but positive temperature, below which the resistivity decreases as the temperature increases. One interesting aspect of such a phenomenon, is that it has been measured in samples of copper with iron impurities at a concentration as small as 0.0005 [Ko05], which raises the question of how such a minute perturbation can produce such an effect. Kondo introduced a toy model in 1964, see (II.2.1) below, to understand such a phenomenon, and computed electronic scattering amplitudes at third order in the Born approximation scheme [Ko64], and found that the effect may stem from an antiferromagnetic coupling between the impurities (called “localized spins” in [Ko64]) and the electrons in the metal. The existence of such a coupling had been proposed by Anderson [An61], who had named the model the “$s - d$ model”.

Kondo’s theory attracted great attention and its scaling properties and connection to $1D$ Coulomb gases were understood [Dy69, An70, AYH70] (the obstacle to a complete understanding of the model (with $\lambda_0 < 0$) being what would later be called the growth of a relevant coupling) when in a seminal paper, published in 1975 [Wi75], Wilson addressed and solved the problem by constructing a sequence of Hamiltonians that adequately represent the system on ever increasing length scales. Using ideas from his formulation of the renormalization group, Wilson showed, by a combination of numerical and perturbative methods, that only few (three) terms in each Hamiltonian, need to be studied in order to account for the Kondo effect (or rather, a related effect on the magnetic susceptibility of the impurities, see below).

The strong-coupling nature of the effect manifests itself in Wilson’s formalism by the presence of a non-trivial fixed point in the renormalization group flow, at which the corresponding effective theory behaves in a way that is qualitatively different from the non-interacting one. Wilson has studied the system around the non-trivial fixed point by perturbative expansions, but the intermediate regime (in which perturbation theory breaks down) was studied by numerical methods. In fact, when using renormalization group techniques to study systems with non-trivial fixed points, oftentimes one cannot treat strong-coupling regimes analytically. The hierarchical $s-d$ model, which will be discussed below, is an exception to this rule: indeed, we will show that the physical properties of the model can be obtained by iterating an *explicit* map, computed analytically, and called the *beta function*, whereas, in the current state of the art, the beta function for the full (non-hierarchical) $s-d$ model can only be computed numerically.

In this paper, we present a hierarchical version of the $s-d$ model, whose renormalization group flow equations can be written out *exactly*, with no need for perturbative methods, and show that the flow admits a non-trivial fixed point. In this model, the transition from the fixed point can be studied by iterating an *explicit* map, which allows us to compute reliable numerical values for the *Kondo temperature*, that is the temperature at which the Kondo effect emerges, which is related to the number of iterations required to reach the non-trivial fixed point from the trivial one. This temperature has been found to obey the expected scaling relations, as predicted in [Wi75].

It is worth noting that the $s-d$ model (or rather a linearized continuum version of it) was shown to be exactly solvable by Andrei [An80] at $h=0$, as well as at $h\neq 0$, [AFL83], using Bethe Ansatz, who proved the existence of a Kondo effect in that model. The aim of the present work is to show how the Kondo effect can be understood as coming from a non-trivial fixed point in a renormalization group analysis (in the context of a hierarchical model) rather than a proof of the existence of the Kondo effect, which has already been carried out in [An80, AFL83].

II.2. $s-d$ model and main results

Consider a *1-dimensional* Fermi gas of spin-1/2 "electrons", and a spin-1/2 Fermionic "impurity", with *no* interactions. It is well known that:

1. the magnetic susceptibility of the impurity diverges as $\beta=\frac{1}{k_BT}\to\infty$ while
2. the total susceptibility per particle of the electron gas (*i.e.* the response to a field acting on the whole sample) [Ki76] is finite at zero temperature.

The question that will be addressed in this work is whether a small coupling of the impurity Fermion with the electron gas can change this behavior, that is whether the susceptibility of the impurity interacting with the electrons diverges or not.

To that end we will study a model inspired by the $s-d$ Hamiltonian, which is an operator that acts on $\mathcal{F}_L\otimes\mathbb{C}^2$ in which $\mathcal{F}_L$ is the Fock space of a length-L chain of spin-1/2 Fermions (the electrons) and $\mathbb{C}^2$ is the state space for the two-level impurity. The expression of the Hamiltonian, in the presence of a magnetic field of amplitude h in the direction $\boldsymbol{\omega}\equiv(\boldsymbol{\omega}_1,\boldsymbol{\omega}_2,\boldsymbol{\omega}_3)$, is

$$\begin{aligned}
&H_K = H_0+V_0+V_h =: H_0+V\\
&H_0 = \sum_{\alpha\in\{\uparrow,\downarrow\}}\sum_{x=-L/2}^{L/2-1} c^+_\alpha(x)\left(\left(-\frac{\Delta}{2}-1\right)c^-_\alpha\right)(x)\\
&V_0 = -\lambda_0\sum_{\substack{j=1,2,3\\ \alpha_1,\alpha_2}} c^+_{\alpha_1}(0)\sigma^j_{\alpha_1,\alpha_2}c^-_{\alpha_2}(0)\,\tau^j\\
&V_h = -h\sum_{j=1,2,3}\boldsymbol{\omega}_j\tau^j
\end{aligned}\tag{II.2.1}$$

where λ_0 is the interaction strength and

1. $c^{\pm}_{\alpha}(x)$ $\alpha =\uparrow,\downarrow$ are creation and annihilation operators acting on the electrons,

2. $\sigma^j \equiv \tau^j$, $j = 1, 2, 3$, are the Pauli matrices, (τ^j acts on the impurity),

3. x is on the unit lattice and $-L/2$, $L/2$ are identified (periodic boundary),

4. $\Delta f(x) = f(x+1) - 2f(x) + f(x-1)$ is the discrete Laplacian,

5. $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3)$ is the direction of the field, which is a norm-1 vector.

6. the -1 term in H_0 is the chemical potential, set to -1 (half-filling) for convenience.

The model will be said to exhibit a *Kondo effect* if, no matter how small the coupling λ_0 is, as long as it is *antiferromagnetic* (*i.e.* $\lambda_0 < 0$), the susceptibility *remains finite and positive as* $\beta \to \infty$ *and continuous as* $h \to 0$, while it diverges in presence of a ferromagnetic (*i.e.* $\lambda_0 > 0$) coupling.

Remark: In the present work, the *Kondo effect* is defined as an effect on the susceptibility of the impurity, and not on the resistivity of the electrons of the chain, which, we recall, was Kondo's original motivation [Ko64]. The reason for this is that the magnetic susceptibility of the impurity is easier to compute than the resistivity of the chain, but still exhibits a non-trivial effect, as discussed by Wilson [Wi75].

Here the same questions will be studied in a hierarchical model defined below. The interest of this model is that various observables can be computed by iterating a map, which is explicitly computed and called the "beta function", involving few (six) variables, called "running couplings". The possibility of computing the beta function exactly for general Fermionic hierarchical models has been noticed and used in [Do91].

Remark: The hierarchical $s - d$ model *will not be an approximation* of (II.2.1). It is a model that illustrates a simple mechanism for the control of the growth of relevant operators in a theory exhibiting a Kondo effect.

The reason why the Kondo effect is not easy to understand is that it occurs in the strong-coupling regime in which the impurity susceptibility in the interacting model is qualitatively different from its non-interacting counterpart. In the sense of the renormalization group it exhibits several "relevant", "marginal" and "irrelevant" running couplings: this makes any naive perturbative approach hopeless because all couplings become large (*i.e.* at least of O(1)) at large scale, no matter how small the interaction is, as long as $\lambda_0 < 0$, and thus leave the perturbative regime. It is among the simplest cases in which asymptotic freedom *does not occur*. Using the fact that the beta function of the hierarchical model can be computed exactly, its strong-coupling regime can easily be investigated.

In the sections below, we will define the hierarchical $s - d$ model and show numerical evidence for the following claims (in principle, such claims could be proved using computer-assisted methods, though, since the numerical results are very clear and stable, it may not be worth the trouble).

If the interactions between the electron spins and the impurity are antiferromagnetic (*i.e.* $\lambda_0 < 0$ in our notations), then

1. The *existence of a Kondo effect* can be proved in spite of the lack of asymptotic freedom and formal growth of the effective Hamiltonian away from the trivial fixed point, *because the beta function can be computed exactly* (in particular non-pertubatively).

2. In addition, there exists an inverse temperature $\beta_K = 2^{n_K(\lambda_0)}$ called the *Kondo* inverse temperature, such that the Kondo effect manifests itself for $\beta > \beta_K$. Asymptotically as

$\lambda_0 \to 0$, $n_K(\lambda_0) = c_1|\lambda_0|^{-1} + O(1)$. The Kondo inverse temperature β_K roughly corresponds to the limit at which perturbation theory breaks down.

3. Denoting the magnetic field by h, if $h > 0$ and $\beta_K h \ll 1$, the flow of the running couplings tends to a trivial fixed point (h-independent but different from 0) which is reached on a scale $r(h)$ which, asymptotically as $h \to 0$, is $r(h) = c_r \log h^{-1} + O(1)$.

The picture is completely different in the ferromagnetic case, in which the susceptibility diverges at zero temperature and the flow of the running couplings is not controlled by the non trivial fixed point.

Remark: Unlike in the model studied by Wilson [Wi75], the $T = 0$ nontrivial fixed point is *not* infinite in the hierarchical $s-d$ model: this shows that the Kondo effect can, in some models, be somewhat subtler than a rigid locking of the impurity spin with an electron spin [No74].

Technically this is one of the few cases in which functional integration for Fermionic fields is controlled by a non-trivial fixed point and can be performed rigorously and applied to a concrete problem.

Remark:

1. It is worth stressing that in a system consisting of two classical spins with coupling λ_0 the susceptibility at 0 field is $4\beta(1+e^{-2\beta\lambda_0})^{-1}$, hence it vanishes at $T = 0$ in the antiferromagnetic case and diverges in the ferromagnetic and in the free case. Therefore this simple model does *not* exhibit a Kondo effect.

