Kondo effect in the hierarchical $s-d$ model

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The $s-d$ model describes a chain of spin-1/2 electrons interacting magnetically with a two-level impurity. It was introduced to study the Kondo effect, in which the magnetic susceptibility of the impurity remains finite in the 0-temperature limit as long as the interaction of the impurity with the electrons is anti-ferromagnetic. A variant of this model was introduced by Andrei, which he proved was exactly solvable via Bethe Ansatz. A hierarchical version of Andrei's model was studied by Benfatto and the authors. In the present letter, that discussion is extended to a hierarchical version of the $s-d$ model. The resulting analysis is very similar to the hierarchical Andrei model, though the result is slightly simpler.

The $s-d$ model was introduced by Anderson [An61] and used by Kondo [Ko64] to study what would subsequently be called the *Kondo effect*. It describes a chain of electrons interacting with a fixed spin-1/2 magnetic impurity. One of the manifestations of the effect is that when the coupling is anti-ferrmoagnetic, the magnetic susceptibility of the impurity remains finite in the 0-temperature limit, whereas it diverges for ferromagnetic and for vanishing interactions.

A modified version of the $s-d$ model was introduced by Andrei [An80], which was shown to be exactly solvable by Bethe Ansatz. In [BGJ15], a hierarchical version of Andrei's model was introduced and shown to exhibit a Kondo effect. In the present letter, we show how the argument can be adapted to the $s-d$ model.

We will show that in the hierarchical $s-d$ model, the computation of the susceptibility reduces to iterating an *explicit* map relating 6 *running coupling constants* (rccs), and that this map can be obtained by restricting the flow equation for the hierarchical Andrei model [BGJ15] to one of its invariant manifolds. The physics of both models are therefore very closely related, as had already been argued in [BGJ15]. This is particularly noteworthy since, at 0-field, the flow in the hierarchical Andrei model is relevant, whereas it is marginal in the hierarchical $s-d$ model, which shows that the relevant direction carries little to no physical significance.

The $s-d$ model [Ko64] represents a chain of non-interacting spin-1/2 fermions, called *electrons*, which interact with an isolated spin-1/2 *impurity* located at site 0. The Hilbert space of the system is $\mathcal{F}_L\otimes\mathbb{C}^2$ in which $\mathcal{F}_L$ is the Fock space of a length-L chain of spin-1/2 fermions (the electrons) and $\mathbb{C}^2$ is the state space for the two-level impurity. The Hamiltonian, in the presence of a magnetic field of amplitude h in the direction $\boldsymbol\omega\equiv(\boldsymbol\omega_1,\boldsymbol\omega_2,\boldsymbol\omega_3)$, is

$$\begin{aligned}
H_K &= H_0+V_0+V_h =: H_0+V\\
H_0 &= \sum_{\alpha\in\{\uparrow,\downarrow\}}\sum_{x=-L/2}^{L/2-1} c^+_\alpha(x)\left(-\frac{\Delta}{2}-1\right)c^-_\alpha(x)\\
V_0 &= -\lambda_0\sum_{\substack{j=1,2,3\\ \alpha_1,\alpha_2}} c^+_{\alpha_1}(0)\sigma^j_{\alpha_1,\alpha_2}c^-_{\alpha_2}(0)\,\tau^j\\
V_h &= -h\sum_{j=1,2,3}\boldsymbol\omega_j\tau^j
\end{aligned}\tag{1}$$

where λ_0 is the interaction strength, Δ is the discrete Laplacian $c^\pm_\alpha(x)$, $\alpha=\uparrow,\downarrow$ are creation and annihilation operators acting on *electrons*, and $\sigma^j=\tau^j$, $j=1,2,3$, are Pauli matrices. The operators τ^j act on the *impurity*. The boundary conditions are taken to be periodic.

In the *Andrei model* [An80], the impurity is represented by a fermion instead of a two-level system, that is the Hilbert space is replaced by $\mathcal{F}_L \otimes \mathcal{F}_1$, and the Hamiltonian is defined by replacing τ^j in (1) by $d^+\tau^j d^-$ in which $d^\pm_\alpha(x)$, $\alpha =\uparrow,\downarrow$ are creation and annihilation operators acting on the impurity.