2. In the exactly solvable XY model, which can be shown to be equivalent to a spin-less analogue of (II.2.1), the susceptibility can be shown to diverge in the $\beta \to \infty$ limit, see [BGJ15, appendix 7] for details. The XY model therefore does *not* exhibit a Kondo effect either.

II.3. Functional integration in the $s-d$ model

In [Wi75], Wilson studies the $s-d$ model using renormalization group techniques in a Hamiltonian context. In the present work, our aim is to reproduce, in a simpler model, analogous results using a formalism based on functional integrals.

In this section, we give a rapid review of the functional integral formalism we will use, following [BG90, Sh94]. We will not attempt to reproduce all technical details, since it will merely be used as an inspiration for the definition of the hierarchical model in section II.4.

We introduce an extra dimension, called *imaginary time*, and define new creation and annihilation operators:

$$c_\alpha^\pm(x,t) := e^{tH_0} c_\alpha^\pm(x) e^{-tH_0}, \tag{II.3.1}$$

for $\alpha \in \{\uparrow,\downarrow\}$, to which we associate anti-commuting *Grassmann variables*:

$$c_\alpha^\pm(x,t) \longmapsto \psi_\alpha^\pm(x,t). \tag{II.3.2}$$

Functional integrals are expressed as "Gaussian integrals" over the Grassmann variables (this means that all integrals will be defined and evaluated via the "Wick rule"):

$$\int P(d\psi)\cdot \ := \ \int \prod_\alpha P(d\psi_\alpha)\ \cdot \tag{II.3.3}$$

$P(d\psi_\alpha)$ are Gaussian measures whose covariance (also called *propagator*) is defined by

$$g_\alpha(x-x',t-t') := \left\{\begin{array}{ll} \dfrac{\mathrm{Tr}\, e^{-\beta H_0} c^-_\alpha(x,t) c^+_\alpha(x',t')}{\mathrm{Tr}\, e^{-\beta H_0}} & \text{if } t > t' \\[2ex] -\dfrac{\mathrm{Tr}\, e^{-\beta H_0} c^+_\alpha(x',t') c^-_\alpha(x,t)}{\mathrm{Tr}\, e^{-\beta H_0}} & \text{if } t \leqslant t' \end{array}\right. . \tag{II.3.4}$$

By a direct computation [BG90, (2.7)], we find that in the limit $L,\beta\to\infty$, if $e(k) := (1-\cos k)-1 \equiv -\cos k$ (assuming the Fermi level is set to 1, *i.e.* the Fermi momentum to $\pm\frac{\pi}{2}$) then

$$g_\alpha(\xi,\tau) = \int \frac{dk_0 dk}{(2\pi)^2}\,\frac{e^{-ik_0(\tau+0^-)-ik\xi}}{-ik_0+e(k)}. \tag{II.3.5}$$

If β, L are finite, $\int \frac{dk_0dk}{(2\pi)^2}$ in (II.3.5) has to be understood as $\frac{1}{\beta}\sum_{k_0}\frac{1}{L}\sum_k$, where k_0 is the "Matsubara momentum" $k_0 = \frac{\pi}{\beta}+\frac{2\pi}{\beta}n_0$, $n_0\in\mathbb{Z}$, $|n_0|\leqslant\frac{1}{2}\beta$, and k is the linear momentum $k=\frac{2\pi}{L}n$, $n\in[-L/2,L/2-1]\cap\mathbb{Z}$.

In the functional representation, the operator V of (II.2.1) is substituted with the following function of the Grassmann variables (II.3.2):

$$V(t) := -\lambda_0 \sum_{\substack{j=1,2,3\\ \alpha_1,\alpha_2}} \psi^+_{\alpha_1}(0,t)\sigma^j_{\alpha_1,\alpha_2}\psi^-_{\alpha_2}(0,t)\,\tau^j - h\sum_{j=1,2,3}\boldsymbol{\omega}_j\tau^j. \tag{II.3.6}$$

$V(t)$ is thus defined as an operator on the Hilbert space of the impurity, that is, on $\mathbb{C}^2$, with coefficients that are linear combinations of Grassmann fields. Notice that $V(t)$ only depends on the fields located at the site $x=0$. This is important because it will allow us to reduce the problem to a 1-dimensional one [AY69, AYH70].

The average of a physical observable F localized at $x=0$, which is a polynomial in the fields $\psi^\pm_\alpha(0,t)$, will be denoted by

$$\langle F\rangle_K := \frac{1}{Z}\mathrm{Tr}\sum_{n=0}^{\infty}(-1)^n\int_{0<t_1<\cdots<t_n<\beta} dt_1\cdots dt_n \int P_0(d\psi)\,V(t_1)\cdots V(t_n)\,F, \tag{II.3.7}$$

in which $P_0(d\psi)$ is the Gaussian Grassmannian measure over the fields $\psi^\pm_\alpha(0,t)$ localized at the site 0 and with propagator $g_\alpha(0,\tau)$, the trace is taken over the Hilbert space of the impurity ($\mathbb{C}^2$), and Z is a normalization factor called the *partition function*:

$$Z := \mathrm{Tr}\sum_{n=0}^{\infty}(-1)^n\int_{0<t_1<\cdots<t_n<\beta} dt_1\cdots dt_n\int P_0(d\psi)\,V(t_1)\cdots V(t_n). \tag{II.3.8}$$

The propagators can be split into scales by introducing a smooth cutoff function χ which is different from 0 only on $(\frac{1}{4},1)$ and, denoting $N_\beta := \log_2\beta$, is such that $\sum_{m=-N_\beta}^{\infty}\chi(2^{-2m}z^2)=1$ for all $|z|\in[\frac{\pi}{\beta},N_\beta]$. Let

$$\begin{aligned} g^{[m]}(0,t) &:= \sum_{\omega\in\{-,+\}}\int \frac{dk_0dk}{(2\pi)^2}\frac{e^{-ik_0(t+0^-)}}{-ik_0+e(k)}\chi(2^{-2m}((k-\omega\pi/2)^2+k_0^2)) \\ g^{[\mathrm{uv}]}(0,t) &:= g(0,t) - \sum_{m=-N_\beta}^{m_0} g^{[m]}(0,t). \end{aligned} \tag{II.3.9}$$

where m_0 is an integer of order one (see below).

Remark: The $\omega=\pm$ label refers to the "quasi particle" momentum ωp_F, where p_F is the Fermi momentum. The usual approach [BG90, Sh94] is to decompose the field ψ into quasi-particle fields:

$$\psi^\pm_\alpha(0,t) = \sum_{\omega=\pm}\psi^\pm_{\omega,\alpha}(0,t), \tag{II.3.10}$$

indeed, the introduction of quasi particles [BG90, Sh94], is key to separating the oscillations on the Fermi scale p_F^{-1} from the propagators thus allowing a "naive" renormalization group analysis of Fermionic models in which multiscale phenomena are important (as in the theory of the ground state of interacting Fermions [BG90, BGe94], or as in the $s-d$ model). In this case, however, since the fields are evaluated at $x=0$, such oscillations play no role, so we will not decompose the field.

We set m_0 to be small enough (*i.e.* negative enough) so that $2^{m_0}p_F \leqslant 1$ and introduce a first *approximation*: we neglect $g^{[\mathrm{uv}]}$ and replace $e(k)$ in (II.3.5) by its first order Taylor expansion around ωp_F, that is by ωk. As long as m_0 is small enough, for all $m \leqslant m_0$ the supports of the two functions $\chi(2^{-2m}((k-\omega\pi/2)^2+k_0^2))$, $\omega=\pm1$, which appear in the first of (II.3.9) do not intersect, and approximating $e(k)$ by ωk is reasonable. We shall hereafter fix $m_0=0$ thus avoiding the introduction of a further length scale and keeping only two scales when no impurity is present.

Since we are interested in the *infrared* properties of the system, we consider such approximations as minor and more of a *simplification* rather than an approximation, since the ultraviolet regime is expected to be trivial because of the discreteness of the model in the operator representation.

After this approximation, the propagator of the model reduces to

$$g^{[m]}(0,t)=\sum_{\omega\in\{-,+\}}\int\frac{dk_0dk}{(2\pi)^2}\frac{e^{-ik_0(t+0^-)}}{-ik_0+\omega k}\chi(2^{-2m}(k^2+k_0^2)). \tag{II.3.11}$$

and satisfies the following *scaling* property:

$$g^{[m]}(0,t)=2^mg^{[0]}(0,2^mt). \tag{II.3.12}$$

The Grassmannian fields are similarly decomposed into scales:

$$\psi_\alpha^\pm(0,t)=\sum_{m=-N_\beta}^{0}2^{\frac{m}{2}}\psi_\alpha^{[m]\pm}(0,2^{-m}t) \tag{II.3.13}$$

where $\psi_\alpha^{[m]}(0,t)$ is assigned the following propagator:

$$\int P_0(d\psi^{[m]})\psi_\alpha^{[m]-}(0,t)\psi_{\alpha'}^{[m]+}(0,t')=:\delta_{\alpha,\alpha'}g^{[0]}(0,2^m(t-t')). \tag{II.3.14}$$

Remark: by (II.3.12) this is equivalent to stating that the propagator associated to the $\psi^{[m]}$ field is $2^{-m}g^{[m]}$.

Finally, we define

$$\psi_\alpha^{[\leqslant m]\pm}(0,t):=\sum_{m'=-N_\beta}^{m}2^{\frac{m'}{2}}\psi_\alpha^{[m']\pm}(0,t). \tag{II.3.15}$$

Notice that the function $g^{[0]}(0,t)$ decays faster than any power as t tends to ∞ (as a consequence of the smoothness of the cut-off function χ), so that at any fixed scale $m\leqslant 0$, fields $\psi^{[m]}$ that are separated in time by more than 2^{-m} can be regarded as (almost) independent.

The decomposition into scales allows us to express the quantities in (II.3.7) inductively (see (II.3.17)). For instance the partition function Z is given by

$$Z=\prod_{m=-N_\beta}^{0}c^{[m]} \tag{II.3.16}$$

where, for $N_\beta < m \leqslant 0$,

$$c^{[m-1]}W^{[m-1]}(\psi^{[\leqslant m-1]}) := \int P(d\psi^{[m]})\, W^{[m]}(\psi^{[\leqslant m]})$$
$$W^{[0]}(\psi^{[\leqslant 0]}) := \sum_{n=0}^{\infty}(-1)^n \int_{0<t_1<\cdots<t_n<\beta} dt_1\cdots dt_n \int P_0(d\psi)\, V(t_1)\cdots V(t_n) \tag{II.3.17}$$

in which $c^{[m-1]} \in \mathbb{R}$ and the constant term (i.e. the term that neither involves fields nor Pauli matrices) of $W^{[m-1]}$ is 1.

II.4. Hierarchical $s-d$ model

In this section, we define a hierarchical $s-d$ model, localized at $x=0$ (the location of the impurity), inspired by the discussion in the previous section and the remark that the problem of the Kondo effect is reduced there to the evaluation of a functional integral over the fields $\psi(x,t)$ *with* $x \equiv 0$. The hierarchical model is a model that is represented using a functional integral, that shares a few features, namely the scaling properties, with the functional integral described in section II.3, which are essential to the Kondo effect. Therein the fields $\psi^{[m]}$ evaluated at $x=0$ are assumed to be constant in t on scale $2^{-m}, m = 0,-1,-2\ldots$, and the propagator $g^{[m]}(0,\tau)$ with large Matsubara momentum k_0 are neglected ($g^{[uv]}=0$ in (II.3.9)).

The hierarchical $s-d$ model is defined by introducing a family of *hierarchical fields* and specifying a *propagator* for each pair of fields. The average of any monomial of fields is then computed using the Wick rule.

As a preliminary step, we pave the time axis $\mathbb{R}$ with boxes of size 2^{-m} for every $m \in \{0,-1,\ldots,-N_\beta\}$. To that end, we define the set of *boxes on scale* m as

$$\mathcal{Q}_m := \left\{[i2^{|m|},(i+1)2^{|m|})\right\}_{i=0,1,\cdots,2^{N_\beta-|m|}-1}^{m=0,-1,\ldots} \tag{II.4.1}$$

Given a box $\Delta \in \mathcal{Q}_m$, we define t_Δ as the center of Δ; conversely, given a point $t \in \mathbb{R}$, we define $\Delta^{[m]}(t)$ as the (unique) box on scale m that contains t.

A naive approach would then be to define the hierarchical model in terms of the fields $\psi_{t_\Delta}^{[m]}$, and neglect the propagators between fields in different boxes, but, as we will see below, such a model would be trivial (all propagators would vanish because of Fermi statistics).

Instead, we further decompose each box into two *half boxes*: given $\Delta \in \mathcal{Q}_m$ and $\eta \in \{-,+\}$, we define

$$\Delta_\eta := \Delta^{[m+1]}(t_\Delta + \eta 2^{-m-2}) \tag{II.4.2}$$

for $m<0$ and similarly for $m=0$. Thus Δ_- is the lower half of Δ and Δ_+ the upper half.

The elementary fields used to define the hierarchical $s-d$ model will be *constant on each half-box* and will be denoted by $\psi_\alpha^{[m]\pm}(\Delta_\eta)$ for $m \in \{0,-1,\cdots,-N_\beta\}$, $\Delta \in \mathcal{Q}_m$, $\eta \in \{-,+\}$, $\alpha \in \{\uparrow,\downarrow\}$.

We now define the propagators associated to ψ. The idea is to define propagators that are *similar* [Wi65, Wi70, Dy69], in a sense made more precise below, to the non-hierarchical propagators defined in (II.3.4). Bearing that in mind, we compute the value of the non-hierarchical propagators between fields at the centers of two half boxes: given a box $\Delta \in \mathcal{Q}_0$ and $\eta \in \{-,+\}$, let $\delta := 2^{-1}$ denote the distance between the centers of Δ_- and Δ_+, we get

$$g^{[0]}(0,\eta\delta) = \eta \sum_{\omega=\pm} \int \frac{dk dk_0}{(2\pi)^2} \frac{k_0 \sin(k_0\delta)}{k_0^2+k^2}\chi(k^2+k_0^2) := \eta a \tag{II.4.3}$$

in which a and b are constants. We define the hierarchical propagators, drawing inspiration from (II.4.3). In an effort to make computations more explicit, we set $a = b \equiv 1$ and define

$$\left\langle \psi_\alpha^{[m]-}(\Delta_{-\eta})\psi_\alpha^{[m]+}(\Delta_\eta)\right\rangle := \eta \tag{II.4.4}$$

for $m \in \{0, -1, \cdots, -N_\beta\}$, $\eta \in \{-, +\}$, $\Delta \in \mathcal{Q}_m$, $\alpha \in \{\downarrow, \uparrow\}$. All other propagators are 0. Note that if we had not defined the model using half boxes, all the propagators in (II.4.3) would vanish, and the model would be trivial.

In order to link back to the non-hierarchical model, we define the following quantities: for all $t \in \mathbb{R}$,

$$\psi_\alpha^\pm(0,t) := \sum_{m=-N_\beta}^{0} 2^{\frac{m}{2}}\psi_\alpha^{[m]\pm}(\Delta^{[m+1]}(t)) \tag{II.4.5}$$

(recall that $m \leqslant 0$ and $\Delta^{[m]}(t) \supset \Delta^{[m+1]}(t)$). The hierarchical model for the on-site Kondo effect so defined is such that the propagator on scale m between two fields vanishes unless both fields belong to the same box and, at the same time, to two different halves within that box. In addition, given t and t' that are such that $|t - t'| > 2^{-1}$, there exists one and only one scale $m_{(t-t')}$ that is such that $\Delta^{[m_{(t-t')}]}(t) = \Delta^{[m_{(t-t')}]}(t')$ and $\Delta^{[m_{(t-t')}+1]}(t) \neq \Delta^{[m_{(t-t')}+1]}(t')$. Therefore $\forall (t,t') \in \mathbb{R}^2$, $\forall (\alpha, \alpha') \in \{\uparrow, \downarrow\}^2$,

$$\langle\, \psi_\alpha^-(0,t)\psi_{\alpha'}^+(0,t')\,\rangle = \delta_{\alpha,\alpha'}2^{m_{(t-t')}}\mathrm{sign}(t-t'). \tag{II.4.6}$$

The non-hierarchical analog of (II.4.6) is (we recall that $\langle\,\cdot\,\rangle_K$ was defined in (II.3.7))

$$\langle\, \psi_\alpha^-(0,t)\psi_{\alpha'}^+(0,t')\,\rangle_K = \delta_{\alpha,\alpha'}\sum_{m'=-N_\beta}^{0} 2^{m'}g^{[0]}(0, 2^{m'}(t-t')) \tag{II.4.7}$$

from which we see that the hierarchical model boils down to neglecting the m' that are "wrong", that is those that are different from $m_{(t-t')}$, and approximating $g_\psi^{[m]}$ by $\mathrm{sign}(t - t')$.

The physical observables F considered here will be polynomials in the hierarchical fields; their averages, by analogy with (II.3.7) and (II.3.8), will be

$$\begin{gathered}\frac{1}{Z}\mathrm{Tr}\sum_{n=0}^{\infty}(-1)^n\int_{0<t_1<\cdots<t_n<\beta} dt_1\cdots dt_n\ \langle V(t_1)\cdots V(t_n)\,F\rangle\,,\\ Z := \mathrm{Tr}\sum_{n=0}^{\infty}(-1)^n\int_{0<t_1<\cdots<t_n<\beta} dt_1\cdots dt_n\ \langle V(t_1)\cdots V(t_n)\rangle\end{gathered} \tag{II.4.8}$$

(in which $\langle\,\cdot\,\rangle$ is computed using the Wick rule and (II.4.4)) and, similarly to (II.3.6),

$$V(t) := -\lambda_0 \sum_{\substack{j=1,2,3\\ \alpha_1,\alpha_2}} \psi_{\alpha_1}^+(0,t)\sigma_{\alpha_1,\alpha_2}^j\psi_{\alpha_2}^-(0,t)\,\tau^j - h\sum_{j=1,2,3}\boldsymbol{\omega}_j\tau^j \tag{II.4.9}$$

in which $\psi_\alpha^\pm(0,t)$ is now defined in (II.4.5).

Note that since the model defined above only involves fields localized at the impurity site, that is at $x = 0$, we only have to deal with 1-dimensional Fermionic fields. *This does not mean* that the lattice supporting the electrons plays no role: on the contrary it will show up, and in an essential way, because the "dimension" of the electron field will be different from that of the impurity, as made already manifest by the factor $2^m \underset{m\to-\infty}{\longrightarrow} 0$ in (II.4.6).

Clearly several properties of the non-hierarchical propagators, (II.3.11), are not reflected in (II.4.6), though their *scaling* was preserved. However it will be seen that even so simplified the model exhibits a "Kondo effect" in the sense outlined in section II.1.

II.5. Beta function for the partition function.

In this section, we show how to compute the partition function Z of the hierarchical $s-d$ model (see (II.4.8)), and introduce the concept of a *renormalization group flow* in this context. We will first restrict the discussion to the $h=0$ case, in which $V=V_0$; the case $h\neq 0$ is discussed in section II.6.