The partition function $Z = \mathrm{Tr}\, e^{-\beta H_K}$ can be expressed formally as a functional integral:

$$Z = \mathrm{Tr} \int P(d\psi) \sum_{n=0}^{\infty} (-1)^n \int_{0<t_1<\cdots<t_n<\beta} dt_1 \cdots dt_n \, \mathcal{V}(t_1)\cdots\mathcal{V}(t_n) \tag{2}$$

in which $\mathcal{V}(t)$ is obtained from V by replacing $c^\pm_\alpha(0)$ in (1) by a *Grassmann* field $\psi^\pm_\alpha(0,t)$, $P(d\psi)$ is a *Gaussian Grassmann measure* over the fields $\{\psi^\pm_\alpha(0,t)\}_{t,\alpha}$ whose *propagator* (*i.e. covariance*) is, in the $L\to\infty$ limit,

$$g(t,t') = \frac{1}{(2\pi)^2} \int dk dk_0 \frac{e^{ik_0(t-t')}}{ik_0 - \cos k},$$

and the trace is over the state-space of the spin-1/2 impurity, that is a trace over $\mathbb{C}^2$.

We will consider a *hierarchical* version of the $s-d$ model. The hierarchical model defined below is *inspired* by the $s-d$ model in the same way as the hierarchical model defined in [BGJ15] was inspired by the Andrei model. We will not give any details on the justification of the definition, as such considerations are entirely analogous to the discussion in [BGJ15].

The model is defined by introducing a family of *hierarchical fields* and specifying a *propagator* for each pair of fields. The average of any monomial of fields is then computed using the Wick rule.

Assuming $\beta = 2^{N_\beta}$ with $N_\beta = \log_2 \beta \in \mathbb{N}$, the time axis $[0,\beta)$ is paved with boxes (*i.e.* intervals) of size 2^{-m} for every $m \in \{0,-1,\ldots,-N_\beta\}$: let

$$\mathcal{Q}_m := \left\{ [i2^{|m|}, (i+1)2^{|m|}) \right\}_{i=0,1,\cdots,2^{N_\beta-|m|}-1}^{m=0,-1,\ldots} \tag{3}$$

Given a box $\Delta \in \mathcal{Q}_m$, let t_Δ denote the center of Δ, and given a point $t \in R$, let $\Delta^{[m]}(t)$ be the (unique) box on scale m that contains t. We further decompose each box $\Delta \in \mathcal{Q}_m$ into two *half boxes*: for $\eta \in \{-,+\}$, let

$$\Delta_\eta := \Delta^{[m+1]}(t_\Delta + \eta 2^{-m-2}) \tag{4}$$

for $m \le 0$. Thus Δ_- can be called the "lower half" of Δ and Δ_+ the "upper half".

The elementary fields used to define the hierarchical $s-d$ model will be *constant on each half-box* and will be denoted by $\psi^{[m]\pm}_\alpha(\Delta_\eta)$ for $m \in \{0,-1,\cdots,-N_\beta\}$, $\Delta \in \mathcal{Q}_m$, $\eta \in \{-,+\}$, $\alpha \in \{\uparrow,\downarrow\}$.

The propagator of the hierarchical $s-d$ model is defined as

$$\left\langle \psi^{[m]-}_\alpha(\Delta_{-\eta}) \psi^{[m]+}_\alpha(\Delta_\eta) \right\rangle := \eta \tag{5}$$

for $m \in \{0,-1,\cdots,-N_\beta\}$, $\Delta \in \mathcal{Q}_m$, $\eta \in \{-,+\}$, $\alpha \in \{\uparrow,\downarrow\}$. The propagator of any other pair of fields is set to 0.

Finally, we define

$$\psi^\pm_\alpha(t) := \sum_{m=0}^{-N_\beta} 2^{\frac{m}{2}} \psi^{[m]\pm}_\alpha(\Delta^{[m+1]}(t)). \tag{6}$$

The partition function for the hierarchical $s-d$ model is

$$Z = \mathrm{Tr} \left\langle \sum_{n=0}^{\infty} (-1)^n \int_{0<t_1<\cdots<t_n<\beta} dt_1 \cdots dt_n \, \mathcal{V}(t_1)\cdots\mathcal{V}(t_n) \right\rangle \tag{7}$$

in which the $\psi^\pm_\alpha(0,t)$ in $\mathcal V(t)$ have been replaced by the $\psi^\pm_\alpha(t)$ defined in (6):

$$\mathcal V(t):=-\lambda_0\sum_{\substack{j=1,2,3\\ \alpha_1,\alpha_2}}\psi^+_{\alpha_1}(t)\sigma^j_{\alpha_1,\alpha_2}\psi^-_{\alpha_2}(t)\,\tau^j-h\sum_{j=1,2,3}\boldsymbol\omega_j\tau^j. \tag{8}$$

This concludes the definition of the hierarchical $s-d$ model.