The computation is carried out in an inductive fashion by splitting the averages in (II.4.8) into partial averages over the fields on scale m. Given $m\in\{0,-1,\cdots,-N_\beta\}$, we define $\langle\cdot\rangle_m$ as the partial average over $\psi_\alpha^{[m]\pm}(\Delta_\eta)$ for $\alpha\in\{\uparrow,\downarrow\}$, $\Delta\in\mathcal Q_m$ and $\eta\in\{-,+\}$, as well as

$$\psi_\alpha^{[\leqslant m]\pm}(\Delta_\eta):=\frac{1}{\sqrt2}\psi_\alpha^{[\leqslant m-1]\pm}(\Delta)+\psi_\alpha^{[m]\pm}(\Delta_\eta) \tag{II.5.1}$$

and for $\Delta\in\mathcal Q_{-m}$, $m<-N_\beta$,

$$\psi_\alpha^{[\leqslant m]}(\Delta_\eta):=0. \tag{II.5.2}$$

Notice that the fields $\psi_\alpha^{[\leqslant m-1]\pm}(\Delta)$ play (temporarily) the role of *external fields* as they do not depend on the index η, and are therefore independent of the half box in which the *internal fields* $\psi_\alpha^{[\leqslant m]\pm}(\Delta_\eta)$ are defined. In addition, by iterating (II.5.1), we can rewrite (II.4.5) as

$$\psi_\alpha^{\pm}(t)\equiv\psi_\alpha^{[\leqslant 0]\pm}(\Delta^{[1]}(t)). \tag{II.5.3}$$

We then define, for $m\in\{0,-1,\cdots,-N_\beta\}$,

$$\begin{gathered}c^{[m-1]}W^{[m-1]}(\psi^{[\leqslant m-1]}):=\left\langle W^{[m]}(\psi^{[\leqslant m]})\right\rangle_m\\ W^{[0]}(\psi^{[\leqslant 0]}):=\sum_{n=0}^{\infty}(-1)^n\int_{0<t_1<\cdots<t_n<\beta}dt_1\cdots dt_n\,V(t_1)\cdots V(t_n)\end{gathered} \tag{II.5.4}$$

in which $c^{[m-1]}\in\mathbb R$ is a constant and the constant term in $W^{[m-1]}$ is 1. By a straightforward induction, we then find that Z is given again by (II.3.16) with the present definition of $c^{[m]}$ (see (II.5.4)).

We will now prove by induction that the hierarchical $s-d$ model defined above is *exactly solvable*, in the sense that (II.5.4) can be written out *explicitly* as a *finite* system of equations. To that end it will be shown that $W^{[m]}$ can be parameterized by only two real numbers, $\boldsymbol\ell^{[m]}=(\ell_0^{[m]},\ell_1^{[m]})\in\mathbb R^2$ and, in the process, the equation relating $\boldsymbol\ell^{[m]}$ and $\boldsymbol\ell^{[m-1]}$ (called the *beta function*) will be computed:

$$W^{[m]}(\psi^{[\leqslant m]})=\prod_{\Delta\in\mathcal Q_m}\prod_{\eta=\pm}\left(1+\sum_{n=0}^{1}\ell_n^{[m]}O_{n,\eta}^{[\leqslant m]}(\Delta)\right) \tag{II.5.5}$$

where

$$O_{0,\eta}^{[\leqslant m]}(\Delta):=\frac12\mathbf A_\eta^{[\leqslant m]}(\Delta)\cdot\boldsymbol\tau,\quad O_{1,\eta}^{[\leqslant m]}(\Delta):=\frac12\mathbf A_\eta^{[\leqslant m]}(\Delta)^2 \tag{II.5.6}$$

in which $\mathbf A^{[\leqslant m]}$ is a vector of polynomials in the fields, whose j-th component for $j\in\{1,2,3\}$ is

$$A_\eta^{[\leqslant m]j}(\Delta):=\sum_{(\alpha,\alpha')\in\{\uparrow,\downarrow\}^2}\psi_\alpha^{[\leqslant m]+}(\Delta_\eta)\sigma^j_{\alpha,\alpha'}\psi_{\alpha'}^{[\leqslant m]-}(\Delta_\eta) \tag{II.5.7}$$

We first show that (II.5.5) holds for $m=0$. Indeed, we can rewrite

$$\sum_{n=0}^{\infty}(-1)^n\int_{0<t_1<\cdots<t_n<\beta}dt_1\cdots dt_n\,V(t_1)\cdots V(t_n)=\prod_{\Delta\in\mathcal Q_0}\prod_{\eta=\pm}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2^nn!}V(t_{\Delta_\eta})^n\right) \tag{II.5.8}$$

and $V(t_{\Delta_\eta}) = -2\lambda_0 O_{0,\eta}^{[\leqslant 0]}(\Delta)$. Furthermore,

$$A_\eta^{[\leqslant 0]i} A_\eta^{[\leqslant 0]j} = 2\delta_{i,j}\psi_\uparrow^{[\leqslant 0]+}\psi_\downarrow^{[\leqslant 0]+}\psi_\uparrow^{[\leqslant 0]-}\psi_\downarrow^{[\leqslant 0]-} = \frac{1}{3}\delta_{i,j}\mathbf{A}_\eta^{[\leqslant 0]}\cdot\mathbf{A}_\eta^{[\leqslant 0]} \tag{II.5.9}$$

which, by the anti-commutation of the Grassmann variables $\psi_\alpha^{[\leqslant 0]\pm}$ implies that a product containing three or more $A_\eta^{[\leqslant 0]i}$ vanishes, as well as

$$(O_{0,\eta}^{[\leqslant 0]})^2 = \frac{1}{2}O_{1,\eta}^{[\leqslant 0]}. \tag{II.5.10}$$

Therefore, by injecting (II.5.10) into (II.5.8), we find that $W^{[0]}$ can be written as in (II.5.5) with $\boldsymbol{\ell}^{[0]} = (\lambda_0, \lambda_0^2/4)$.

We then compute $W^{[m-1]}$ using (II.5.4) and show that it can be written as in (II.5.5). We first notice that the propagator in (II.4.4) is diagonal in Δ, and does not depend on the value of Δ, therefore, we can split the averaging over $\psi^{[m]}(\Delta_\pm)$ for different Δ. We thereby find that

$$\left\langle \prod_{\Delta\in\mathcal{Q}_m}\prod_{\eta=\pm}\left(1+\sum_{n=0}^{1}\ell_n^{[m]}O_{n,\eta}^{[\leqslant m]}(\Delta)\right)\right\rangle_m = \prod_{\Delta\in\mathcal{Q}_m}\left\langle\prod_{\eta=\pm}\left(1+\sum_{n=0}^{1}\ell_n^{[m]}O_{n,\eta}^{[\leqslant m]}(\Delta)\right)\right\rangle_m. \tag{II.5.11}$$

We then compute the average, which is a somewhat long computation, although finite (see appendix II.A1 for the main shortcuts). We find that

$$\left\langle\prod_{\eta=\pm}\left(1+\sum_{n=0}^{1}\ell_n^{[m]}O_{n,\eta}^{[\leqslant m]}(\Delta)\right)\right\rangle_m = C^{[m]}\left(1+\sum_{n=0}^{1}\ell_n^{[m-1]}O_n^{[\leqslant m-1]}(\Delta)\right) \tag{II.5.12}$$

with (in order to reduce the size of the following equation, we dropped all $^{[m]}$ from the right side)

$$\begin{aligned} C^{[m]} &= 1+\frac{3}{2}\ell_0^2+9\ell_1^2 \\ \ell_0^{[m-1]} &= \frac{1}{C^{[m]}}\left(\ell_0+3\ell_0\ell_1-\ell_0^2\right) \\ \ell_1^{[m-1]} &= \frac{1}{C^{[m]}}\left(\frac{1}{2}\ell_1+\frac{1}{8}\ell_0^2\right) \end{aligned} \tag{II.5.13}$$

This concludes the proof of (II.5.5), and provides an explicit map, defined in (II.5.13) and which we denote by $\mathcal{R}$, that is such that $\boldsymbol{\ell}^{[m]} = \mathcal{R}^{|m|}\boldsymbol{\ell}^{[0]}$. Finally, the $c^{[m]}$ appearing in (II.5.4) is given by

$$c^{[m]} = (C^{[m+1]})^{2^{N_\beta+m}}. \tag{II.5.14}$$

The dynamical system defined by the map $\mathcal{R}$ in (II.5.13) admits two non trivial fixed points, at $\boldsymbol{\ell} = \mathbf{0}$ and $\boldsymbol{\ell} = \boldsymbol{\ell}^*$ (see appendix II.A3 for a proof) with

$$\ell_0^* = -x_0\frac{1+5x_0}{1-4x_0}, \quad \ell_1^* = \frac{x_0}{3} \tag{II.5.15}$$

where $x_0 \approx 0.15878626704216...$ is the real root of $4-19x-22x^2-107x^3 = 0$. A numerical analysis shows that, if the initial data λ_0 is small and < 0, then the flow converges to $\boldsymbol{\ell}^*$, whereas it converges to $\mathbf{0}$ if $\lambda_0 \geqslant 0$.

Remark: Proving that the flow converges to $\boldsymbol{\ell}^*$ analytically is complicated by the somewhat contrived expression of $\boldsymbol{\ell}^*$. It is however not difficult to prove that if the flow converges, then it must go to $\boldsymbol{\ell}^*$ (see appendix II.A3). Since the numerical iterations of the flow converge quite clearly, we will not attempt a full proof of the convergence to the fixed point.

II.6. Beta function for the Kondo effect

In this section, we discuss the Kondo effect in the hierarchical model: *i.e.* the phenomenon that as soon as the interaction is strictly repulsive (*i.e.* $\lambda_0 < 0$) the susceptibility of the impurity at zero temperature remains positive and finite, although it can become very large for small coupling. The problem will be rigorously reduced to the study of a dynamical system, extending the map $\boldsymbol{\ell} \to \mathcal{R}\boldsymbol{\ell}$ in (II.5.13). The value of the susceptibility follows from the iterates of the map, as explained below. The computation will be performed numerically; a rigorous computer assisted analysis of the flow appears possible, but we have not attempted it because the results are very stable and clear.

We introduce a magnetic field of amplitude $h \in \mathbb{R}$ and direction $\boldsymbol{\omega} \in \mathcal{S}_2$ (in which $\mathcal{S}_2$ denotes the 2-sphere) acting on the impurity. As a consequence, the potential V becomes

$$V_h(t) = V_0(t) - h \sum_{j\in\{1,2,3\}} \boldsymbol{\omega}_j \tau^j \tag{II.6.1}$$

The corresponding partition function is denoted by Z_h and the free energy of the system by $f_h := -\beta^{-1} \log Z_h$. The *impurity susceptibility* is then defined as

$$\chi(h,\beta) := \frac{\partial^2 f_h}{\partial h^2}. \tag{II.6.2}$$

The h-dependent potential and the constant term, *i.e.* $W_h^{[m]}$ and $c_h^{[m]}$, are then defined in the same way as in (II.5.4), in terms of which,

$$f_h = -\frac{1}{\beta} \sum_{m=-N_\beta}^{0} \log c_h^{[m]}. \tag{II.6.3}$$

We compute $c_h^{[m]}$ in the same way as in section II.5. Because of the extra term in the potential in (II.6.1), the number of running coupling constants increases to six: indeed we prove by induction that $W_h^{[m]}$ is parametrized by six real numbers, $\boldsymbol{\ell}_h^{[m]} = (\ell_{0,h}^{[m]}, \ell_{1,h}^{[m]}, \ell_{4,h}^{[m]}, \ell_{5,h}^{[m]}, \ell_{6,h}^{[m]}, \ell_{7,h}^{[m]}) \in \mathbb{R}^6$ (the numbering is meant to recall that in [BGJ15]):

$$W_h^{[m]}(\psi^{[\leqslant m]}) = \prod_{\Delta\in\mathcal{Q}_m} \prod_{\eta=\pm} \left(\sum_n \ell_{n,h}^{[m]} O_{n,\eta}^{[\leqslant m]}(\Delta) \right) \tag{II.6.4}$$

where $O_{n,\eta}^{[\leqslant m]}(\Delta)$ for $n \in \{0,1\}$ was defined in (II.5.6) and

$$\begin{gathered} O_{4,\eta}^{[\leqslant m]}(\Delta) := \frac{1}{2}\mathbf{A}_\eta^{[\leqslant m]}(\Delta)\cdot\boldsymbol{\omega}, \quad O_{5,\eta}^{[\leqslant m]}(\Delta) := \frac{1}{2}\boldsymbol{\tau}\cdot\boldsymbol{\omega}, \\ O_{6,\eta}^{[\leqslant m]}(\Delta) := \frac{1}{2}\Big(\mathbf{A}_\eta^{[\leqslant m]}(\Delta)\cdot\boldsymbol{\omega}\Big)\Big(\boldsymbol{\tau}\cdot\boldsymbol{\omega}\Big), \quad O_{7,\eta}^{[\leqslant m]}(\Delta) := \frac{1}{2}\Big(\mathbf{A}_\eta^{[\leqslant m]}(\Delta)\cdot\mathbf{A}_\eta^{[\leqslant m]}(\Delta)\Big)\Big(\boldsymbol{\tau}\cdot\boldsymbol{\omega}\Big). \end{gathered} \tag{II.6.5}$$

We proceed as in section II.5. For $m = 0$, we write $W_h^{[m]}(\psi^{[\leqslant 0]})$ as in (II.6.4) with

$$\begin{aligned} &C = \cosh(\tilde h), \quad \ell_0^{[0]} = \frac{1}{C}\frac{\lambda_0}{\tilde h}\sinh(\tilde h), \quad \ell_1^{[0]} = \frac{1}{C}\frac{\lambda_0^2}{12\tilde h}(\tilde h\cosh(\tilde h) + 2\sinh(\tilde h)) \\ &\ell_4^{[0]} = \frac{1}{C}\lambda_0\sinh(\tilde h), \quad \ell_5^{[0]} = \frac{2}{C}\sinh(\tilde h), \quad \ell_6^{[0]} = \frac{1}{C}\frac{\lambda_0}{\tilde h}(\tilde h\cosh(\tilde h) - \sinh(\tilde h)) \\ &\ell_7^{[0]} = \frac{1}{C}\frac{\lambda_0^2}{12\tilde h^2}(\tilde h^2\sinh(\tilde h) + 2\tilde h\cosh(\tilde h) - 2\sinh(\tilde h)) \end{aligned} \tag{II.6.6}$$

where $\tilde h := h/2$ (see appendix II.A4 for a proof).

For $m < 0$, the computation follows the same general lines as in section II.5, although the computation is longer, but can be performed easily using a computer ([Ja15], see appendix II.A6). The result of the computation is a map $\widetilde{\mathcal{R}}$ which maps $\ell_{n,h}^{[m]}$ to $\ell_{n,h}^{[m-1]}$, as well as the expression for the constant $C_h^{[m]}$. Their explicit expression is somewhat long, and is deferred to (II.A2.1).

By (II.5.14) and (II.6.3), we rewrite (II.6.2) as

$$\chi(h,\beta) = \sum_{m=-N_\beta}^{0} 2^m \Big(\frac{\partial_h^2 C_h^{[m]}}{C_h^{[m]}} - \frac{(\partial_h C_h^{[m]})^2}{(C_h^{[m]})^2} \Big). \tag{II.6.7}$$

In addition, the derivatives of $C_h^{[m]}$ can be computed exactly using the flow in (II.A2.1): indeed $\partial_h C_h^{[m]} = \partial_{\boldsymbol{\ell}} C_h^{[m]} \cdot \partial_h \boldsymbol{\ell}_h^{[m]}$ and similarly for $\partial_h^2 C_h^{[m]}$, and $\partial_h \boldsymbol{\ell}_h^{[m]}$ can be computed inductively by deriving $\widetilde{\mathcal{R}}(\boldsymbol{\ell})$:

$$\partial_h \boldsymbol{\ell}_h^{[m-1]} = \partial_{\boldsymbol{\ell}} \widetilde{\mathcal{R}}(\boldsymbol{\ell}_h^{[m]}) \cdot \partial_h \boldsymbol{\ell}_h^{[m]}, \tag{II.6.8}$$

and similarly for $\partial_h^2 \boldsymbol{\ell}_h^{[m]}$. Therefore, using (II.A2.1) and its derivatives, we can inductively compute $\chi(\beta, h)$.

By a numerical study which produces results that are stable and clear we find the following results.

1 - If $\lambda_0 \equiv \ell_0 < 0$ and $h = 0$, then the flow tends to a nontrivial, λ-independent, fixed point $\boldsymbol{\ell}^*$ (see figure II.6.1).

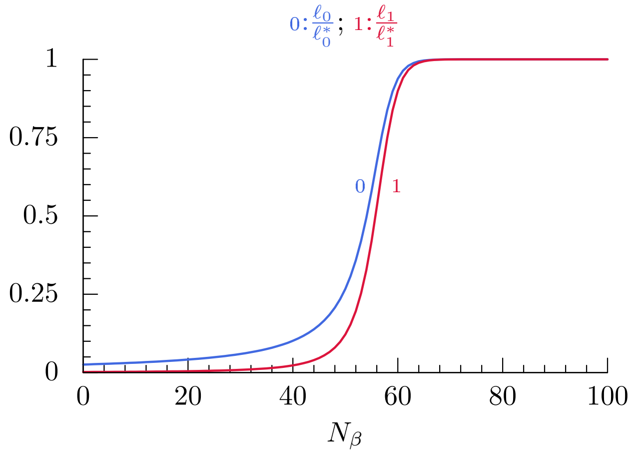

fig II.6.1: Plot of $\frac{\boldsymbol{\ell}}{\boldsymbol{\ell}^*}$ as a function of the iteration step N_β for $\lambda_0 \equiv \ell_0 = -0.02$. The *marginal* coupling ℓ_0 (number 0, blue) tends to its fixed value, closely followed by the irrelevant coupling ℓ_1 (number 1, red).

We define $n_j(\lambda_0)$ for $j = 0, 1$ as the step of the flow at which the right-discrete derivative of ℓ_j/ℓ_j^* with respect to the step N_β is largest. The reason for this definition is that, as λ_0 tends to 0, the flow of ℓ_j tends to a step function, so that for each component j the scale n_j is a good measure of the number of iterations needed for that component to reach its fixed value. The *Kondo temperature* β_K is defined as $2^{n_0(\lambda_0)}$, and is the temperature at which the non-trivial fixed point is reached by all components. For small λ_0, we find that (see figure II.A5.1), for $j = 0, 1, 3$,

$$n_j(\lambda_0) = c_0 |\lambda_0|^{-1} + O(1), \quad c_0 \approx 1. \tag{II.6.9}$$

2 - There are two fixed points: $\boldsymbol{\ell}^*$ and $\boldsymbol{\ell}_0^* := (0, 0)$ (see figure II.6.2).

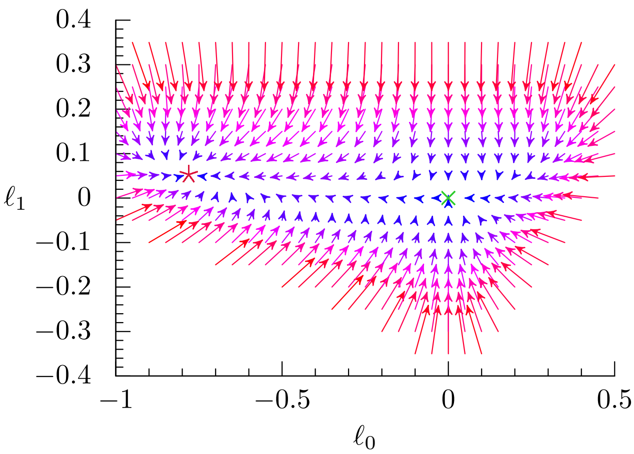

fig II.6.2: Phase diagram of the flow. There are two fixed points: $\boldsymbol{\ell}^*$ (which is linearly stable and represented by a red star) and $\boldsymbol{\ell}_0^*$ (which has one linearly unstable direction and one quadratically marginal and is represented by a green cross).