We will now show how to compute the partition function (7) using a renormalization group iteration. We first rewrite

$$\sum_{n=0}^\infty(-1)^n\int_{0<t_1<\cdots<t_n<\beta}dt_1\cdots dt_n\,\mathcal V(t_1)\cdots\mathcal V(t_n)=\prod_{\Delta\in\mathcal Q_0}\prod_{\eta=\pm}\left(\sum_{n=0}^\infty\frac{(-1)^n}{2^nn!}\mathcal V(t_{\Delta_\eta})^n\right) \tag{9}$$

and find that

$$\sum_{n=0}^\infty\frac{(-1)^n}{2^nn!}\mathcal V(t_{\Delta^{[0]}_\eta})^n=C\left(1+\sum_p\ell_p^{[0]}O^{[\le0]}_{p,\eta}(\Delta^{[0]})\right) \tag{10}$$

with

$$\begin{gathered}O^{[\le0]}_{0,\eta}(\Delta):=\tfrac12\mathbf A^{[\le0]}_\eta(\Delta)\cdot\boldsymbol\tau,\quad O^{[\le0]}_{1,\eta}(\Delta):=\tfrac12\mathbf A^{[\le0]}_\eta(\Delta)^2,\\ O^{[\le0]}_{4,\eta}(\Delta):=\tfrac12\mathbf A^{[\le0]}_\eta(\Delta)\cdot\boldsymbol\omega,\quad O^{[\le0]}_{5,\eta}(\Delta):=\tfrac12\boldsymbol\tau\cdot\boldsymbol\omega,\\ O^{[\le0]}_{6,\eta}(\Delta):=\tfrac12(\mathbf A^{[\le0]}_\eta(\Delta)\cdot\boldsymbol\omega)(\boldsymbol\tau\cdot\boldsymbol\omega),\quad O^{[\le0]}_{7,\eta}(\Delta):=\tfrac12(\mathbf A^{[\le0]}_\eta(\Delta)^2)(\boldsymbol\tau\cdot\boldsymbol\omega)\end{gathered} \tag{11}$$

(the numbering is meant to recall that in [BGJ15]) in which $\boldsymbol\tau=(\tau^1,\tau^2,\tau^3)$ and $\mathbf A^{[\le0]}_\eta(\Delta)$ is a vector of polynomials in the fields whose j-th component for $j\in\{1,2,3\}$ is

$$A^{[\le0]j}_\eta(\Delta):=\sum_{(\alpha,\alpha')\in\{\uparrow,\downarrow\}^2}\psi^{[\le0]+}_\alpha(\Delta_\eta)\sigma^j_{\alpha,\alpha'}\psi^{[\le0]-}_{\alpha'}(\Delta_\eta) \tag{12}$$

$\psi^{[\le0]\pm}_\alpha:=\sum_{m\le0}2^{\frac m2}\psi^{[m]\pm}_\alpha$, and

$$\begin{gathered}C=\cosh(\tilde h),\quad \ell_0^{[0]}=\frac1C\frac{\lambda_0}{\tilde h}\sinh(\tilde h),\quad \ell_1^{[0]}=\frac1C\frac{\lambda_0^2}{12\tilde h}(\tilde h\cosh(\tilde h)+2\sinh(\tilde h))\\ \ell_4^{[0]}=\frac1C\lambda_0\sinh(\tilde h),\quad \ell_5^{[0]}=\frac2C\sinh(\tilde h),\quad \ell_6^{[0]}=\frac1C\frac{\lambda_0}{\tilde h}(\tilde h\cosh(\tilde h)-\sinh(\tilde h))\\ \ell_7^{[0]}=\frac1C\frac{\lambda_0^2}{12\tilde h^2}(\tilde h^2\sinh(\tilde h)+2\tilde h\cosh(\tilde h)-2\sinh(\tilde h))\end{gathered} \tag{13}$$

in which $\tilde h:=h/2$.