When the running coupling constants are at $\boldsymbol{\ell}^*$, the susceptibility remains finite as $\beta \to \infty$ and positive, whereas when they are at $\boldsymbol{\ell}_0^*$, it grows linearly with β.

In addition, when $\lambda_0 < 0$ the flow escapes along the unstable direction towards $\boldsymbol{\ell}^*$ after $n_K(\lambda_0)$ steps. The susceptibility is therefore finite for $\lambda_0 < 0$ (see figure II.6.3 (which may be compared to the exact solution [AFL83, figure 3])).

If $\lambda_0 > 0$, then the flow approaches $\boldsymbol{\ell}_0^*$ from the $\lambda_0 > 0$ side, which is marginally stable, so the flow never leaves the vicinity of $\boldsymbol{\ell}_0^*$ and the susceptibility diverges as $\beta \to \infty$.

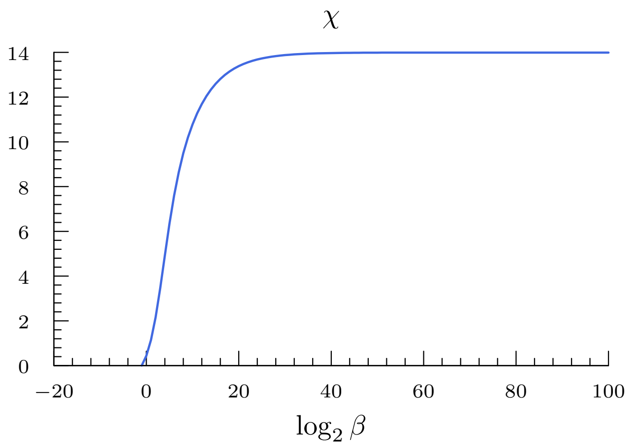

fig II.6.3: plot of $\chi(\beta, 0)$ as a function of $\log_2 \beta$ for $\lambda_0 = -0.28$.

3 - We now discuss the flow at $h > 0$ and address the question of continuity of the susceptibility in h as $h \to 0$. If $\lambda_0 < 0$ and $h \ll \beta_K^{-1} = 2^{-n_K(\lambda_0)}$, ℓ_0, ℓ_1 first behave similarly to the $h = 0$ case and tend to the same fixed point $\boldsymbol{\ell}^*$ and stay there until ℓ_4 through ℓ_7 become large enough, after which the flow tends to a fixed point in which, $\ell_5 = 2$ and $\ell_j = 0$ for $j \neq 5$ (see figure II.6.4).

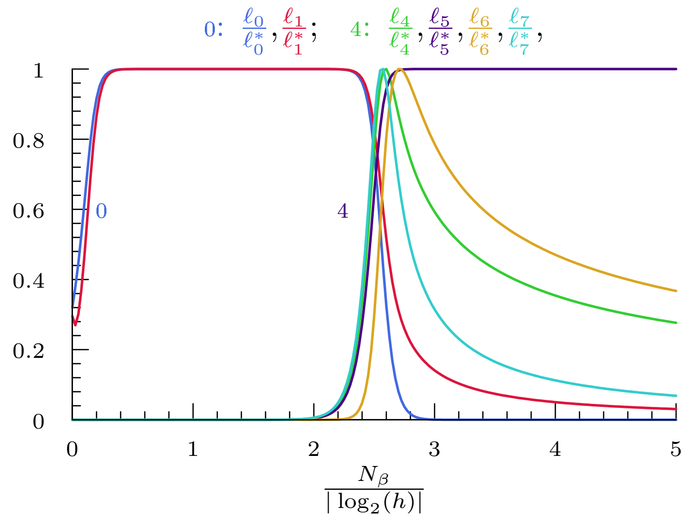

fig II.6.4: plot of $\frac{\boldsymbol{\ell}}{\boldsymbol{\ell}^*}$ as a function of the iteration step N_β for $\lambda_0 = -0.25$ and $h = 2^{-40}$. Here ℓ_0^*, ℓ_1^* are the components of the non-trivial fixed point $\boldsymbol{\ell}^*$ and ℓ_4^* through ℓ_7^* are the values reached by ℓ_4 through ℓ_7 of largest absolute value. The flow behaves similarly to that at $h = 0$ until ℓ_4 through ℓ_7 become large, at which point the couplings decay to 0, except for ℓ_5.

Setting the initial conditions for the flow as $\ell_j = \ell_j^*$ for $j = 0, 1$ and $\ell_5 = 2\tanh(\tilde{h})$, we define $r_j(h)$ for $j = 0, 1, 4, 5, 6, 7$ as the step of the flow at which the discrete derivative of ℓ_j/ℓ_j^* is respectively smallest (that is most negative) and largest. Thus $r_j(h)$ measures when the flow leaves $\boldsymbol{\ell}^*$. We find that (see figure II.A5.2) for small h,

$$r_j(h) = c_r \log_2 h^{-1} + O(1), \quad c_r \approx 2.6. \tag{II.6.10}$$

Note that the previous picture only holds if $r_j(h) \gg \log_2(\beta_K)$, that is $\beta_K h \ll 1$.

The susceptibility at $0 < h \ll \beta_K^{-1}$ is continuous in h as $h \to 0$ (see figure II.6.5). This, combined with the discussion in point (2) above, implies that the hierarchical $s-d$ model exhibits a Kondo effect.

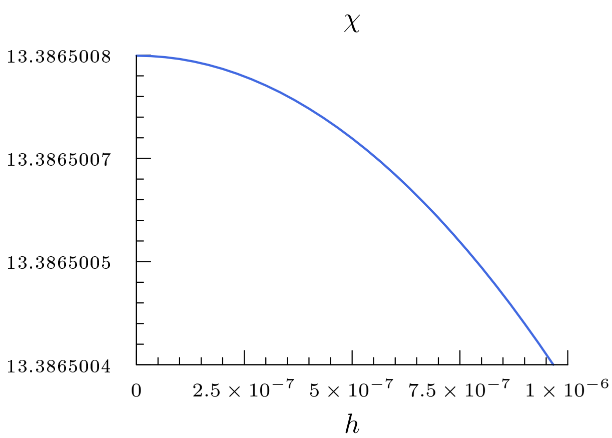

fig II.6.5: plot of $\chi(\beta, h)$ for $h \leqslant 10^{-6}$ at $\lambda_0 = -0.28$ and $\beta = 2^{20}$ (so that the largest value for βh is ~ 1).

II.7. Concluding remarks

1 - The hierarchical $s-d$ model defined in section II.4 is a well defined statistical mechanics model, for which the partition function and correlation functions are unambiguously defined and finite as long as β is finite. In addition, since the magnetic susceptibility of the impurity can be rewritten as a correlation function:

$$\chi(\beta,0)=\beta\left\langle(\boldsymbol{\omega}\cdot\boldsymbol{\tau})^2\right\rangle_{h=0}\,, \tag{II.7.1}$$

$\chi(\beta,0)$ is a thermodynamical quantity of the model.

2 - The qualitative behavior of the renormalization group flow is unchanged if all but the relevant and marginal running coupling constants (*i.e.* six constants out of nine) of the beta functions of section II.4,II.5 are neglected (*i.e.* set to 0 at every step of the iteration). In particular, we still find a Kondo effect.

3 - In the hierarchical model defined in section II.4, quantities other than the magnetic susceptibility of the impurity can be computed, although all observables must only involve fields localized at $x=0$. For instance, the response to a magnetic field acting on all sites of the Fermionic chain as well as the impurity cannot be investigated in this model, since the sites of the chain with $x\neq 0$ are not accounted for.

3-1 - We have attempted to extend the definition of the hierarchical model to allow observables on the sites of the chain at $x\neq 0$, by paving the space-time plane with square boxes (instead of paving the time axis with intervals, see section II.4), defining hierarchical fields for each quarter box and postulating a propagator between them by analogy with the non-hierarchical model. The magnetic susceptibility of the impurity is defined as the response to a magnetic field acting on every site of the chain and on the impurity, to which the susceptibility of the non-interacting chain is subtracted. We have found, iterating the flow numerically, that for such a model *there is no Kondo effect*, that is the impurity susceptibility diverges as β when $\beta\to\infty$.

3-2 - A second approach has yielded better results, although it is not completely satisfactory. The idea is to incorporate the effect of the magnetic field h acting on the Fermionic chain into the propagator of the non-hierarchical model, after which the potential V only depends on the site at $x=0$, so that the hierarchical model can be defined in the same way as in section II.4 but with *an h-dependent propagator*. In this model, we have found that *there is a Kondo effect*.

Appendices

II.A1. Some identities.

In this appendix, we state three relations used to compute the flow equation (II.5.13), which follow from a patient algebraic meditation:

$$\begin{aligned} \left\langle A_1^{j_1} A_2^{j_2} \right\rangle &= \delta_{j_1,j_2}\Big(2+\frac{1}{3}\mathbf{a}^2\Big) - 2\,a^{j_1,j_2}\delta_{j_1\neq j_2}\, s_{t_2,t_1} \\ \left\langle A_1^{j_1} A_1^{j_2} A_2^{j_3} \right\rangle &= 2\,a^{j_3}\,\delta_{j_1,j_2} \\ \left\langle A_1^{j_1} A_1^{j_2} A_2^{j_3} A_2^{j_4} \right\rangle &= 4\delta_{j_1,j_2}\delta_{j_3,j_4} \end{aligned} \tag{II.A1.1}$$

where the lower case $\mathbf{a}$ denotes $\langle\,\mathbf{A}_1\,\rangle \equiv \langle\,\mathbf{A}_2\,\rangle$ and $a^{j_1,j_2} = \langle\,\psi_1^+\sigma^{j_1}\sigma^{j_2}\psi_1^-\,\rangle = \langle\,\psi_2^+\sigma^{j_1}\sigma^{j_2}\psi_2^-\,\rangle$.

II.A2. Complete beta function

The beta function for the flow described in section II.6 is

$$\begin{aligned} C^{[m]} &= 1+\frac{3}{2}\ell_0^2+\ell_0\ell_6+9\ell_1^2+\frac{\ell_4^2}{2}+\frac{\ell_5^2}{4}+\frac{\ell_6^2}{2}+9\ell_7^2 \\ \ell_0^{[m-1]} &= \frac{1}{C^{[m]}}\left(\ell_0-\ell_0^2+3\ell_0\ell_1-\ell_0\ell_6\right) \\ \ell_1^{[m-1]} &= \frac{1}{C^{[m]}}\left(\frac{\ell_1}{2}+\frac{\ell_0^2}{8}+\frac{\ell_0\ell_6}{12}+\frac{\ell_4^2}{24}+\frac{\ell_5\ell_7}{4}+\frac{\ell_6^2}{24}\right) \\ \ell_4^{[m-1]} &= \frac{1}{C^{[m]}}\left(\ell_4+\frac{\ell_0\ell_5}{2}+3\ell_0\ell_7+3\ell_1\ell_4+\frac{\ell_5\ell_6}{2}+3\ell_6\ell_7\right) \\ \ell_5^{[m-1]} &= \frac{1}{C^{[m]}}\left(2\ell_5+2\ell_0\ell_4+36\ell_1\ell_7+2\ell_4\ell_6\right) \\ \ell_6^{[m-1]} &= \frac{1}{C^{[m]}}\left(\ell_6+\ell_0\ell_6+3\ell_1\ell_6+\frac{\ell_4\ell_5}{2}+3\ell_4\ell_7\right) \\ \ell_7^{[m-1]} &= \frac{1}{C^{[m]}}\left(\frac{\ell_7}{2}+\frac{\ell_0\ell_4}{12}+\frac{\ell_1\ell_5}{4}+\frac{\ell_4\ell_6}{12}\right) \end{aligned} \tag{II.A2.1}$$

in which we dropped the $^{[m]}$ exponent on the right side. By considering the linearized flow equation (around $\ell_j = 0$), we find that ℓ_0, ℓ_4, ℓ_6 are *marginal*, ℓ_2 *relevant* and ℓ_1, ℓ_7 *irrelevant*. The consequent linear flow is *very different* from the full flow discussed in section II.6.

II.A3. Fixed points at $h = 0$

We first compute the fixed points of (II.5.13). If $\ell_0 = 0$, then $\ell_1 = 0$. If $\ell_0 \neq 0$, then the equation for the fixed point of (II.5.13) becomes

$$\left\{ \begin{aligned} &3\ell_0^2+2\ell_0+6\ell_1(3\ell_1-1)=0 \\ &\ell_1(1+18\ell_1^2)+\ell_0^2(3\ell_1-\tfrac{1}{4})=0. \end{aligned} \right. \tag{II.A3.1}$$

In particular, $\ell_1(1-12\ell_1)>0$, so that

$$\ell_0 = \pm 2\sqrt{\frac{\ell_1(1+18\ell_1^2)}{1-12\ell_1}} \tag{II.A3.2}$$

which we inject into (II.A3.1) to find that $\ell_0<0$ and

$$1-\frac{35}{4}(3\ell_1)+\frac{27}{2}(3\ell_1)^2-\frac{19}{4}(3\ell_1)^3+107(3\ell_1)^4=0. \tag{II.A3.3}$$

Finally, we notice that $\frac{1}{12}$ is a solution of (II.A3.3), which implies that

$$4-19(3\ell_1)-22(3\ell_1)^2-107(3\ell_1)^3=0 \tag{II.A3.4}$$

which has a unique real solution. Finally, we find that if ℓ_1 satisfies (II.A3.4), then

$$2\sqrt{\frac{\ell_1(1+18\ell_1^2)}{1-12\ell_1}} = 3\ell_1\frac{1+15\ell_1}{1-12\ell_1}. \tag{II.A3.5}$$

We have therefore shown that (II.5.13) has two fixed points:

$$\boldsymbol{\ell}_0^* := (0,0), \quad \boldsymbol{\ell}^* := \left(-x_0\frac{1+5x_0}{1-4x_0}, \frac{x_0}{3}\right). \tag{II.A3.6}$$

In addition, it follows from (II.5.13) that, if $\lambda_0<0$, then (recall that $\ell_0^{[0]}=\lambda_0$ and $\ell_1^{[0]}=\lambda_0^2/2$)

$$\ell_0^{[m]}<0, \quad 0\leqslant \ell_1^{[m]}<\frac{1}{12} \tag{II.A3.7}$$

for all $m\leqslant 0$, which implies that the set $\{\boldsymbol{\ell}\mid \ell_0<0,\ \ell_1\geqslant 0\}$ is stable under the flow. In addition, if $\ell_0^{[m]}>-\frac{2}{3}$, then $\ell_0^{[m-1]}<\ell_0^{[m]}$, so that the flow cannot converge to $\boldsymbol{\ell}_0^*$. Therefore if the flow converges, then it converges to $\boldsymbol{\ell}^*$.

II.A4. Initial condition for the flow

In this appendix, we prove (II.6.6), starting from (II.5.8):

$$W_h^{[0]} = \prod_{\Delta\in\mathcal{Q}_0}\prod_{\eta=\pm}\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left(\lambda_0 O_{0,\eta}^{[0]}(\Delta)+hO_{5,\eta}^{[0]}(\Delta)\right)^n\right) \tag{II.A4.1}$$

In order to alleviate the notation, we will drop all $_\eta$, $^{[0]}$ and (Δ). We write

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\lambda_0O_0+hO_5\right)^n = &\sum_{n=0}^{\infty}\frac{h^n}{n!}O_5^n\\ &+\sum_{n=1}^{\infty}\frac{\lambda_0h^{n-1}}{n!}\sum_{k=0}^{n-1}O_5^kO_0O_5^{n-1-k}\\ &+\sum_{n=2}^{\infty}\frac{\lambda_0^2h^{n-2}}{n!}\sum_{k_1=0}^{n-2}\sum_{k_2=0}^{n-k_1-2}O_5^{k_1}O_0O_5^{k_2}O_0O_5^{n-2-k_1-k_2}\end{aligned} \tag{II.A4.2}$$

We have

$$\begin{aligned}&O_0^2=\tfrac{1}{2}O_1, \quad O_5^2=\tfrac{1}{4}, \quad O_0O_5+O_5O_0=O_4, \quad O_1O_5=O_5O_1=\tfrac{1}{2}O_7,\\ &O_0O_4=O_4O_0=\tfrac{1}{6}O_7, \quad O_0O_6=O_6O_0=\tfrac{1}{6}O_1, \quad O_5O_7=O_7O_5=\tfrac{1}{2}O_1\end{aligned} \tag{II.A4.3}$$

We will now use (II.A4.3) to compute each term in (II.A4.2)

1 - By (II.A4.3), if n is even, then $O_5^n = 1/2^n$, and if n is odd, then $O_5^n = O_5/2^{n-1}$. Therefore

$$\sum_{n=0}^{\infty} \frac{h^n}{n!} O_5^n = \cosh(\tilde{h}) + 2\sinh(\tilde{h}) O_5 \tag{II.A4.4}$$

where $\tilde{h} := h/2$.

2 - We now turn our attention to $O_5^k O_0 O_5^{n-1-k}$.

- If n and k are even, then $O_5^k O_0 O_5^{n-1-k} = O_0 O_5/2^{n-2}$.
- If n is even and k is odd, then $O_5^k O_0 O_5^{n-1-k} = O_5 O_0/2^{n-2}$.
- If n is odd and k is even, then $O_5^k O_0 O_5^{n-1-k} = O_0/2^{n-1}$.
- If n and k are odd, then $O_5^k O_0 O_5^{n-1-k} = O_5 O_0 O_5/2^{n-3} = -O_0/2^{n-1} + O_6/2^{n-2}$.

Therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\lambda_0 h^{n-1}}{n!} \sum_{k=0}^{n-1} O_5^k O_0 O_5^{n-1-k} &= \sum_{n \text{ odd}} \frac{\lambda_0 h^{n-1}}{2^{n-1} n!} (O_0 + (n-1) O_6) \\
&\quad + \sum_{n \geqslant 2 \text{ even}} \frac{\lambda_0 h^{n-1}}{2^{n-1}(n-1)!} O_4 \\
&= \lambda_0 \left(\cosh(\tilde{h}) O_6 + \frac{\sinh(\tilde{h})}{\tilde{h}} (O_0 - O_6) + \sinh(\tilde{h}) O_4 \right).
\end{aligned} \tag{II.A4.5}$$

3 - Finally, we compute $O_5^{k_1} O_0 O_5^{k_2} O_0 O_5^{n-2-k_1-k_2} =: X_{n,k_1,k_2}$.