By a straightforward induction, we find that the partition function (7) can be computed by defining

$$C^{[m]}\mathcal W^{[m-1]}(\Delta^{[m]}):=\left\langle\prod_\eta\left(\mathcal W^{[m]}(\Delta^{[m]}_\eta)\right)\right\rangle_m \tag{14}$$

in which $\langle\cdot\rangle_m$ denotes the average over $\psi^{[m]}$, $C^{[m]}>0$ and

$$\mathcal W^{[m-1]}(\Delta^{[m]})=1+\sum_p\ell_p^{[m]}O_p^{[\le m]}(\Delta^{[m]}) \tag{15}$$

in terms of which

$$Z=C^{2|\mathcal Q_0|}\prod_{m=-N(\beta)+1}^0(C^{[m]})^{|\mathcal Q_{m-1}|} \tag{16}$$

in which $|\mathcal{Q}_m| = 2^{N(\beta)-|m|}$ is the cardinality of $\mathcal{Q}_m$. In addition, similarly to [BGJ15], the map relating $\ell_p^{[m]}$ to $\ell_p^{[m-1]}$ and $C^{[m]}$ can be computed explicitly from (14):

$$\begin{aligned}
C^{[m]} &= 1 + \frac{3}{2}\ell_0^2 + \ell_0\ell_6 + 9\ell_1^2 + \frac{\ell_4^2}{2} + \frac{\ell_5^2}{4} + \frac{\ell_6^2}{2} + 9\ell_7^2 \\
\ell_0^{[m-1]} &= \frac{1}{C}\left(\ell_0 - \ell_0^2 + 3\ell_0\ell_1 - \ell_0\ell_6\right) \\
\ell_1^{[m-1]} &= \frac{1}{C}\left(\frac{\ell_1}{2} + \frac{\ell_0^2}{8} + \frac{\ell_0\ell_6}{12} + \frac{\ell_4^2}{24} + \frac{\ell_5\ell_7}{4} + \frac{\ell_6^2}{24}\right) \\
\ell_4^{[m-1]} &= \frac{1}{C}\left(\ell_4 + \frac{\ell_0\ell_5}{2} + 3\ell_0\ell_7 + 3\ell_1\ell_4 + \frac{\ell_5\ell_6}{2} + 3\ell_6\ell_7\right) \\
\ell_5^{[m-1]} &= \frac{1}{C}\left(2\ell_5 + 2\ell_0\ell_4 + 36\ell_1\ell_7 + 2\ell_4\ell_6\right) \\
\ell_6^{[m-1]} &= \frac{1}{C}\left(\ell_6 + \ell_0\ell_6 + 3\ell_1\ell_6 + \frac{\ell_4\ell_5}{2} + 3\ell_4\ell_7\right) \\
\ell_7^{[m-1]} &= \frac{1}{C}\left(\frac{\ell_7}{2} + \frac{\ell_0\ell_4}{12} + \frac{\ell_1\ell_5}{4} + \frac{\ell_4\ell_6}{12}\right)
\end{aligned} \tag{17}$$

in which the $^{[m]}$ have been dropped from the right hand side.

The flow equation (17) can be recovered from that of the hierarchical Andrei model studied in [BGJ15] (see in particular [BGJ15, (C1)] by restricting the flow to the invariant submanifold defined by

$$\ell_2^{[m]} = \frac{1}{3}, \quad \ell_3^{[m]} = \frac{1}{6}\ell_1^{[m]}, \quad \ell_8^{[m]} = \frac{1}{6}\ell_4^{[m]}. \tag{18}$$

This is of particular interest since $\ell_2^{[m]}$ is a relevant coupling and the fact that it plays no role in the $s-d$ model indicates that it has little to no physical relevance.

The qualitative behavior of the flow is therefore the same as that described in [BGJ15] for the hierarchical Andrei model. In particular the susceptibility, which can be computed by deriving $-\beta^{-1}\log Z$ with respect to h, remains finite in the 0-temperature limit as long as $\lambda_0 < 0$, that is as long as the interaction is anti-ferromagnetic.

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[An61] P.W. Anderson - *Localized magnetic states in metals*, Physical Review, Vol. 124, n. 1, p. 41-53, 1961, doi:10.1103/PhysRev.124.41.

[An80] N. Andrei - *Diagonalization of the Kondo Hamiltonian*, Physical Review Letters, Vol. 45, n. 5, 1980, doi:10.1103/PhysRevLett.45.379.

[BGJ15] G. Benfatto, G. Gallavotti, I. Jauslin - *Kondo effect in a Fermionic hierarchical model*, to appear in the Journal of Statistical Physics, 2015, doi:10.1007/s10955-015-1378-7.

[Ko64] J. Kondo - *Resistance minimum in dilute magnetic alloys*, Progress of Theoretical Physics, Vol. 32, n. 1, 1964, doi:10.1143/PTP.32.37.