- If n is even, k_1 is even and k_2 is even, then $X_{n,k_1,k_2} = O_1/2^{n-1}$.
- If n is even, k_1 is odd and k_2 is even, then $X_{n,k_1,k_2} = O_1/2^{n-1}$.
- If n is even, k_1 is even and k_2 is odd, then $X_{n,k_1,k_2} = -O_1/(3 \cdot 2^{n-1})$.
- If n is even, k_1 is odd and k_2 is odd, then $X_{n,k_1,k_2} = -O_1/(3 \cdot 2^{n-1})$.
- If n is odd, k_1 is even and k_2 is even, then $X_{n,k_1,k_2} = O_7/2^{n-1}$.
- If n is odd, k_1 is odd and k_2 is even, then $X_{n,k_1,k_2} = O_7/2^{n-1}$.
- If n is odd, k_1 is even and k_2 is odd, then $X_{n,k_1,k_2} = -O_7/(3 \cdot 2^{n-1})$.
- If n is odd, k_1 is odd and k_2 is odd, then $X_{n,k_1,k_2} = -O_7/(3 \cdot 2^{n-1})$.

Therefore

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{\lambda_0^2 h^{n-2}}{n!} \sum_{k_1=0}^{n-2} \sum_{k_2=0}^{n-k_1-2} X_{n,k_1,k_2} &= \sum_{n \geqslant 2 \text{ even}} \frac{\lambda_0^2 h^{n-2}}{3 \cdot 2^n n!} n(n+1) O_1 + \sum_{n \geqslant 3 \text{ odd}} \frac{\lambda_0^2 h^{n-2}}{3 \cdot 2^n n!} (n-1)(n+2) O_7 \\
&= \frac{\lambda_0^2}{12} \left(\left(\cosh(\tilde{h}) + 2\frac{\sinh(\tilde{h})}{\tilde{h}} \right) O_1 + \left(\sinh(\tilde{h}) + \frac{2}{\tilde{h}} \left(\cosh(\tilde{h}) - \frac{\sinh(\tilde{h})}{\tilde{h}} \right) \right) O_7 \right).
\end{aligned} \tag{II.A4.6}$$

This concludes the proof of (II.6.6)

II.A5. Asymptotic behavior of $n_j(\lambda_0)$ and $r_j(h)$

In this appendix, we show plots to support the claims on the asymptotic behavior of $n_j(\lambda_0)$ (see (II.6.9), figure II.A5.1) and $r_j(h)$ (see (II.6.10), figure II.A5.2). The plots below have error bars which are due to the fact that $n_j(\lambda_0)$ and $r_j(h)$ are integers, so their value could be off by ± 1.

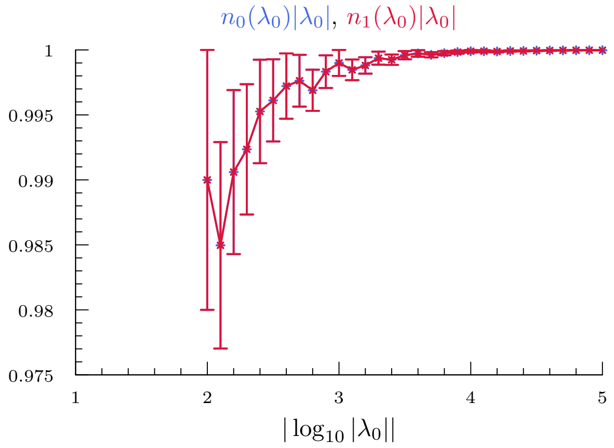

fig II.A5.1: plot of $n_j(\lambda_0)|\lambda_0|$ for $j=0$ (blue, color online) and $j=1$ (red) as a function of $|\log_{10}|\lambda_0||$. This plot confirms (II.6.9).

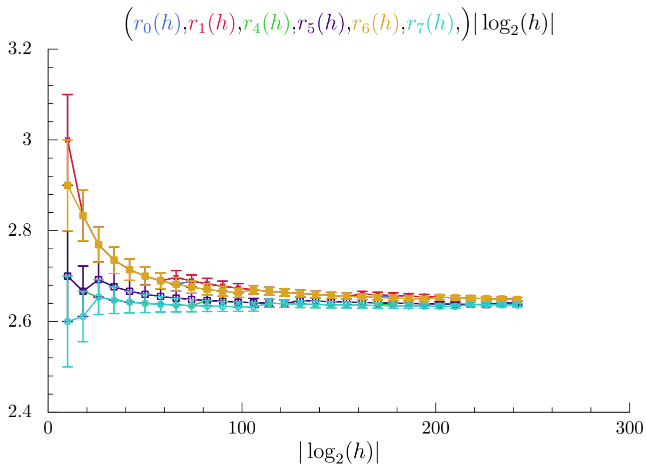

fig II.A5.2: plot of $r_j(h)|\log_2(h)||$ as a function of $|\log_2(h)|$. This plot confirms (II.6.10).

II.A6. `meankondo`: a computer program to compute flow equations

The computation of the flow equation (II.A2.1) is quite long, but elementary, which makes it ideally suited for a computer. We therefore wrote a program, called `meankondo` [Ja15], and used it to carry out this computation. One interesting feature of `meankondo` is that it has been designed in a *model-agnostic* way, that is, unlike its name might indicate, it is not specific to the $s-d$ model and can be used to compute and manipulate flow equations for a wide variety of Fermionic hierarchical models. It may therefore be useful to anyone studying such models, so we have thoroughly documented its features and released the source code under an Apache 2.0 license. See `http://ian.jauslin.org/software/meankondo` for details.

References

[An61] P.W. Anderson - *Localized magnetic states in metals*, Physical Review, Vol. 124, n. 1, p. 41-53, 1961, doi:10.1103/PhysRev.124.41.

[AY69] P.W. Anderson, G. Yuval - *Exact results in the Kondo problem: equivalence to a classical one-dimensional Coulomb gas*, Physical Review Letters, Vol. 23, n. 2, p. 89-92, 1969, doi:10.1103/PhysRevLett.23.89.

[An70] P.W. Anderson - *A poor man's derivation of scaling laws for the Kondo problem*, Journal of Physics C: Solid State Physics, Vol. 3, p. 2436, 1970, doi:10.1088/0022-3719/3/12/008.

[AYH70] P.W. Anderson, G. Yuval, D.R. Hamann - *Exact results in the Kondo problem - II. Scaling theory, qualitatively correct solution, and some new results on one-dimensional classical statistical mechanics*, Physical Review B, Vol. 1, n. 11, p. 4464-4473, 1970, doi:10.1103/PhysRevB.1.4464.

[An80] N. Andrei - *Diagonalization of the Kondo Hamiltonian*, Physical Review Letters, Vol. 45, n. 5, 1980, doi:10.1103/PhysRevLett.45.379.

[AFL83] N. Andrei, K. Furuya, J.H. Lowenstein - *Solution of the Kondo problem*, Reviews of Modern Physics, Vol. 55, n. 2, p. 331-402, 1983, doi:10.1103/RevModPhys.55.331.

[BG90] G. Benfatto, G. Gallavotti - *Perturbation theory of the Fermi surface in a quantum liquid - a general quasiparticle formalism and one-dimensional systems*, Journal of Statistical Physics, Vol. 59, n. 3-4, p. 541-664, 1990, doi:10.1007/BF01025844.

[BGe94] G. Benfatto, G. Gallavotti, A.Procacci, B. Scoppola - *Beta function and Schwinger functions for a many Fermions system in one dimension - Anomaly of the Fermi surface*, Communications in Mathematical Physics, Vol. 160, p. 93-171, 1994, doi:10.1007/BF02099791.

[BGJ15] G. Benfatto, G. Gallavotti, I. Jauslin - *Kondo effect in a Fermionic hierarchical model*, to appear in the Journal of Statistical Physics, 2015, doi:10.1007/s10955-015-1378-7.

[Do91] T.C. Dorlas - *Renormalization group analysis of a simple hierarchical fermion model*, Communications in Mathematical Physics, Vol. 136, p. 169-194, 1991, doi:10.1007/BF02096796.

[Dy69] F.J. Dyson - *Existence of a phase-transition in a one-dimensional Ising ferromagnet*, Communications in Mathematical Physics, Vol. 12, p. 91-107, 1969, doi:10.1007/BF01645907.

[GJ15] G. Gallavotti, I. Jauslin - *Kondo effect in the hierarchical $s-d$ model*, to appear in the Journal of Statistical Physics, 2015, doi:10.1007/s10955-015-1370-2.

[Ja15] `meankondo`, developped by I. Jauslin, http://ian.jauslin.org/software/meankondo

[Ki76] C. Kittel - *Introduction to solid state physics*, Wiley&Sons, 1976.

[Ko64] J. Kondo - *Resistance minimum in dilute magnetic alloys*, Progress of Theoretical Physics, Vol. 32, n. 1, 1964, doi:10.1143/PTP.32.37.

[Ko05] J. Kondo - *Sticking to my bush*, Journal of the Physical Society of Japan, Vol. 74, n. 1, p. 1-3, 2005, doi:10.1143/JPSJ.74.1.

[No74] P. Nozières - *A "Fermi-liquid" description of the Kondo problem at low temperatures*, Journal of Low Temperature Physics, Vol. 17, n. 1-2, p. 31-42, 1974, doi:10.1007/BF00654541.

[Sh94] R. Shankar - *Renormalization group approach to interacting fermions*, Reviews of Modern Physics, Vol. 66, n. 1, p. 129-192, 1994, doi:10.1103/RevModPhys.66.129.

[Wi65] K.G. Wilson - *Model Hamiltonians for Local Quantum Field Theory*, Physical Review, Vol. 140, n. 2B, p. B445-B457, 1965, doi:10.1103/PhysRev.140.B445.

[Wi70] K.G. Wilson - *Model of coupling-constant renormalization*, Physical Review D, Vol. 2, n. 8, p. 1438-1472, 1970, doi:10.1103/PhysRevD.2.1438.

[Wi75] K.G. Wilson - *The renormalization group: Critical phenomena and the Kondo problem*, Reviews of Modern Physics, Vol. 47, n. 4, 1975, doi:10.1103/RevModPhys.47.773.